Lowness properties and highness properties

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References

- September BSL: Survey "Calibrating randomness" (DHNT) and "Computability and randomness: Open questions" (Miller/N)
- My survey "Eliminating concepts" in IMS Singapore volume: concentrates on K-trivials and lowness properties
- Upcoming books by Downey/Hirschfeldt (Algorithmic Randomness and Complexity, Springer) and by Nies (Computability and Randomness, OUP); available on respective web sites. My one at

  http://www.cs.auckland.ac.nz/nies/Niesbook.pdf.

$K(y)$

- A *machine* is a partial recursive function $M : \{0,1\}^* \mapsto \{0,1\}^*$.
- M is *prefix free* if its domain is an antichain under inclusion of strings.

Let $(M_d)_{d\geq 0}$ be an effective listing of all prefix free machines. The is given by

$$U(0^d 1\sigma) = M_d(\sigma).$$

Let U^B be the standard universal prefix free oracle machine. $U^B(\sigma) = y$ means σ is a description of y with oracle B.

Let

$$K^B(y) = \min\{|\sigma| : U^B(\sigma) = y\}.$$

Thus, $K^B(y)$ is the length of a shortest prefix free description of y, with oracle B.

K-trivial sets

A set A is K-trivial if there is $c \in \mathbb{N}$ such that

$$\forall n \; K(A \upharpoonright n) \leq K(0^n) + c$$

(Chaitin, 1975).

- By Schnorr's theorem, Z is ML-random if for each n, $K(Z \upharpoonright n)$ is near its maximal value $n + K(0^n)$.
- K-trivial means far from ML-random, because $K(A \upharpoonright n)$ is minimal (all up to constants).

Solovay (1976) constructed a noncomputable K-trivial.

Improving this, the cost function construction gives a noncomputable c.e. example. In fact, promptly simple.

Cost function construction

Downey, Hirschfeldt, Nies, Stephan 2002 have given a short "definition" of a noncomputable c.e. K-trivial set, which had been anticipated by various researchers (Kummer, Zambella). We use the "cost function"

$$\boxed{c(x, s) = \sum_{x<y\leq s} 2^{-K_s(y)}}.$$

This determines a non-computable set A:

$A_s = A_{s-1} \cup \{x : \exists e$

$W_{e,s} \cap A_{s-1} = \emptyset$	we haven't met e-th non-computability requirement
$x \in W_{e,s}$	we can meet it, via x
$x \geq 2e$	make A co-infinite
$c(x, s) \leq 2^{-(e+2)}\}$	ensure A is K-trivial.

Properties of the K-trivials

The K-trivial sets are all Δ^0_2 (Chaitin 76) and closed under $\oplus$ (DHNS, 2002). Hirschfeldt proved they are Turing incomplete.

(N, 2004) shows: the K trivial sets form an ideal $\mathcal{K}$, with the following properties

- $\mathcal{K}$ is the downward closure of its c.e. members
- each $A \in \mathcal{K}$ is super-low: $A' \leq_{\mathrm{tt}} \emptyset'$.

Equivalent properties

The following are equivalent to K triviality of A:

- low for ML-random (introduced by Zambella 1990): $\mathsf{MLRand}^A = MLR$
- basis for ML-random (introduced by Kucera 1987): $A \leq_T Z$ where Z is in MLRand^A
- low for K (Muchnik 1995): $\exists b \, \forall y \; K^A(y) \geq K(y) - b$.

K-trivial $\Rightarrow$ low for K is hardest. Decanter method, with golden run.

Question 1 *Study K-trivial for time bounded versions of K.*

Almost complete sets

Now for the new stuff.

- B is almost complete if $\emptyset'$ is K-trivial relative to B.
- That is, there is $c \in \mathbb{N}$ such that
$$\forall n \; K^B(\emptyset' \upharpoonright n) \leq K^B(0^n) + c$$
- We will see that such a set is super-high: $\emptyset'' \leq_{\mathrm{tt}} B'$.
- First I discuss why there exists an almost complete $B <_T \emptyset'$.

Inverting a c.e. operator

Theorem 2 (Jockusch/Shore 1983) *For each c.e. operator W, there is a c.e. set B such that*

$$W^B \oplus B \equiv_T \emptyset'.$$

- Apply this to the c.e. operator $B \mapsto W^B$ given by doing the cost function construction relative to the oracle B,
- in this way, obtain an almost complete c.e. set $B <_T \emptyset'$.

LR reducibility

Let $A \leq_{LR} B$ if

$$\forall Z\ (Z \notin \mathsf{MLRand}^A \ \Rightarrow\ Z \notin \mathsf{MLRand}^B).$$

This means: if A can see that Z is nonrandom, then so can B.

- Clearly $\leq_T \Rightarrow \leq_{LR}$.
- By relativizing a result from Nies 2004, we have

$$B \text{ almost complete } \Leftrightarrow\ \emptyset' \oplus B \leq_{LR} B \ \Rightarrow\ \emptyset' \leq_{LR} B.$$

- In particular, for Δ^0_2 sets B, we have

$$B \text{ almost complete } \Leftrightarrow\ \emptyset' \leq_{LR} B.$$

Almost complete is not closed upwards

J. Miller proved: almost complete is NOT closed upward under $\leq_T$.

- Let $B <_T \emptyset'$ be almost complete such that $\emptyset'$ is promptly simple in B.
- Then it is low cuppable in B, so there is $G \geq_T B$ such that
$$\emptyset' \oplus G \equiv_T B' \equiv_T G'.$$
- But if $\emptyset'$ is K-trivial in G then $\emptyset' \oplus G$ is K-trivial in G, so
$$(\emptyset' \oplus G)' \equiv_T G'.$$

In particular, $\emptyset' \leq_{LR} B \leq_T G$ and hence $\emptyset' \leq_{LR} G$, while G is not almost complete.

How different are those classes?

- Soft question: understand this difference better.
- For instance, Simpson has asked whether the sets $\geq_{LR} \emptyset'$ are the upward closure of the almost complete sets.

Equivalent characterizations of the class of sets $\geq_{LR} \emptyset'$

- Kjos-Hanssen proved that for each B,

$$\emptyset' \leq_{LR} B \;\Leftrightarrow\; B \text{ is positive measure dominating.}$$

- Binns, Kjos-Hanssen, Miller, Solomon have shown recently that pm domination is equivalent to the apparently stronger uniform a.e. domination (both introduced by Dobrinen and Simpson). In the same paper, they prove $\leq_{LR}$ is equivalent to the apparently stronger $\leq_{LK}$, via related methods.

So, just as the K trivials, the class of sets $\geq_{LR} \emptyset'$ has several equivalent characterizations. And for Δ^0_2 sets B, it is even the same as almost complete.

Super-high

Simpson proved

$$\emptyset' \leq_{LR} B \; \Rightarrow \emptyset'' \leq_{\mathrm{tt}} B'.$$

The weaker result that an almost complete B satisfies $\emptyset'' \leq_{\mathrm{tt}} B'$ is in (N 2004).

Question 3 *Study the c.e.* $\leq_{LR}$ *degrees. Are they dense?*

Direct construction of a c.e. uniformly a.e. dominating

Recall that we obtained an almost complete c.e. $B <_T \emptyset'$ via Jockusch-Shore pseudojump inversion. This is great, but a bit indirect, and also not very flexible. For instance, can one avoid the cone above an incomputable c.e. A? (Currently open.)

Cholak, Greenberg and Miller (2005) build a c.e. $B <_T \emptyset'$ that is u.a.e. dominating, using a $\emptyset''$ construction. This B is almost complete.

G. Barmpalias succeeded in making B half of a minimal pair.

Direct construction of a c.e. almost complete

Nies and Shore have a direct construction of a c.e. almost complete $B <_T \emptyset'$. We build KC-set L^B relative to B, and meet the global requirement

$$\forall m \exists \rho \in L^B \ [\rho = \langle K^B(m) + 1, \emptyset' \upharpoonright m \rangle].$$

- ρ is called a request for m. If $\emptyset' \upharpoonright m$ changes, then the request becomes inappropriate. Usually we now change B to get rid of it, but this may be prevented by an incompleteness requirement $N_e : C \neq \Phi_e^B$ (we also enumerate C to make B incomplete).
- We have to minimize the weight of those inappropriate requests we cannot reject. For this, N_e has to obey a cost function.

- N_e acts via a witness x targeted for C, and also a number w. It hopes that $\emptyset' \upharpoonright w+1$ has settled. The cost for L^B of holding all requests for $m > w$ is half of

$$c^B(w)[s] = \sum_{w<m<s} 2^{-K^B(m)[s]}.$$

- N_e is allowed to act if $c^B(w)[s] \leq 3^{-e}$. If $\emptyset' \upharpoonright w+1$ changes after all, then nothing is lost for L^B. Eventually, on the least w where the cost IS low (for the final B), it succeeds.

- N_e also takes over requests previously held by N_j, $j > e$, which is fine because $\sum_{j>e} 3^{-e} = 2^{-e}/2$.

Using this direct construction of an almost complete:

Theorem 4 *There is a c.e. K-trivial A and a c.e. almost complete B such that $A \not\leq_T B$.*

Minimal pairs?

Making a minimal pair of c.e. almost complete sets looks harder. So far, this is open.

The closest result is due to Shore (unpublished):

Theorem 5 *There is a minimal pair* B_0, B_1 *of c.e. sets such that* $\emptyset'' \leq_{\mathrm{tt}} B_i'$ *for both* i.

Note that there is a minimal pair of u.a.e. dominating sets, applying an observation of Simpson to the forcing construction of Cholak, Greenberg and Miller: in fact for any A there is a u.a.e. dominating B which forms a min pair with A, via avoiding the countably many upper cones given by all incomputable sets $C \leq_T A$.

ML-random almost complete

A result analogous to Jockusch-Shore can be proved for ML-randomness.

Theorem 6 (N 2006) *For each c.e. operator W, there is a ML-random set $Z \leq \emptyset'$ such that*

$$W^Z \oplus Z \equiv_T \emptyset'.$$

See Thm. 4.17 in draft of my book, available on the web, finally.

Corollary 7 *There is a ML-random almost complete set $Z <_T \emptyset'$.*

No direct construction is available at this point.

Infima of ML random sets

- There is no minimal pair of almost complete ML random sets
- In fact, some c.e. set is below all of them! (see next slide).
- Also, below any two Δ^0_2 ML-random sets, there is an incomputable c.e. set (Kucera).
- However, there is nothing below *all* the Δ^0_2 ML randoms. (Not even below all high ones, according to Downey +Miller, also claimed previously by Kucera. Proofs not available, unfortunately.)

The class $\mathcal{L}$

We will restrict ourselves to lowness properties involving only c.e. sets A, for a while. Let

$$\mathcal{L} = \{A \ : A \ c.e. \ \& \ \ \forall Z$$
$$Z \text{ ML-random, almost complete } \ \Rightarrow \ A \leq_T Z\}.$$

- Hirschfeldt proved that there is a promptly simple set in $\mathcal{L}$.
- By the previous corollary, each $A \in \mathcal{L}$ is ML-coverable, namely, there is ML-random Z such that
$$A <_T Z <_T \emptyset'.$$
- Each c.e. ML-coverable set is a basis for ML-randomness, and hence K-trivial.
- Thus $\mathcal{L}$ is a subideal of the c.e. K-trivials (possibly equal).

ML-noncuppable

A c.e. set A is ML-cuppable if

$$A \oplus Z \equiv_T \emptyset' \text{ for some ML-random } Z <_T \emptyset'.$$

Being *not* ML-cuppable implies being K-trivial (if A is not K trivial and low, then A cups with the ML-random set $\Omega^A <_T \emptyset'$. If not low, apply Sacks splitting.)

Theorem 8 (N, PAMS 2005) *There is a promptly simple set which is not ML-cuppable.*

$\mathcal{L} \subseteq$ ML-noncuppable

The reason for this inclusion is that each potential ML-random cupping partner Z of a K-trivial A is almost complete:

$$\emptyset' \leq_T A \oplus Z,\ A \in \mathcal{K},\ Z \ \text{ ML-random } \Rightarrow\ Z \text{ almost complete.}$$

This takes a 4-line proof involving the van Lambalgen Theorem (Hirschfeldt 2005).

Is $\mathcal{L}$ a proper subideal of $\mathcal{K}$?

Despite all this, we don't know whether $\mathcal{L}$ is a proper subideal of the c.e. K-trivials. In other words,

Question 9 *Is there a K trivial c.e. A and an almost complete ML random Z such that*

$$A \not\leq_T Z?$$

Lower bounds for Σ^0_3 null classes

The existence proof for $\mathcal{L}$ has been both simplified and generalized. See end of my "Eliminating concepts" paper, Lecture Notes of IMS workshop, 2005.

Theorem 10 (Hirschfeldt, Miller) *Let $\mathcal{C}$ be a Σ^0_3 null class. Then there is a promptly simple A such that $A \leq_T Z$ for each ML-random $Z \in \mathcal{C}$.*

The proof is a simple cost function argument. (If $\mathcal{C}$ is the Π^0_2 class $\{Z\}$, for a ML-random Δ^0_2 set Z, then the proof turns into Kučera's proof that there is a promptly simple A below Z.)

- Apply this to the Σ^0_3 null class $\mathcal{C}$ = almost complete in order to obtain a promptly simple $A \in \mathcal{L}$.
- We can even take the larger class $\mathcal{C} = \{B : \emptyset'$ is jump traceable relative to $B\}$. Here A is jump traceable if there is a c.e. trace (T_e) such that $\Phi_e(e)\downarrow \Rightarrow \; \Phi_e(e) \in T_e$.

  This is the same as super-high for Δ^0_2 sets B.
- Or again, we could take the Σ^0_3 class of sets $\geq_{LK} \emptyset'$ (equivalently, $\geq_{LR} \emptyset'$).

Π^0_2 (or weak 2-) randomness

Z is Π^0_2-random (usually called weakly 2-random) if Z is in no Π^0_2 null class. Each such Z is ML-random. In fact,

Z is Π^0_2-random $\Leftrightarrow$ Z is ML-random and $Z, \emptyset'$ form minimal pair.

"$\Rightarrow$ " is easy.

"$\Leftarrow$": if Z is ML random but not Π^0_2-random, let $\mathcal{C}$ be the null Π^0_2 class showing this. Then there is c.e. incomputable $A \leq Z$.

According to Joe Miller, to be Z ML-random and $Z \mid_T \emptyset'$ is strictly weaker than to be Π^0_2-random.

Low for Π^0_2-randomness

A is low for Π^0_2-randomness if

$$\Pi^0_2\text{-random relative to } A = \Pi^0_2\text{-random.}$$

Theorem 11 (Downey,N,Weber,Yu 2005)

- *There is a c.e. noncomputable set that is low for Π^0_2-randomness*
- *each low for Π^0_2-randomness set is low for K.*

same as K trivial

Theorem 12 (N; Binns, Kjos-H, Miller, Solomon independently) *Each K-trivial set is low for Π^0_2-randomness.*

The two proofs are very different.

- Nies' proof used the golden run machinery
- Miller & friends' proof is measure theoretic. They show low for ML-randomness $\Rightarrow$ low for Π^0_2-randomness

SJT

- Figueira, N, Stephan (2004) introduced the following strengthening of super-lowness:
- For each order function h (that is computable, nondecreasing, unbounded) A' has an approximation that changes at most $h(x)$ times at x.
- This implies strongly jump traceable (defined similarly as a strengthening of jump traceable). For c.e. A, the two notions are equivalent.
- They build a c.e. noncomputable such set, via a construction that resembles the cost function construction.
- A is strongly jump traceable $\Leftrightarrow$ A is "lowly" for C, i.e., for every order function h and almost every x,

$$C(x) \leq C^A(x) + h(C^A(x)).$$

A proper subclass of $\mathcal{K}$

- Downey and Greenberg have proved that each c.e. set strongly jump-traceable set is K-trivial
- Cholak, Downey, Greenberg have proved the c.e. SJT sets form a proper subclass of $\mathcal{K}$.
- it is open if this also holds for Δ_2^0 SJT sets.

Effective descriptive set theory

Π^1_1 sets of numbers are a high-level analog of c.e. sets, where the steps of an effective enumeration are recursive ordinals. Hjorth and Nies (Proc. LMS, ta) have studied the analogs of K and of ML-randomness based on Π^1_1-sets.

- The Kraft-Chaitin theorem and Schnorr's Theorem still hold, but the proofs takes considerable extra effort because of limit stages
- There is a Π^1_1 set of numbers which is K-trivial (in this new sense) and not hyperarithmetic.

Low(ML)=hyperarithmetic

Theorem 13 *If A is low for Π^1_1-ML-random, then A is hyperarithmetic.*

First we show that $\omega_1^A = \omega_1^{CK}$. This is used to prove that A is in fact K-trivial at some $\eta < \omega_1^{CK}$, namely

$$\forall n\ K_\eta(A \restriction n) \leq K_\eta(n) + b.$$

Then A is hyperarithmetic, by the same argument Chaitin used in the c.e. case to show that K-trivial sets are Δ^0_2

Π^1_1-classes

- A class $\mathcal{C} \subseteq 2^\omega$ is Π^1_1 iff there is a functional Ψ such that for each Z, Ψ^Z is a (code for a) linear order with domain ω, and

$$Z \in \mathcal{C} \iff \Psi^Z \text{ wellordered.}$$

- Think of α, the length of Ψ^Z as the stage when Z enters $\mathcal{C}$. Note that

$$\alpha < \omega_1^{CK}(Z) < \omega_1.$$

- $\mathcal{C}$ is Δ^1_1 if $\mathcal{C}$ and $2^\omega - \mathcal{C}$ are Π^1_1.

The ultimate in convenience

Let Γ be a "point class" (such as arithmetical, Δ^1_1, Π^1_1, defined for both sets of numbers and sets of reals)

- Z is Γ ML-random if $Z \notin \bigcap_n U_n$, for any sequence (U_n) of open sets, coded by a set in Γ, such that $\mu U_n \leq 2^{-n}$.
- Z is Γ random (without ML in it) if $Z \notin \mathcal{C}$, for any null class in Γ.

Remark: For Γ= arithmetical and $\Gamma = \Delta^1_1$, those two notions are the same: in fact, for each null Δ^1_1-class $\mathcal{C}$ one can find a Γ test $\{U_i\}_{i \in \mathbb{N}}$ such that $\mathcal{C} \subseteq \bigcap_i U_i$.

We will trace the history of some of those randomness notions.

Δ^1_1 random was suggested by Martin-Löf

- In a little–known paper (1970), Martin-Löf suggested the Δ^1_1-classes of measure 0 as tests: Z is Δ^1_1-random if Z is in no null Δ^1_1-class.
- By the remark above, Π^1_1-ML-random implies Δ^1_1-random.
- Δ^1_1-random is the effective descriptive set theory analog of both computably random and Schnorr random.
- There is a Δ^1_1-random Z of slowly growing initial segment complexity (in sense of $K_{\Pi^1_1}$). Thus Z is not Π^1_1-ML-random.

A very strong randomness notion

Sacks (1990) in Exercise 2.5.IV suggested the Π^1_1 null classes as tests. This is the strongest randomness notion we have seen so far. The exercise was to separate this from Δ^1_1-random.

The exact relationship is:

Z is Π^1_1-random $\Leftrightarrow$ Z is Δ^1_1-random and $Z \not\geq_h \mathcal{O}$ (or equivalently, $\omega_1^Z = \omega_1^{CK}$).

Other facts on Π^1_1-randomness

- Since each Π^1_1-random Z satisfies $\omega_1^Z = \omega_1^{CK}$, the Π^1_1-ML-random set $\Omega_{\Pi^1_1}$ is not Π^1_1-random.
- By Gandy's basis theorem, some strongly random set satisfies $\mathcal{O}^Z \leq_T \mathcal{O}$.
- Analog of van Lambalgen's Theorem

Theorem 14 (Kechris 75; Hjorth,N 2004) *There is a greatest Π^1_1-class $Q \subseteq 2^\omega$ of measure 0. Thus Q is a universal test.*

Associated lowness notions

- Chong, N and Yu: there is a perfect class of sets that are low for Δ^1_1-randomness. In fact each Sacks generic does.
  - Conditions: perfect hyperarithmetical trees,
  - decide: Σ^1_1 statements.
- This contrasts with the Nies result that the only low for computably random sets are the computable ones.
- We show

  low for Δ^1_1 random $\Leftrightarrow$ Δ^1_1 traceable $\Leftrightarrow$ Pi^1_1 traceable.

Low for Π^1_1-randomness

It is not known if there is a nonhyperarithmetical set that is low for Π^1_1-randomness. Recall Z is Π^1_1-random $\Leftrightarrow$ Z is Δ^1_1-random and $Z \not\geq_h \mathcal{O}$.

A is random cuppable if $\exists Z$ Π^1_1-random $A \oplus Z \geq_h \mathcal{O}$.

In very recent work of Harrington, N, Slaman:

A is low for Π^1_1-randomness $\Leftrightarrow$ A is low for Δ^1_1-randomness and A is random noncuppable.