New directions in computability and randomness

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Luminy workshop 2009

Some areas of possible future research

- Cost functions
- Classes related to the K-trivials
- Partial relativization and weak reducibilities
- Randomness notions between 2-random and 1-random (such as Demuth random, or Schnorr random relative to $\emptyset'$)
- Randomness via computational complexity theory
- Randomness via higher descriptive set theory
- Randomness/lowness with infinite objects other than subsets of $\mathbb{N}$ (such as continuous functions)

Part 1

Cost functions

What cost functions are good for

Cost functions are a great tool for analyzing certain classes of Δ^0_2 sets.
Mostly, these classes are lowness properties such as being K-trivial, or strongly jump traceable.

Cost functions help a lot to understand the following results:

- Each K-trivial set is Turing below a c.e. K-trivial set (Nies).
- Each null Σ^0_3 class of ML-random sets has a simple Turing lower bound. Moreover, this lower bound is obtained via an injury-free construction (Hirschfeldt, Miller).
- Each strongly jump traceable c.e. set is Turing below each ω-c.e. ML-random set (Greenberg, Nies).

Definition of cost functions

Definition 1

A cost function is a computable function

$$c : \mathbb{N} \times \mathbb{N} \rightarrow \{x \in \mathbb{Q} \colon \; x \geq 0\}.$$

We view $c(x, s)$ as the cost of changing $A(x)$ at stage s.

Obeying a cost function

Recall that A is Δ^0_2 iff $A \le_T \emptyset'$ iff $A(x) = \lim_s A_s(x)$ for a computable approximation $(A_s)_{s\in\mathbb{N}}$ (Limit Lemma).

Definition 2

The computable approximation $(A_s)_{s\in\mathbb{N}}$ obeys a cost function c if

$$\infty > \sum_{x,s} c(x,s)\, [\![x < s \ \&\ x \text{ is least s.t. } A_{s-1}(x) \neq A_s(x)]\!].$$

We write $A \models c$ (A obeys c) if some computable approximation of A obeys c.

Usually we use this to construct some auxiliary object of finite "weight", such as a bounded request set (aka Kraft-Chaitin set), or a Solovay test.

Basic existence theorem

For a cost function $c : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$, let $\widehat{c}(x) = \sup_s c(x, s)$.
We say that c has the limit condition if $\lim_x \widehat{c}(x) = 0$.

Theorem 3 (Various authors)

If a cost function c satisfies the limit condition, then some (promptly) simple set A obeys c.

Proof. Let W_e be the e-th c.e. set. If W_e is infinite we want some $x \in W_e$ to enter A. We define a computable enumeration $(A_s)_{s\in\mathbb{N}}$ as follows. Let $A_0 = \emptyset$. For $s > 0$,

$A_s = A_{s-1} \cup \{x : \exists e$

$W_{e,s} \cap A_{s-1} = \emptyset$	We haven't met e-th simplicity requirement.
$x \in W_{e,s}$	We can meet it via x.
$x \geq 2e$	We make A co-infinite.
$c(x, s) \leq 2^{-e}\}$.	We ensure that A obeys c.

□

Cost function characterization of the K-trivials

The standard cost function $c_{\mathcal{K}}$ is given by

$$c_{\mathcal{K}}(x, s) = \sum_{i=x+1}^{s} 2^{-K_s(i)}.$$

Theorem 4 (Nies 05)

A is K-trivial $\Leftrightarrow$
some computable approximation of A obeys $c_{\mathcal{K}}$.

Corollary 5

For each K-trivial A there is a c.e. K-trivial set $D \geq_T A$.

D is the change set $\{\langle x, i\rangle : A(x) \text{ changes at least } i \text{ times}\}$.
One verifies that D obeys $c_{\mathcal{K}}$ as well.
Actually, this works for any cost function in place of $c_{\mathcal{K}}$.

Proving Kučera's Theorem with a cost function

Theorem 6 (Kučera 1986)

Suppose Y is a ML-random Δ^0_2 set. Then some promptly simple set A is Turing below Y.

The proof can be phrased in the language of cost functions.
Let $c_Y(x,s) = 2^{-x}$ for each $x \geq s$. If $x < s$, and $e < x$ is least such that $Y_{s-1}(e) \neq Y_s(e)$, let

$$c_Y(x,s) = \max(c_Y(x,s-1), 2^{-e}).$$

Since Y is Δ^0_2, the cost function c_Y satisfies the limit condition.

Proposition 7 (Greenberg and Nies)

If the Δ^0_2 set A obeys c_Y, then $A \leq_T Y$ with use function bounded by the identity.

Some promptly simple A obeys c_Y. So $A \leq_T Y$.

Strongly jump traceable sets

- A computably enumerable trace with bound h is a uniformly computably enumerable sequence $(T_x)_{x \in \mathbb{N}}$ such that $|T_x| \leq h(x)$ for each x.
- Let $J^A(e)$ be the value at e of a universal A-partial computable function. (For instance, let $J^A(e) \simeq \Phi_e^A(e)$ where Φ_e is the e-th Turing functional.)
- The set A is called strongly jump traceable if for each order function h, there is a c.e. trace $(T_x)_{x \in \mathbb{N}}$ with bound h such that, whenever $J^A(x)$ it is defined, we have

$$J^A(x) \in T_x$$

- *SJT* will denote the class of c.e. strongly jump traceable sets.
- There is an incomputable set in *SJT* by Figueira, Nies, Stephan (2004).

Open questions on cost functions I

Usually we are given a class $\mathcal{D}$ and a cost function c such that $A \models c \Rightarrow A \in \mathcal{D}$. The question is what else is in $\mathcal{D}$.

Question 8

Let Y be a ML-random Δ^0_2 set.

- *Is there a c.e. $D \leq_T Y$ such that $D \not\leq_{wtt} Y$ (and hence $D \not\models c_Y$)?*
- *If $B \leq_T Y$ is c.e., is there a c.e. $A \models c_Y$ such that $B \leq_T A$?*

Question 9

Let Y be a Demuth random Δ^0_2 set. If A is c.e. and $A \leq_T Y$, is A strongly jump traceable?

We have shown that $A \models c_Y \Rightarrow A$ is s.j.t.

Some open questions on cost functions II

There is a ML-random Δ^0_2 set Y such that any c.e. $A \leq_T Y$ is s.j.t. (Greenberg, Hirschfeldt, Nies, to appear). The following would be stronger.

Question 10

Let c be a c.f. with the limit condition. Is there a ML-random Δ^0_2 *set* Y *such that for each c.e. set* A*, if* $A \leq_T Y$ *then* $A \models c$*?*

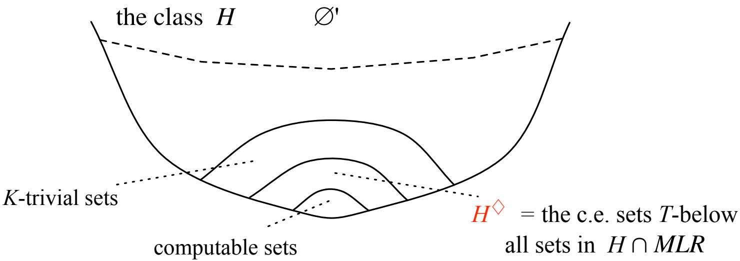
Part 2

Classes related to the K-trivials

Diamond Classes

$2^{\mathbb{N}}$ denotes Cantor space with the uniform (coin-flip) measure.
For a null class $\mathcal{H} \subseteq 2^{\mathbb{N}}$, we define

$\mathcal{H}^{\Diamond}$ = the c.e. sets A Turing below each ML-random set in $\mathcal{H}$.



- The larger $\mathcal{H}$ is, the smaller is $\mathcal{H}^{\Diamond}$.
- $\mathcal{H}^{\Diamond}$ induces an ideal in the computably enumerable Turing degrees.

A lowness property and its dual highness property

- Recall that $Z \subseteq \mathbb{N}$ is low if $Z' \leq_T \emptyset'$, and Z is high if $\emptyset'' \leq_T Z'$.
- These classes are "too big": we have

$$\text{(low)}^\diamondsuit = \text{(high)}^\diamondsuit = \text{computable}.$$

  (For instance, (high)$^\diamondsuit$= computable because there is a minimal pair of high ML-random sets.)
- So we will try somewhat smaller classes, replacing $\leq_T$ by the stronger truth-table reducibility $\leq_{tt}$.

Definition 11 (Mohrherr 1986)

A set Z is superlow if $Z' \leq_{tt} \emptyset'$. Z is superhigh if $\emptyset'' \leq_{tt} Z'$.

A random set can be superlow (low basis theorem). It can also be superhigh but Turing incomplete (Kučera coding).

These diamond classes characterize SJT

The following theorems say that a c.e. set A is strongly jump traceable iff it is computed, in a specific sense, by many ML-random oracles.

Theorem 12 (Greenberg, Hirschfeldt and Nies (to appear))

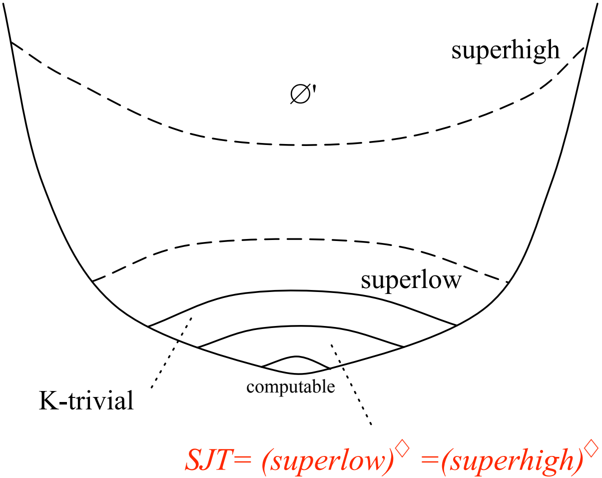
$$\text{SJT} = \text{superlow}^{\Diamond}.$$

That is, a c.e. set A is strongly jump traceable $\Leftrightarrow$ A is Turing below each superlow ML-random set.

Theorem 13 (Nies, improved version in G'berg, H'feldt, N)

$$\text{SJT} = \text{superhigh}^{\Diamond}.$$

Diagram: *SJT* means computed by many oracles



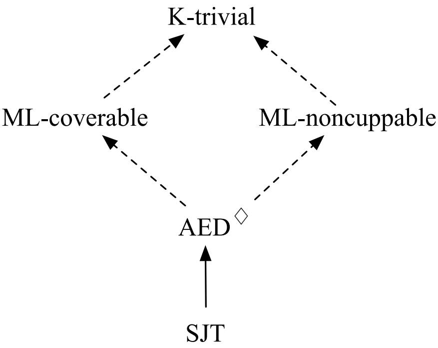
Open questions on classes between *SJT* and *K*-trivial

- No natural classes are currently known to lie properly between *SJT* and *K*-trivial
- A good candidate is (AED)$^{\Diamond}$. Here AED is the class of almost everywhere dominating sets Z of Dobrinen and Simpson: for almost all sets X, each function $f \leq_T X$ is dominated by a function $g \leq_T Z$. For the highness properties, there are proper implications

$$\text{Turing-complete} \Rightarrow \text{AED} \Rightarrow \text{superhigh.}$$

- For the corresponding diamond classes, Greenberg and Nies proved that *SJT* is properly contained in (AED)$^\Diamond$.
- They built a single benign cost function c such that $A \models c$ implies $A \in$(AED)$^\Diamond$.
- However, (AED)$^\Diamond$ may coincide with K-trivial.
- This would imply that the classes ML-coverable and ML-noncuppable also coincide with K-trivial.

Classes of c.e. sets between SJT and K-trivial



(The dashed arrows may be coincidences.)

- A is ML-coverable if $A \leq_T Y$ for some ML-random $Y \not\geq_T \emptyset'$.
- A is ML-noncuppable if
  $\emptyset' \leq_T A \oplus Y$ for ML-random Y implies $\emptyset' \leq_T Y$.

Some references for Parts 1 and 2

Downey and Greenberg. Each SJT is K-trivial.

N. Greenberg and A. Nies. Benign cost functions and lowness properties. Submitted.

N. Greenberg, D. Hirschfeldt and A. Nies. Characterizing the strongly jump-traceable sets via randomness. To appear.

A. Nies. Calculus of cost functions. To appear.

A. Nies. Computability and randomness, Oxford, 2009. Sections 5.3, 8.4, 8.5.

Part 3

Partial relativization and weak reducibilities

Partial relativization

Let $\mathcal{C} \subseteq 2^{\mathbb{N}}$ be a relativizable class (for instance, a lowness property, saying that a set is close to being computable).
Usually the relation "$A \in \mathcal{C}^B$" is not transitive.
For instance, if $\mathcal{C}$ is the usual lowness $A' \leq_T \emptyset'$, we have

$$\mathcal{C}^B = \{A\colon\ (A \oplus B)' \leq_T B'\}.$$

Take low sets A, D such that $\emptyset' \equiv_T A \oplus D$, then

$$A \in \mathcal{C}^{\emptyset},\ \emptyset \in \mathcal{C}^D, \text{ but } A \notin \mathcal{C}^D.$$

To obtain transitivity, one relativizes only partially.
We will say that A has property $\mathcal{C}$ by B, or plop B.
For instance, if $\mathcal{C}$ is lowness we have

$$\mathcal{C}^{by\ B} = \{A\colon\ A' \leq_T B'\},$$

and the relation "$A \in \mathcal{C}^{by\ B}$" is transitive.

Dictionary

plop |pläp|

noun

a short sound as of a small, solid object dropping into water without a splash.

verb (**plopped** , **plop•ping**)

fall or cause to fall with such a sound : [intrans.] *the stone plopped into the pond* [trans.] | *she plopped a sugar cube into the cup.*

• (**plop oneself down**) sit or lie down gently but clumsily : *he plopped himself down on the nearest chair.*

ORIGIN early 19th cent.: imitative.

Definition of weak reducibility

A preordering $\le_W$ on $2^{\mathbb{N}}$ is called weak reducibility if

- $\le_W$ is Σ^0_n for some n as a relation on sets;
- $A \le_T B$ implies $A \le_W B$;
- $X' \not\le_W X$ for each set X.

The idea is that B does not know everything about A (as in the case of $A \le_T B$), only a certain aspect of what A can do.
For instance, let
$A \le_{\mathrm{cdom}} B \Leftrightarrow$ each A-computable function is dominated by a B-computable function. B knows how quickly the functions computed by A grow.
A further example: $A \le_{LK} B \Leftrightarrow \forall x\ K^B(x) \le^+ K^A(x)$.

How a relativizable class yields weak reducibilities

Given a relativizable class $\mathcal{C}$, there are two ways to obtain a weak reducibility $A \leq_W B$:

- $A \leq_W B \Leftrightarrow A \in \mathcal{C}^{by\,B}$,
  for the right type of partial relativization.
- $A \leq_W B \Leftrightarrow \mathcal{C}^A \subseteq \mathcal{C}^B$.
  That is, apply the first to the class $\{X \colon \mathcal{C}^X \subseteq \mathcal{C}\}$.

Associated lowness and highness properties

For each weak reducibility $\leq_W$ we have

- a lowness property $Z \leq_W \emptyset$,
- a highness properties $\emptyset' \leq_W Z$.

They are disjoint by the last condition.

An further example of a weak reducibility due to Nies (2005) is

$$A \leq_{LR} B \Leftrightarrow \text{ each } B\text{-random set is } A\text{-random.}$$

- The associated lowness property is being low for random.
- The highness property is equivalent to being uniformly almost everywhere dominating, by Kjos-Hanssen, Miller, Solomon (to appear).

Jump traceability

- J^X denotes the universal pc functional with oracle X. An order function is a nondecreasing, unbounded, computable function.
- A computably enumerable trace with bound h is a uniformly computably enumerable sequence $(T_x)_{x \in \mathbb{N}}$ such that $|T_x| \leq h(x)$ for each x.
- A is called jump traceable if there is a c.e. trace $(T_e)_{e \in \mathbb{N}}$ for J^A, and an order function h such that $|T_e| \leq h(e)$ for each e.

Plopping jump traceability

Definition 14 (Simpson, 2006, implicitly)

A is jump traceable plop B, written $A \leq_{JT} B$, if there is a c.e. trace $(T_e)_{e \in \mathbb{N}}$ relative to B for J^A, and an order function h such that $|T_e| \leq h(e)$ for each e.

In contrast, to define jump traceable relative to B, one would require the existence of a B-c.e. trace for $J^{A \oplus B}$ instead of J^A, but the bound for this trace need only be computable in B.

The rules of thumb for plopping successfully:

- write A instead of $A \oplus B$ (in the right places)
- leave computable bounds in peace.

The weak reducibility $\leq_{JT}$

It is not hard to show that $\leq_{JT}$ is a Σ^0_3 relation on sets, that $A \leq_T B$ implies $A \leq_{JT} B$, and that $A' \not\leq_{JT} A$.

Proposition 15

The relation $\leq_{JT}$ is transitive.

Proof. Suppose A is jump traceable by B via a trace $(S_n)_{n\in\mathbb{N}}$ with computable bound g, and B is jump traceable by C via a trace $(T_i)_{i\in\mathbb{N}}$ with a computable bound h. There is a computable function β such that

$$J^B(\beta(\langle n,k\rangle)) \simeq \text{the } k\text{-th element enumerated into } S_n.$$

Let $V_n = \bigcup_{k<g(n)} T_{\beta(\langle n,k\rangle)}$, then $\# V_n \leq g(n) \cdot h(\beta(\langle n, g(n)\rangle))$ and A is jump traceable by C via the trace $(V_n)_{n\in\mathbb{N}}$. □

Which theorems/proofs survive a partial relativization?

Theorem 16 (Nies 05)

Lowness for ML-randomness is the same as lowness for prefix-free complexity K.

This becomes: $\leq_{LR}$ is equivalent to $\leq_{LK}$,
by Kjos, Miller, Solomon, to appear. A different proof is needed, though.

Theorem 17 (Figueira, N, Stephan 07)

Let C be plain descriptive string complexity. Then A is jump traceable $\Leftrightarrow \forall x \left[C(x) \leq^+ C^A(x) + h(C^A(x))\right]$ for some order function h.

Plopping the proof, this becomes:
$A \leq_{JT} B \Leftrightarrow$
$\forall x \left[C^B(x) \leq^+ C^A(x) + h(C^A(x))\right]$ for some order function h.

A theorem that cannot be plopped

Theorem 18 (Nies 2002)

Let A be c.e. Then
A is jump traceable $\Leftrightarrow$ A is superlow (i.e., $A' \leq_{tt} \emptyset'$).

Let $A = \emptyset'$ and try to plop this theorem to a set B. We have

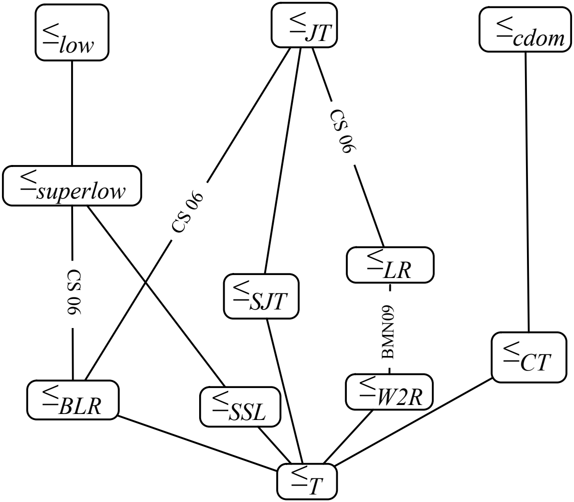
$$\emptyset' \leq_{JT} B \;\Rightarrow\; \emptyset'' \leq_{tt} B' \text{ (B is superhigh)}$$

by a result of Simpson.
The converse direction, however, fails: there is a superhigh jump traceable set B (Kjos-H and Nies). Then $B \not\geq_{JT} \emptyset'$.

Table of some weak reducibilities

Weak reducibility	Lowness property	Highness prop.
$\leq_T$	computable	$\geq_T \emptyset'$
$\leq_{LR} \Leftrightarrow \leq_{LK}$	$\mathrm{Low}(\text{MLR}) =$ low for K	u.a.e.d
$\leq_{JT}$	jump traceable	$\geq_{JT} \emptyset'$
$A' \leq_{tt} B'$	superlow	superhigh
$A' \leq_T B'$	low	high
$\leq_{CT}$	comp. traceable	$\geq_T \emptyset'$
$\leq_{\mathrm{cdom}}$	comp. dominated	$\geq_T \emptyset'$
$\leq_{BLR}$ (Cole & Simpson)	jump tr. & superlow	$\geq_{JT} \emptyset'$

Diagram of weak reducibilities



Results of G. Barmpalias, Selwyn Ng

We say that A is a base for $\leq_W$ if $A \leq_W Z$ for some Z that is ML-random relative to A.

- (Ng) For both $\leq_{SJT}$ and $\leq CT$, the cone below $\emptyset'$ has size continuum.
- (Ng) The only bases for $\leq_{JT}$ are the jump traceable sets.
- (Barmpalias) Some set A is a base for $\leq_{LR}$ but not low for randomness.

Directions of study for weak reducibilities

- Degree theoretic questions: existence of minimal degrees, of minimal pairs, of nontrivial suprema.
- The cardinality of single degrees, and lower cones. Each LR degree countable (Nies 2005 showed this for $\leq_{LK}$. Now use Kjos/Miller/Solomon that $\leq_{LK} \Leftrightarrow \leq_{LR}$), while the LR lower cone below $\emptyset'$ (and in fact below each non-GL_2) is uncountable (Barmpalias, Lewis, Soskova).
- Implications between weak redu's. For instance, does $\leq_{SJT}$ imply $\leq_{LR}$?
- Theory of Borel equivalences. For instance, is $\equiv_{LR}$ Borel complete for countable Borel equivalence relations?

Some references for Part 3

Barmpalias et al. papers on $\leq_{LR}$

G. Barmpalias, J. Miller, A. Nies. Randomness notions and partial relativization. To appear.

S. Ng, Thesis.

A. Nies. Computability and randomness, Oxford, 2009. Section 8.4

Part 4

Randomness lower down

Complexity theory: k-traces

Definition 19

Let $k \geq 1$. A *k-trace* is a sequence $(T_x)_{x \in \Sigma^*}$ of subsets of Σ^* such that

- $|T_x| = k$ for each x
- The function $x \to$ (code for) T_x is in P.

$(T_x)_{n \in \mathbb{N}}$ is a trace for the function $f\colon \Sigma^* \to \Sigma^*$ if $f(x) \in T_x$ for each x.
We say that A is *k-traceable* if each function $f \in \mathrm{P}^A$ has a k-trace.

Supersparse sets are 2-traceable

Definition 20 (Ambos-Spies 1986)

Let $f : \mathbb{N} \mapsto \mathbb{N}$ be a strictly increasing, time constructible function. A is f-super sparse if

- $A \subseteq \{0^{f(i)} : i \in \mathbb{N}\}$
- Some machine determines $A(0^{f(i-1)})$ in time $O(f(i))$.

Let f be the iteration of the function $n \rightarrow 2^n$. Ambos-Spies constructed an f-supersparse set in EXPTIME $-$ P.

Theorem 21 (Ambos-Spies 1986)

Each f-supersparse set is 2-traceable.

Question 22

Is each k-traceable set low for polynomial randomness [polynomial Schnorr randomness]?