Local Compactness for Computable Polish Metric Spaces is Π^1_1-complete

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Abstract. We show that the property of being locally compact for computable Polish metric spaces is Π^1_1 complete. We verify that local compactness for Polish metric spaces can be expressed by a sentence in $L_{\omega_1,\omega}$.

1 Introduction

Computable model theory is a well-established field of research that studies effectiveness aspects of countable structures and of model-theoretic concepts. Structures occurring in mathematics are often of size the continuum; in particular, separable complete metric spaces (also called Polish) play a central role in analysis, measure theory, and other areas. In order to carry out studies similar to computable model theory in the metric setting, computable Polish metric spaces have been introduced. Recall that a pseudo-metric satisfies symmetry and the triangle inequality, but allows pairs of distinct points to have distance 0.

Definition 1. (i) We represent a Polish metric space as follows. A point $V = \langle v_{i.k} \rangle_{i,k \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ is a *distance matrix* if V is a pseudo-metric on $\mathbb{N}$. Let M_V denote the completion of the corresponding pseudo-metric space. In M_V we have a distinguished dense sequence of points $\langle p_i \rangle$ and present the space by giving their distances.
(ii) Let $\langle \phi_e \rangle$ be an effective listing of the rational-valued partial computable functions. A *computable presentation* of a Polish metric space is a distance matrix as in (i) where $|v_{i,k} - \phi_e(\langle i,k,t \rangle)| \leq 2^{-t}$. We call e an index for the space, and write V_e for the distance matrix given by ϕ_e in case ϕ_e is total.

Melnikov and Nies [2] studied the complexity of isomorphism for compact computable metric spaces. They showed that being compact is a Π^0_3 property of (an index for) a computable metric space, and that the complexity of isomorphism is Π^0_2 within that Π^0_3 class.

They also studied the complexity of a more general class. Recall that a topological space is *locally compact* if every point has a compact neighbourhood. For computable Polish metric spaces, this property is Π^1_1: one has to express that for every point x, there is a positive rational r such that the closed ball B of radius

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r around x is compact (which is Π^0_3 in a Cauchy name for x by relativizing the bound for compactness mentioned above).

Melnikov and Nies [2] asserted in Proposition 9 that being locally compact is Π^1_1 complete. Unfortunately, the proof sketch for the Π^1_1-hardness given there was incorrect (the reduction introduced is not Borel). In this note we give a proof of that result.

One can ask for the descriptive complexity of other important classes of (computable) Polish metric space. As an example we mention connectedness, where the only bound known is the trivial one, namely Π^1_2. To be compact and connected (i.e., a continuum) is Π^0_3 by [2, Proposition 11].

2 Main Result

We need a few well-known facts from topology. Firstly, let X be a Hausdorff space, $Y \subseteq X$ be a subspace, and $K \subseteq Y$. Then K is compact in Y iff K is compact in X. Next, suppose also that Y is dense in X. If V is open in X and $V \cap Y \subseteq K$ where $K \subseteq Y$ is compact, then $V \subseteq K$. (Otherwise, $V - K \neq \emptyset$ is open in X, so that $Y \cap (V - K) \neq \emptyset$, a contradiction.)

Lemma 2. *Let X be a Hausdorff space. Suppose Y is dense in X. If Y is locally compact as a subspace of X, then Y is open in X.*

Proof. Let $z \in Y$. There is compact $K \subseteq Y$ such that

$$\exists V \text{ open in } X \, [z \in V \cap Y \subseteq K].$$

Then $V \subseteq K \subseteq Y$.

Lemma 3. *Let X be a compact space, Y dense in X. Then*

$$Y \text{ is locally compact} \Leftrightarrow Y \text{ is open in } X.$$

Proof. $\Rightarrow$: This follows from previous lemma.

$\Leftarrow$: Let $v \in Y$. As a compact space, X is regular. So there are open sets U, V such that $v \in U$ and $X - Y \subseteq V$. Then $X - V \subseteq Y$ is a compact neighbourhood of v. This shows the lemma.

It is easy for a closed subset of $I_{\mathbb{Q}} = \mathbb{Q} \cap [0, 1]$ to be non-compact: for instance, the set of members of any sequence converging to an irrational is closed but not compact. On the other hand, for any successor ordinal α, the range of an embedding of a countable well-order of type α into $\mathbb{Q}$ is a compact set of Cantor-Bendixson (CB)-rank α. By an index for a computable subset R of $I_{\mathbb{Q}}$ we mean a number e such that ϕ_e, interpreted as a function $I_{\mathbb{Q}} \to \mathbb{N}$, is the characteristic function of S. We write $R = R_e$. The following is a straightforward effectivization of the classic result of Hurewicz from descriptive set theory (e.g. [1, Exercise 27.4]) that the compact subsets of $I_{\mathbb{Q}}$ form a Π^1_1-complete set. For detail on the effective version, see the proof of the main result in [3]. Let $O \subseteq \omega$ denote a Π^1_1 complete set.

Fact 4. *The set of indices for compact computable subsets of $I_{\mathbb{Q}}$ is Π^1_1-complete. Moreover, there is a computable function g such that $R_{g(e)}$ is closed in $I_{\mathbb{Q}}$ for each e, and $e \in O \leftrightarrow R_{g(e)}$ is compact.*

We will effectively assign to a closed subset of $I_{\mathbb{Q}}$ a Polish metric space in order to show:

Theorem 5. *(i) $\{V\colon M_V$ is locally compact $\}$ is properly Π^1_1.*
(ii) $\{i\colon M_{V_i}$ is locally compact $\}$ is a Π^1_1-complete set.

Proof. Write $\mathcal{N} = [0,1] - \mathbb{Q}$ (as this space is homeomorphic to Baire space ${}^{\omega}\omega$). Since $\mathbb{Q}$ is F_σ, by Alexandrov's result (see [1, 3.11]) we have a compatible complete metric on $\mathcal{N}$ given by

$$d(x,y) = |x-y| + \sum_{k=0}^{\infty} \min\{2^{-k-1}, |\frac{1}{|x-q_k|} - \frac{1}{|y-q_k|}|\},$$

where $\langle q_k \rangle_{k\in\mathbb{N}}$ list $I_{\mathbb{Q}}$ without repetitions in some effective way.

For topological space X and sets $S \subseteq Y \subseteq X$, denote by $C_Y(S) = \bar{S} \cap Y$ the closure of S in Y with the subspace topology.

(i) We describe the coding procedure turning a closed subset of $I_{\mathbb{Q}}$ into a representation of a Polish metric space as above so that compactness of the subset corresponds to local compactness of the space. Let R be a closed subset of $I_{\mathbb{Q}}$. For each $v \in R$ the set $\Theta(R)$ contains a certain sequence of irrationals converging to v, as follows:

$$\Theta(R) = [0,1] \cap \{q_k - 2^{-m}\sqrt{2}\colon\ q_k \in R \wedge m \geq k\}.$$

To obtain a representation of $\Theta(R)$ as a Polish metric space, let $\Delta(R)$ be a sequence that lists $[0,1] \cap \{q_k - 2^{-m}\sqrt{2}\colon\ m \geq k\}$ without repetitions. Let $v_{i,j} = d(\Delta(R)_i, \Delta(R)_j)$ where d is the distance on $\mathcal{N}$ defined above. Recall Definition 1 and note that $(v_{i,j})_{i,j\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ is a metric on $\mathbb{N}$ and $M_V \cong C_{\mathcal{N}}(\Theta(R))$.

Claim. $C_{[0,1]}(\Theta(R)) = R \,\dot{\cup}\, C_{\mathcal{N}}(\Theta(R))$.

The inclusion "$\supseteq$" is clear. For the inclusion "$\subseteq$", suppose that $x \in C_{[0,1]}(\Theta(R))$. There are sequences of numbers $\langle k_t \rangle_{t\in\mathbb{N}}$ and $\langle m_t \rangle_{t\in\mathbb{N}}$ such that $q_{k_t} - 2^{-m_t}\sqrt{2} \in \Theta(R)$ and $x = \lim_{t\to\infty}[q_{k_t} - 2^{-m_t}\sqrt{2}]$. Clearly $q_{k_t} \in R$ for each t.

If the sequence $\langle k_t \rangle$ is bounded, by passing to a subsequence we may assume that it is constant. Then $m_t \to \infty$, so $x \in R$.

Otherwise, by passing to a subsequence we may assume that $k_t \to \infty$. Then $m_t \to \infty$ because we chose $m \geq k$ in the definition of $\Theta(R)$, and so $x = \lim_t q_{k_t}$. If $x \in \mathbb{Q}$ then $x \in R$ because R is closed. Otherwise, $x \in C_{\mathcal{N}}(\Theta(R))$. This proves the claim.

Claim. Let $R \subseteq I_{\mathbb{Q}}$ be closed. Then

$$R \text{ is compact} \Leftrightarrow C_{\mathcal{N}}(\Theta(R)) \text{ is locally compact.}$$

First suppose that R is compact. By the first claim,

$$C_{\mathcal{N}}(\Theta(R)) = C_{[0,1]}(\Theta(R)) - R.$$

Since R is closed in $[0, 1]$, the set $C_{\mathcal{N}}(\Theta(R))$ is open in its closure which is a compact set, so it is locally compact.

Now suppose that $C_{\mathcal{N}}(\Theta(R))$ is locally compact. This set is dense in $C_{[0,1]}(\Theta(R))$, so it is open in $C_{[0,1]}(\Theta(R))$ by Lemma 3. By the first claim again, this means that R is closed in $C_{[0,1]}(\Theta(R))$, and hence compact. This establishes (i).

For (ii), we use the function g from Fact 4. Uniformly in e, we can obtain a computable sequence $\Delta(R_{g(e)})$ as above. For any effective listing $\langle x_e \rangle$ of a countable subset of $\mathcal{N}$, the function $e, i \rightarrow d(x_e, x_i)$ is computable. Hence we can determine an index $i = p(e)$ such that $M_{V_i} \cong C_{\mathcal{N}}(\Theta(R_{g(e)}))$. So $e \in O \leftrightarrow M_{V_{p(e)}}$ is locally compact.

3 Expressive Power of $L_{\omega_1,\omega}$

Recall that for a signature S, the language $L_{\omega_1,\omega}(S)$ is the extension of first-order language that allows countable conjunctions and disjunctions over a set of formulas with a shared finite reservoir of free variables. In the setting of countable structures, the class of models for a sentence in $L_{\omega_1,\omega}(S)$ is Borel. In this section we use our main result to show that this fails when countability is replaced by separability in the context of complete metric spaces.

A metric space (M, d) can be turned into a structure in the classical sense by introducing a binary relation $R_q xy$ for each positive rational q, with the intended meaning that $d(x, y) < q$. Let S denote the signature consisting of these relation symbols.

Proposition 6. *Local compactness among Polish metric spaces can be described by an $L_{\omega_1,\omega}(S)$ sentence α.*

Proof. The sentence α expresses

$$\forall x \bigvee\nolimits_{t \in \mathbb{Q}^+} [\overline{B_t(x)} \text{ is compact}].$$

Recall that for a complete metric space M, compactness is the same as total boundedness: for each rational $r > 0$, M is the union of k many balls of radius r for some k. We can use this to express that $M = \overline{B_t(x)}$ is compact by a conjunction over rationals $r > 0$, of a disjunction over the number of balls k, of first-order sentences of $\exists\forall$ type asserting that there are k balls covering M. This shows the proposition.

Thus, the set of Polish metric models of α is not Borel.

Question 7. Determine the descriptive complexity of being connected among [locally compact] Polish metric spaces.

Question 8. Can connectedness be expressed by an $L_{\omega_1,\omega}(S)$ sentence?

As suggested by T. Tsankov, it would also be interesting to study the meaning of the Π^1_1-rank for the class of locally compact spaces.

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