Weak reducibilities

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Lowness properties

A lowness property of a set specifies a sense in which the set is computationally weak. Usually this means that it is not very useful as an oracle.

We require that such a property be closed downward under Turing reducibility; in particular it only depends on the Turing degree of the set.

If a set is computable then it satisfies any lowness property. A set that satisfies a lowness property can be thought of as almost computable in a specific sense.

Examples:

$A' \leq_T \emptyset'$ (usual lowness)

$A' \leq_{tt} \emptyset'$ (superlowness)

Highness properties

Highness properties say that the set is computationally strong. They are closed upward under Turing reducibility. If a set satisfies a highness property it is almost Turing above $\emptyset'$ in a specific sense.

Examples:

$C' \geq_T \emptyset''$ (usual highness)

$C' \geq_{tt} \emptyset''$ (superhighness)

General framework

Lowness and highness properties are often dual to each other.

We suggest a more general framework for such pairs of dual properties.

A reducibility is a preordering on $2^{\mathbb{N}}$ that specifies a way to compare sets with regard to their computational complexity.

We will introduce a notion of weak reducibility $\leq_W$.

Such a reducibility determines a lowness property $A \leq_W \emptyset$ and a dual highness property $C \geq_W \emptyset'$.

For instance, we could define $A \leq_W B$ iff $A' \leq_T B'$, or $A \leq_W B$ iff $A' \leq_{tt} B'$, to get the examples above.

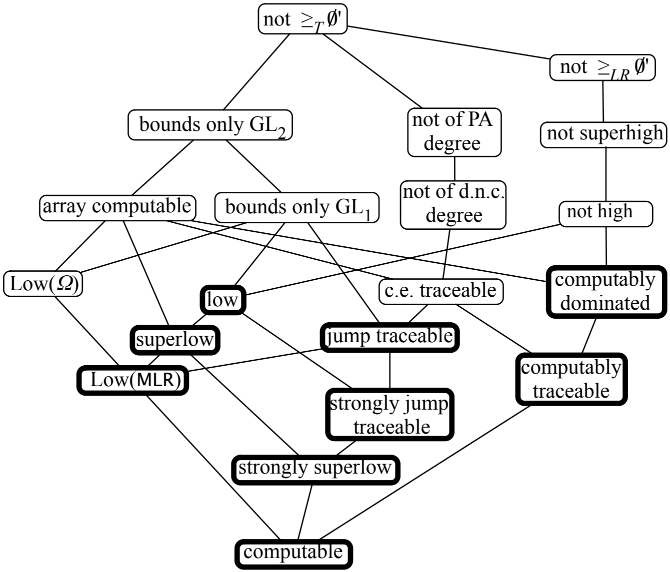
Definition of weak reducibility

A reducibility $\leq_W$ is weak if

- $A \leq_T B$ implies $A \leq_W B$ (as opposed to strong reducibilities like $\leq_{tt}$ that imply $\leq_T$).
- $\leq_W$ is Σ^0_n for some n as a relation on sets (often $n = 3$)
- $X' \not\leq_W X$ for each set X (so the lowness and highness properties are disjoint).

Thus, we want $\leq_W$ to be somewhat close to $\leq_T$; for instance, arithmetical reducibility, defined by $X \leq_{ar} Y \leftrightarrow \exists n \, X \leq_T Y^{(n)}$, does not qualify. Neither does enumeration reducibility.
In general, there are no reduction procedures for a weak reducibility.

The bold-framed properties are given by weak red's



Implications of the weak reducibilities

The inclusions of lowness properties in the diagram extend to inclusions of the weak reducibilities,
with the exception $A \leq_{LR} B \not\Rightarrow A' \leq_{tt} B'$
and the possible exception of $\leq_{SSL} \overset{?}{\Rightarrow} \leq_{SJT}$.

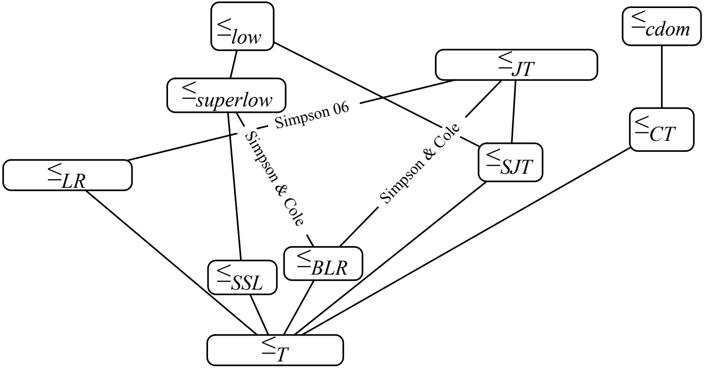


Table of some weak reducibilities

Weak reducibility	Lowness property	Highness prop.
$\leq_T$	computable	$\geq_T \emptyset'$
$\leq_{LR} \Leftrightarrow \leq_{LK}$	$\mathrm{Low(MLR)}$ = low for K	u.a.e.d
$\leq_{JT}$ (jump traceable by)	jump traceable	$\geq_{JT} \emptyset'$
$A' \leq_{tt} B'$	superlow	superhigh
$A' \leq_T B'$	low	high
$\leq_{CT}$	comp. traceable	$\geq_T \emptyset'$
$\leq_{\mathrm{cdom}}$	comp. dominated	$\geq_T \emptyset'$
$\leq_{BLR}$ (Cole & Simpson)	jump tr. & superlow	$\geq_{JT} \emptyset'$ & superhig

For instance, $A \leq_{\mathrm{cdom}} B$ if each A-computable function is dominated by a B-computable function.

BLR reducibility

Cole and Simpson, JML, to appear

- $BLR(X)$= class of functions with an X-recursive approximation, and the number of changes recursively bounded.
- $A \leq_{BLR} B$ if $BLR(A) \subseteq BLR(B)$.
- They show
  $(\star)$ $\leq_{BLR}$ implies both $\leq_{JT}$ and $\leq$ *superlow*.
- For the lowness property they show equality. Simpson also proved equality for the highness property. The converse of $(\star)$ is open.

Directions of study

- The usual degree theoretic questions (e.g. existence of minimal degrees, or minimal pairs)
- cardinality of single degrees/lower cones. For instance each LR degree countable (Nies/Miller) while LR lower cone below $\emptyset'$ (and in fact below each non-GL_2) is uncountable (Barmpalias, Lewis, Soskova). Note that each weak reducibiblity degree structure has cardinality $\geq \omega_1$.
- Apply this to randomness. For instance, see whether
$$A \leq_{CT} B \Leftrightarrow \mathrm{SR}^B \subseteq \mathrm{SR}^A.$$
Here SR^X is the set of Schnorr random "reals" relative to X.

JT reducibility

Recently a question on separating highness properties was answered by using the weak reducibility $\leq_{JT}$.

Definition

(Simpson) A is jump traceable by B, written $A \leq_{JT} B$, if there is a c.e. trace $(T_e)_{e\in\mathbb{N}}$ relative to B for J^A, and an order function h such that $\#T_e \leq h(e)$ for each e.

Being jump traceable *by B* is somewhat different from being jump traceable *relative* to B because we only require the existence of a c.e. trace for the function J^A, not for $J^{A\oplus B}$; on the other hand, the bound for this trace must be computable, not merely computable in B. This “partial relativization” is typical.

$\leq_{JT}$ is a weak reducibility

It is not hard to show that $\leq_{JT}$ is a Σ^0_3 relation on sets, that $A \leq_T B$ implies $A \leq_{JT} B$, and that $A' \not\leq_{JT} A$.

Fact

The relation $\leq_{JT}$ is transitive.

Proof. Suppose A is jump traceable by B via a trace $(S_n)_{n\in\mathbb{N}}$ with computable bound g, and B is jump traceable by C via a trace $(T_i)_{i\in\mathbb{N}}$ with a computable bound h. There is a computable function β such that

$$J^B(\beta(\langle n,k\rangle)) \simeq \text{the } k\text{-th element enumerated into } S_n.$$

Let $V_n = \bigcup_{k<g(n)} T_{\beta(\langle n,k\rangle)}$, then $\#V_n \leq g(n)\cdot h(\beta(\langle n, g(n)\rangle))$ and A is jump traceable by C via the trace $(V_n)_{n\in\mathbb{N}}$. □

Transitivity can be non-trivial to show. For instance, it also works for computable traceability, but not for c.e. traceability.

Separating highness properties

- Cole and Simpson asked whether
  C superhigh $\Leftrightarrow \emptyset'$ is jump traceable by C.
- The answer is NO.
- Mohrherr 84: for every set $A \geq_{tt} \emptyset'$ there is a set C such that $C' \equiv_{tt} A$. The construction makes C jump traceable as noted by Kjos Hanssen .
- If $A = \emptyset''$, this jump inversion for $\equiv_{tt}$ yields a superhigh jump traceable set C.
- Thus $C \leq_{JT} \emptyset$ and hence $\emptyset' \not\leq_{JT} C$.

Two highness properties

$$\begin{aligned} LRH &= \{C \colon \emptyset' \leq_{LR} C\} = \text{u.a.e.d. sets.}, \\ JTH &= \{C \colon \emptyset' \leq_{JT} C\}. \end{aligned}$$

JTH is properly contained in the class of superhigh sets.
However, the two latter classes coincide on the Δ^0_2 sets (Cole and Simpson, extending “Reals which compute little” by Nies).
There is a superhigh Δ^0_2 (even c.e.) set C such that $C <_{LR} \emptyset'$ (pseudo jump inversion)
Hence *LRH* is a proper subclass of *JTH*.

Diamond operator

For a class $\mathcal{H} \subseteq 2^{\mathbb{N}}$ let

$$\mathcal{H}^{\Diamond} = \{A\colon\ A \text{ is c.e. } \& \ \forall Y \in \mathcal{H} \cap \text{MLR}\, [A \leq_T Y]\}.$$

Here MLR is the set of Martin-Löf random sets.
If $\mathcal{H}$ is a null Σ^0_3 class, then $\mathcal{H}^{\Diamond}$ contains a promptly simple set (Hirschfeldt and Miller).
So there is a promptly simple set in $JTH^{\Diamond} \subseteq LRH^{\Diamond}$.

Subclasses of the c.e. K-trivials

Theorem

Consider the following properties of a c.e. set A.

(i) $A \in JTH^{\Diamond}$;

(ii) $A \in LRH^{\Diamond}$;

(iii) *A is ML-coverable, namely, there is a ML-random set $Z \geq_T A$ such that $\emptyset' \not\leq_T Z$;*

(iv) *For each ML-random set Z, if $\emptyset' \leq_T A \oplus Z$ then $\emptyset' \leq_T Z$;*

(v) *A is K-trivial.*

The following implications are known: (i)$\Rightarrow$ (ii)$\Rightarrow$(iii)$\Rightarrow$(v); (ii)$\Rightarrow$(iv)$\Rightarrow$(v).

All these classes are closed downward under $\leq_T$ within the c.e. sets. The classes given by (i), (ii) and (v) are even known to be ideals.

Strong jump traceability beats it all

$JTH^\Diamond$ is the "smallest" class known to contain a promptly simple that may coincide with the c.e. K-trivials.
It actually contains quite a bit.

Theorem

(Greenberg and Nies) Each strongly jump traceable c.e. set A is in $JTH^\Diamond$.

We first prove that each strongly jump traceable c.e. set A obeys each benign cost function (a generalization of the standard cost function used to build a K-trivial.) Then we find a benign c.f. for being in $JTH^\Diamond$.

Note that high$^\Diamond$= computable because $\Omega^{\emptyset'}$ is high and 2-random, so $\Omega^{\emptyset'}$ and Ω form a minimal pair.

Is the class of superhigh sets Σ^0_3?

This would provide an affirmative answer to:

Is there an incomputable set in superhigh$^\Diamond$?

Even if superhigh is not Σ^0_3, it could be that

$$\text{superhigh} \Leftrightarrow \textit{JTH} \text{ for ML-random sets.}$$

I rather expect that superhigh$^\Diamond$ = computable.

References

- Steve Simpson papers
- My book Computability and Randomness, submitted to OUP; available by request
- Forthcoming paper with Greenberg.