Computably enumerable sets below random sets

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Abstract

We use Demuth randomness to study strong lowness properties of computably enumerable sets, and sometimes of Δ^0_2 sets.

(1) A set $A \subseteq \mathbb{N}$ is called a base for Demuth randomness if some set Y Turing above A is Demuth random relative to A. We show that there is an incomputable, computably enumerable base for Demuth randomness, and that each base for Demuth randomness is strongly jump-traceable.

(2) We obtain new proofs that each computably enumerable set below all superlow (superhigh) Martin-Löf random sets is strongly jump traceable, using Demuth tests.

(3) The sets Turing below each ω^2-computably approximable Martin-Löf random set form a proper subclass of the bases for Demuth randomness, and hence of the strongly jump traceable sets.

(4) The c.e. sets Turing below each ω^2-computably approximable Martin-Löf random set satisfy a new, very strong combinatorial lowness property called ω-traceability.

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1. Introduction

1.1. Background

The interaction of computability-theoretic and random-theoretic notions it in the focus of current research. We consider a simple setting which epitomizes this interaction: a computably enumerable set $A \subseteq \mathbb{N}$ that is Turing below random sets $Y \subseteq \mathbb{N}$.

Given a random set Y, it will be difficult to build a computably enumerable (c.e.) set below it that is incomputable. Our intuition is that

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randomness and computable enumerability are incompatible, even opposite properties. There is mathematical evidence for this. For instance, recall that a set $Y \subseteq \mathbb{N}$ is called weakly 2-random if Y is in no null Π^0_2 class. Such a set forms a minimal pair with the halting problem (see [19, Thm. 5.3.16]), and hence cannot have a c.e. incomputable set below.

If we merely require Martin-Löf randomness of Y, then an incomputable, c.e. set A below Y exists in interesting cases. Recall that a Martin-Löf test is an effective sequence $(G_m)_{m\in\mathbb{N}}$ of c.e. open classes in Cantor space 2^ω such that $\lambda G_m \leq 2^{-m}$ for each m. A set Y *passes* this test if $Y \notin G_m$ for some m. Otherwise one says that Y *fails* the test. Y is *Martin-Löf random* (ML-random) if Y passes each ML-test. In the definition of passing a test, equivalently, one could require that $Y \notin G_m$ for almost all m.

Chaitin's Ω is a ML-random real that is Turing equivalent to the halting problem. So *all* c.e. sets are Turing below Ω. On the other hand, we wouldn't expect a "general" ML-random set Y to be above the halting problem. In this case, our intuition formulated above is still partially correct: there may exist an incomputable c.e. set A below Y, but such sets A are very restricted. Recall that a set is Δ^0_2 iff it is Turing below the halting problem $\emptyset'$. The following is a classic result.

Theorem 1.1 (Kučera [14]). *Suppose Y is a Martin-Löf random Δ^0_2 set. Then some computably enumerable, but incomputable set A is Turing below Y.*

The next result says that such a set A is restricted. This was shown by Hirschfeldt et al. in [13]. The K-trivial sets, introduced in [1], are at the same time far from random, and close to computable [18]. For instance, a K-trivial set A is superlow: its jump A' is truth-table below the halting problem.

Theorem 1.2 ([13]). *Let Y be a Martin-Löf random set that is not Turing above the halting problem. Suppose a computably enumerable set A is Turing below Y. Then A is K-trivial.*

A *lowness property* is a property saying a set is close to being computable in a specific sense. Computability theory, which is to a large extent about the complexity of sets, has a vital interest in understanding lowness properties. Before the intrusion of random-theoretic methods about a decade ago, superlowness was the strongest known lowness property such that an incomputable c.e. example exists. Through articles such as [6] and [18], it is now known that the class of K-trivial c.e. sets is much smaller than the superlow c.e. sets. This class is interesting degree-theoretically. For instance, it is closed downward under Turing reducibility $\leq_T$, and closed under effective join $\oplus$. Thus it induces an ideal in the c.e. Turing degrees. This ideal has a Σ^0_3 index set, which is the minimal descriptional index set complexity of a proper class of c.e. Turing degrees.

1.2. Strengthening Kučera's result

Kučera's Theorem 1.1 is our starting point for studying lowness properties of a c.e. set A. To obtain lowness properties stronger than K-triviality, we strengthen the condition related to Kučera's result that $A \leq_T Y$ for some Turing incomplete random set Y. There are two approaches, which are interrelated as we will see later.

(a) Replace the single oracle set Y by a null class $\mathcal{C} \subseteq 2^\omega$ containing some Martin-Löf random set Y not Turing above $\emptyset'$, and require that $A \leq_T Z$ for each ML-random set $Z \in \mathcal{C}$. Intuitively speaking, the larger $\mathcal{C}$ is, the closer to being computable must a c.e. set be that is Turing below all ML-random members of $\mathcal{C}$.

(b) Stay with a single oracle set Y, but require that it satisfy a randomness property stronger than Martin-Löf-randomness. Intuitively speaking, the more random Y is, the closer to being computable must a c.e. set be that is Turing below Y.

Both approaches lead to similar results, which are related to a combinatorial lowness property called strong jump traceability.

An *order function* is a nondecreasing unbounded computable function $h\colon \mathbb{N} \to \mathbb{N}$. We say that a set A is *jump traceable with bound* h if there is a uniformly c.e. sequence of sets $(T_x)_{x\in\mathbb{N}}$ (called a c.e. trace) such that $\#T_x \leq h(x)$ and $J^A(x) \in T_x$ for almost all x such that $J^A(x)$ is defined. We say that A is *strongly jump traceable (s.j.t.)* [8] if A is jump traceable with bound h for each order function h.

The class of c.e., strongly jump traceable sets is denoted by $\mathsf{SJT}_{c.e.}$. This class forms a *proper* subclass of the c.e. K-trivial sets by Cholak, Downey, and Greenberg [2]. It is closed downward under $\leq_T$, and closed under $\oplus$ [2]. Downey and Greenberg [5] showed that in fact each strongly jump traceable set, c.e. or not, is K-trivial.

A taxonomy of lowness properties has been introduced in [20, Section 6] and [11]. Strong jump traceability is an instance of the "weak-as-an-oracle" paradigm. Approach (a) above leads to sets A that are low because they are below many oracles. A third paradigm, inertness, says that the set has a computable approximation with a finite total of changes as measured in terms of a class of cost functions. Several results show the equivalence of lowness properties obtained through different paradigms.

1.3. Approach (a), and diamond classes

We say that that a Δ^0_2 set Y is *ω-computably approximable* (ω-c.a.) if Y has a computable approximation with a computable bound $g(n)$ on the number of times $Y(n)$ changes. These sets are also called ω-r.e., or ω-c.e., in the literature. It is easy to obtain a ML-random ω-c.a. set. Examples are Chaitin's number Ω, or a superlow ML-random set constructed via the (super)low basis theorem. Thus, the following theorem of Greenberg, Hirschfeldt and Nies says that a c.e. set A is strongly jump traceable iff it is Turing below many ML-random oracles. This gives an example of an equivalence of the first, and the second lowness paradigm.

Theorem 1.3 ([12] and [11] together)**.** *Let A be c.e. Then*

$$A \text{ is s.j.t.} \quad \Leftrightarrow \quad A \text{ is Turing below each } \omega\text{-c.a. ML-random set.}$$

The implication "⇒" is due to Greenberg and Nies [12]; the harder converse implication to Greenberg, Hirschfeldt and Nies [11]. In fact, [11] shows the stronger result that any set A (c.e. or not) Turing below each superlow ML-random set is strongly jump traceable.

We say that a set Y is *superhigh* if $\emptyset'' \leq_{\mathrm{tt}} Y'$; this property is, in a sense, dual to superlowness. The following result of [11] is related to Theorem 1.3.

Theorem 1.4. *Let A be c.e. Then*

$$A \text{ is s.j.t} \quad \Leftrightarrow \quad A \text{ is Turing below each superhigh ML-random set.}$$

In each case, the implications from left to right follow a common scheme, relying on the fact that a strongly jump traceable, c.e. set A obeys all so-called benign cost functions (see [12], or [19, Section 5.3]). However, the converse implications, while both following a general framework inspired by the golden run method, are rather ad hoc; it is unclear which property of the given class $\mathcal{C}$ is needed to make them work. Their proofs are also rather technical, in particular the one for superhighness. One aim of this paper is to give uniform and simpler proofs for these implications. We do this by showing that Approach (a) above can in a sense be subsumed under Approach (b). We will elaborate on this in Subsection 1.5. Note that this method only works for c.e. sets A (see Remark 4.6 for more on this).

The following notation is useful. For a class $\mathcal{C} \subseteq 2^\omega$, let $\mathcal{C}^\diamond$ denote the collection of c.e. sets that are computable from all Martin-Löf random sets in $\mathcal{C}$. This "infimum" operator was implicitly introduced in unpublished work of Hirschfeldt and Miller. Each class of the form $\mathcal{C}^\diamond$ induces an ideal in the c.e. Turing degrees. Hirschfeldt and Miller showed that $\mathcal{C}^\diamond$ contains a simple set for each null Σ^0_3 class $\mathcal{C}$ (see [19, 5.3.15]). Since $\{Y\}$ is a Π^0_2, and hence a Σ^0_3 class for each Δ^0_2 set Y, this strengthens Kučera's result.

We can now summarize the above results of [12, 11] by the equations

$$\mathsf{SJT}_{c.e.} = (\omega\text{-c.a.})^\diamond = (\text{superlow})^\diamond = (\text{superhigh})^\diamond.$$

1.4. Approach (b), and bases for randomness notions

In Approach (b) we consider a c.e. set A that is Turing below a set Y satisfying a randomness notion stronger than ML-randomness. Weak 2-randomness doesn't work, because it makes A computable, as observed above. Instead, we will employ various variants of Demuth randomness, a notion between 2-randomness and Martin-Löf randomness that is still compatible with being Turing below $\emptyset'$ (but no longer with being above $\emptyset'$). This notion was introduced and studied in the 1980s by Demuth [3, 4] in connection with differentiability of certain effective functions defined on the unit interval. It remained obscure for a long time, but now begins to stand out for its rich interaction with the computational complexity aspect of sets.

Demuth tests are more general than ML-tests in that one can change for a computably bounded number of times the m-th component (a c.e. open class in Cantor space 2^ω of measure at most 2^{-m}). Let $\mathcal{S}_m$ be the final version of the m-th component. Passing a Demuth test means to be out of *almost all* $\mathcal{S}_m$. See [19, Section 3.6] for more background on Demuth randomness.

Greenberg [10] built a Δ^0_2 Martin-Löf random set Y such that every c.e. set computable from Y is strongly jump traceable. Subsequently, Kučera and Nies [15] showed that any Demuth random Δ^0_2 set Y fulfills this purpose.

Theorem 1.5 ([15]). *Let Y be Demuth random. Let A be a c.e. set such that $A \le_T Y$. Then A is strongly jump traceable.*

Let $\mathcal{C}$ be a randomness notion. The part of this paper related to Approach (b) relies on the following concept. We say that a set $A \subseteq \mathbb{N}$ is a *base for* $\mathcal{C}$ if $A \le_T Y$ for some set $Y \in \mathcal{C}^A$. That is, A can be computed from a set Y that is also random relative to A in the sense of $\mathcal{C}$. Intuitively, such a set should be close to computable, because the class of oracles computing it is large. Mathematical evidence for the correctness of this intuition is given for instance in [13], where it is shown that the bases for Martin-Löf randomness coincide with the K-trivial sets. Theorem 1.2 above is actually a Corollary to this result, because it is not hard to show that if A is c.e. and $A \le_T Y$ for some ML-random set Y that is Turing incomplete, then Y is already ML-random relative to A. See [19, Cor. 5.1.23] for more detail.

The stronger a randomness notion is, the smaller becomes the class of bases for that notion. Let us reconsider weak 2-randomness, a notion somewhat stronger than Martin-Löf randomness where the tests are simply Π^0_2 null classes. If A is a base for weak 2-randomness then A is computable. To see this, note that for each Turing functional Φ, the class of oracles Y such that $\Phi(Y) = A$ is a $\Pi^0_2(A)$ class. If A is incomputable then this class is null, so Y is not weakly 2-random relative to A.

Unlike the case of weak 2-randomness, the bases for Demuth randomness form an interesting proper subclass of the bases for ML-randomness. In

Section 3 we will prove that there is a promptly simple (hence incomputable c.e.) base for Demuth randomness. Thereafter, we show in Theorem 3.2 that each base for Demuth randomness is strongly jump traceable. The containment is proper: new work of Greenberg and Turetsky shows that some strongly jump traceable c.e. set is not a base for Demuth randomness

Our proof that every base for Demuth randomness is s.j.t. is related to the proof of Theorem 1.5 due to [15], but does not need the assumption that the set is computably enumerable. From the proof of our result we obtain an alternative proof of the result in [15].

We say that A is low for a randomness notion $\mathcal{C}$ if $\mathcal{C} = \mathcal{C}^A$. It is easy to see that each set A that is low for ML-randomness is a base for ML-randomness, the reason being that $A \leq_T Y$ for some ML-random set A. The situation for Demuth randomness is quite different: Downey and Ng [7] showed that every set that is low for Demuth randomness must be of hyperimmune-free Turing degree. Given that the s.j.t. sets are Δ^0_2, this means that a set that is low for Demuth randomness cannot be a base for Demuth randomness, unless it is computable. Recent work of Bienvenu, Downey, Greenberg, Nies and Turetsky shows that continuum many sets are low for Demuth randomness.

We remark that the concept of a base has also been studied for notions weaker than Martin-Löf randomness. In [9] it is proved that the bases for Schnorr randomness coincide with the Turing incomplete sets. Bases for computable randomness have not been characterized so far. However, in [13] it is shown that this class includes every Δ^0_2 set that is not diagonally noncomputable, but excludes each set of PA-degree.

1.5. Using Approach (b) to prove the inclusion of diamond classes in $\mathsf{SJT}_{c.e.}$

For each order function h and each Turing functional Φ, we build a Demuth test such that any c.e. set computed via Φ by an oracle passing this test is jump traceable for bound h. This technical result is a spin-off of the proof of Theorem 3.2 that each base for Demuth randomness is strongly jump traceable.

There is no universal Demuth test, so a class $\mathcal{C}$ not containing a Demuth random set can still for each *single test* have a member that passes this test. A class of this type will be called Demuth test-compatible. If A is in $\mathcal{C}^\diamond$ for such a class, then it is strongly jump traceable. Our uniform method for showing that a diamond class is contained in $\mathsf{SJT}_{c.e.}$ proceeds by showing that the class $\mathcal{C}$ in question is Demuth test-compatible; this suffices by the technical result mentioned above. In Section 4 we carry this out for superlowness and superhighness. The proofs use some elements of the proofs in [11]. However, we get away without the infinitely many levels that are due to the golden-run type control mechanisms there.

1.6. Sets below all ω^2-c.a. ML-random sets

Let us recall our basic setting, a c.e. set A Turing below random sets Y. In the last two sections we follow the two approaches outlined above in order to study an extreme lowness property of a set A: being below all ω^2-c.a. ML-random sets.

A set Y is called *ω^2-c.a.* if there is a computable approximation $(Y_s)_{s\in\mathbb{N}}$ such that at each change $Y_s(n) \neq Y_{s-1}(n)$, we can count down along the canonical computable well-order of type ω^2 defined on pairs of natural numbers. We will show that a Demuth random set can be ω^2-c.a.

The ω^2-c.a. sets form a Σ^0_3 class containing the ω-c.a. sets. Thus, the class $(\omega^2\text{-c.a.})^\diamond$ contains a promptly simple set and is contained in $\mathsf{SJT}_{c.e.} = (\omega\text{-c.a.})^\diamond$.

Let MLR denote the class of ML-random sets. For a class $\mathcal{C} \subseteq 2^\omega$ let

$$\mathcal{C}^\square = \{A \colon \forall Y \in \mathcal{C} \cap \mathsf{MLR}[A \leq_T Y]\},$$

i.e., we drop the restriction that A be computably enumerable. In Subsection 5.3 we show that the class $(\omega^2\text{-c.a.})^\square$ is a proper subclass of the bases for Demuth randomness. The main work is to build an ω^2-c.a. set that is Demuth random in each K-trivial set.

For later use we provide the following fact due to [13].

Proposition 1.6. *Each set in $A \in (\omega\text{-c.a.})^\square$ is a base for ML-randomness, and hence K-trivial.*

In fact, A is s.j.t., as already observed at the end of Subsection 1.3.

Proof. It suffices to note that $A \leq_T \Omega_0, \Omega_1$, where Ω_0 (Ω_1) is the sequence of bits of Ω in the even (odd) positions. Now Ω_1 is ML-random in Ω_0, and hence in A. □

In the final Section 6 we consider a possible analog of the equations $\mathsf{SJT}_{c.e.} = (\omega\text{-c.a.})^\diamond = (\text{superlow})^\diamond$ discussed above in the setting of ω^2-approximations. We introduce a stronger form of traceability: in a trace $(T_x)_{x\in\mathbb{N}}$, we can at any stage adjust downwards the bound on the number of elements that are allowed to go in after that stage.

We say that Y is ω^2-low if Y' is ω^2-c.a. We show that for each set A in $(\omega^2\text{-c.a.})^\diamond$, and in fact in the class $(\omega^2\text{-low})^\diamond$, we can trace J^A by such traces, whenever the bound has an "inverse ω-c.a. approximation". To do so, we introduce a more general type of Demuth-tests, where (once again) the number of version changes is merely limited by counting down along a well order of type ω^2. We show that the class of ω^2-low sets is compatible with such tests.

The results can be generalized to the setting of ω^n-approximations. In [16] we plan to carry out these extensions, along with a possible coincidence of $(\omega^n\text{-c.a.})^\diamond$ and the class of c.e. sets that are traceable in an appropriate strong sense.

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2. Preliminaries

We follow the notation in [19]. In particular, for a set $V \subseteq 2^{<\omega}$, we let

$$[V]^{\prec} = \{Z \in 2^{\omega} \colon \exists \sigma \in V\, [\sigma \prec Z]\}.$$

If $\mathcal{W}$ is a measurable class in Cantor space 2^{ω}, then $\lambda \mathcal{W}$ denotes its (product) measure. $\#X$ denotes the cardinality of a set X.

In this section we provide some definitions and simple facts that will be needed later. They are mainly about Demuth tests and superlowness.

2.1. Measure

The following is due to [11].

Lemma 2.1 (Cut-off)**.** *Suppose we are given an effectively open class $\mathcal{W} = [W]^{\prec}$ for some c.e. set W and a positive rational number ϵ. Then we can, uniformly in ϵ and a c.e. index for W, obtain a c.e. index for an effectively open class, which we denote by $\mathcal{W}^{(\leq \epsilon)}$, such that*

$$\mathcal{W}^{(\leq \epsilon)} \subseteq \mathcal{W}; \ \lambda \mathcal{W}^{(\leq \epsilon)} \leq \epsilon; \text{ and if } \lambda \mathcal{W} \leq \epsilon, \text{ then } \mathcal{W}^{(\leq \epsilon)} = \mathcal{W}.$$

This is because we can stop the enumeration of clopen sets $[\sigma]$ into $\mathcal{W}$ when its measure attempts to exceed ϵ.

2.2. Demuth randomness

We begin with the formal definition of Demuth randomness.

Definition 2.2. A *Demuth test* is a sequence of c.e. open sets $(\mathcal{S}_m)_{m \in \mathbb{N}}$ such that $\forall m\, \lambda \mathcal{S}_m \leq 2^{-m}$, and there is a function f such that $\mathcal{S}_m$ is the Σ^0_1 class $[W_{f(m)}]^{\prec}$; furthermore, $f(m) = \lim_s g(m, s)$ for a computable function g such that the size of the set $\{s \colon g(m, s) \neq g(m, s-1)\}$ is bounded by a computable function.

When the approximating function g is understood from the context, we write $\mathcal{S}_m[t]$ for $[W_{g(m,t)}]^{\prec}$ and say that this is the *version* of $\mathcal{S}_m$ at stage t. After adjusting g, we may assume that $\mathcal{S}_m[t] \leq 2^{-m}$ for each t. We often define a Demuth test by specifying the versions.

A set Z *passes* the test if $Z \notin \mathcal{S}_m$ for almost every m. We say that Z is *Demuth random* if Z passes each Demuth test.

To illustrate this tests concept, we sketch a result of Demuth (see [19, Thm. 3.6.26] for more detail).

Theorem 2.3. *Each Demuth random set Z is generalized low$_1$, namely, $Z' \leq_T Z \oplus \emptyset'$.*

Proof. We introduce a function $g \leq_{\mathrm{wtt}} \emptyset'$ and a Demuth test $(\mathcal{S}_m)_{m\in\mathbb{N}}$ such that $g(m) \geq J^Z(m)$ whenever $J^Z(m)$ is defined, for any set Z that passes $(\mathcal{S}_m)_{m\in\mathbb{N}}$. This easily implies that Z is generalized low$_1$.

For each m, let $\mathcal{L}_m$ be the open class $\{Z\colon\ J^Z(m)\downarrow\}$, and let $L_{m,s}$ be the clopen class $\{Z\colon\ J^Z_s(m)\downarrow\}$. We define an auxiliary clopen class C_m. At stage s we define approximations $g_s(m)$ to $g(m)$ and $C_{m,s}$ to C_m, in such a way that the clopen set $C_{m,s}$ contains the oracles Z such that $g_s(m)$ dominates $J^Z(m)$. Whenever at a stage s the measure of the clopen set $L_{m,s} - C_{m,s-1}$ exceeds 2^{-m}, we put this set into $C_{m,s}$ and increase $g(m)$ to the stage number, so that it also dominates the values $J^Z(m)$ for these newly added oracles Z. These changes to the current approximation of C_m, and hence of $g(m)$, can take place at most 2^m times. Thus g is ω-c.a. and C_m stabilizes, whence $\mathcal{S}_m = L_m - C_m$ determines a Demuth test as desired. Note that the version of the m-th component changes at most 2^m times. □

A Demuth test $(\mathcal{S}_m)_{m\in\mathbb{N}}$ is called *monotonic* if $\mathcal{S}_m \supseteq \mathcal{S}_{m+1}$ for each m. A set that passes all monotonic Demuth tests is called *weakly Demuth random*. For instance, if V is an ω-c.a. set, then the Demuth test given by $\mathcal{S}_m = [V \upharpoonright_m]$ is monotonic. Thus, a weakly Demuth random cannot be ω-c.a. Note that the test constructed above cannot be monotonic, because there is a high Δ^0_2 weakly Demuth random set by [15, Thm. 3.1].

Two given Demuth tests can be "covered" by a single one. In our applications, the second given Demuth test will usually be a universal ML-test.

Fact 2.4. *For each pair of Demuth tests $(\mathcal{S}_m)_{m\in\mathbb{N}}$ and $(\mathcal{T}_m)_{m\in\mathbb{N}}$, there is a Demuth test $(\mathcal{U}_k)_{k\in\mathbb{N}}$ such that any set passing $(\mathcal{U}_k)_{k\in\mathbb{N}}$ passes both $(\mathcal{S}_m)_{m\in\mathbb{N}}$ and $(\mathcal{T}_m)_{m\in\mathbb{N}}$.*

Proof. Simply let $\mathcal{U}_k[s] = \mathcal{S}_{k+1}[s] \cup \mathcal{T}_{k+1}[s]$. □

The proof of the following simple lemma uses a trade-off for Demuth tests between the measure of the components, and the number of times their versions change.

Lemma 2.5. *Let $(\mathcal{S}_k)_{k\in\mathbb{N}}$ be a Demuth test. Then there is a Demuth test $(\mathcal{G}_m)_{m\in\mathbb{N}}$ such that $\lambda\mathcal{G}_m \leq 4^{-m}$, and every set that passes $(\mathcal{G}_m)_{m\in\mathbb{N}}$ also passes $(\mathcal{S}_k)_{k\in\mathbb{N}}$.*

Proof. We define the Demuth test $(\mathcal{G}_m)_{m\in\mathbb{N}}$ through the versions of its components at stages t:

$$\mathcal{G}_m[t] = \mathcal{S}_{2m+1}[t] \cup \mathcal{S}_{2m+2}[t].$$

Since $\lambda \mathcal{S}_r[t] \leq 2^{-r}$ for each r, t, we have $\lambda \mathcal{G}_m[t] \leq 4^{-m}$ for each m, t. If h is an order function such that $\mathcal{S}_m[r]$ changes at most $h(r)$ times, then $\mathcal{G}_m[t]$ changes at most $h(2m+2)^2$ times. Clearly $(\mathcal{G}_m)_{m \in \mathbb{N}}$ is a Demuth test as required. □

The following will be used in the proof of Lemma 3.3.

Lemma 2.6. *Given a Demuth test* $(\mathcal{S}_m)_{m\in\mathbb{N}}$*, there is a Demuth test* $(\mathcal{H}_m)_{m\in\mathbb{N}}$ *such that, if a set* Y *passes* $(\mathcal{H}_m)_{m\in\mathbb{N}}$*, then* $0^e 1Y$ *passes* $(\mathcal{S}_m)_{m\in\mathbb{N}}$ *for each* e*.*

Proof. Firstly, by Lemma 2.5 let $(\mathcal{G}_m)_{m\in\mathbb{N}}$ be a Demuth test such that $\lambda \mathcal{G}_m \leq 4^{-m}$ for each m, and every set that passes $(\mathcal{G}_m)_{m\in\mathbb{N}}$ also passes $(\mathcal{S}_m)_{m\in\mathbb{N}}$. Next, let $\mathcal{H}_0 = \mathcal{H}_1 = \emptyset$ and for $m \geq 2$,

$$\mathcal{H}_m = \bigcup_{e \leq m-2} \{Y\colon\ 0^e 1Y \in \mathcal{G}_m\}.$$

Then $\lambda \mathcal{H}_m \leq 4^{-m} \sum_{e=0}^{m-2} 2^{e+1} \leq 2^{-m}$. Clearly $(\mathcal{H}_m)_{m\in\mathbb{N}}$ is a Demuth test as required. □

2.3. Superlowness

If $(A_s)_{s\in\mathbb{N}}$ is a computable enumeration for a c.e. set A, and $(\Gamma_s)_{s\in\mathbb{N}}$ is an effective enumeration of (the graph of) a Turing functional, then we let $\Gamma^A[s] = \Gamma_s^{A_s}$. The following result [15, Lemma 4.1] is a special case of [11, Theorem 3.5]. We include its proof for the sake of completeness.

Lemma 2.7. *Suppose the c.e. set* A *is superlow. Then for each Turing functional* Γ *there is a computable enumeration* $(A_s)_{s\in\mathbb{N}}$ *of* A *and a computable function* g *such that* $g(x)$ *bounds the number of stages* s *such that* $\Gamma^A(x)[s-1]$ *is defined with use* u *and* $A_s \upharpoonright_u \neq A_{s-1} \upharpoonright_u$*.*

In the situation of the lemma, we say the computation $\Gamma^A(x)[s-1]$ is *destroyed* at stage s.

Proof. Let $(\widetilde{A}_s)_{s\in\mathbb{N}}$ be some computable enumeration of A. There is a Turing functional Δ such that for each x and each stage s such that $\Gamma^{\widetilde{A}}(x)[s]\downarrow$, the output of $\Delta^{\widetilde{A}}(x)[s]$ is the stage $t \leq s$ when this computation became defined. Clearly the defined distinct values $\Delta^{\widetilde{A}}(x)[s]$ are increasing in s.

By [17] A is jump-traceable. Thus, there is a c.e. trace $(T_x)_{x\in\mathbb{N}}$ with computable bound g for Δ^A. Define a computable sequence of stages as follows. Let $s_0 = 0$. For $i \geq 0$, let

$$s_{i+1} = \mu s > s_i . \forall x < s_i \, [\Gamma^{\widetilde{A}}(x)[s]\downarrow \longrightarrow \Delta^{\widetilde{A}}(x)[s] \in T_{x,s}].$$

Define a computable enumeration $(A_s)_{s\in\mathbb{N}}$ of A by $A_s(x) = \widetilde{A}_{s_i}(x)$ for $s_i \leq s < s_{i+1}$. For each s such that $\Gamma^A(x)[s]$ is newly defined, a further element must enter T_x. Thus $(A_s)_{s\in\mathbb{N}}$ is a as required. □

3. Bases for Demuth randomness

The following fact will be strengthened in Theorem 5.5, where we show that the class of bases for Demuth randomness contains $\mathcal{C}^\diamond$ for the Σ^0_3 null class $\mathcal{C}$ of ω^2-c.a. sets.

Proposition 3.1. *There is a promptly simple base for Demuth randomness.*

Proof. A result of Demuth (see [19, Section 3.6]) states that there is a Demuth random low set Y. Applying the same result relative to Y, we obtain a set V that is Demuth random and low relative to Y. Thus V is low. Now by a Theorem of Kučera (see [19, Thm. 4.2.1 and Ex. 4.2.7]) there is a promptly simple set $A \leq_T Y, V$. Then V is Demuth random relative to A. □

Theorem 3.2. *Each base for Demuth randomness is strongly jump traceable.*

Proof. The proof is a simpler variant of the proof of Theorem 1.5 due to [15]. Suppose that $A = \Phi^Y$ for a Turing functional Φ and a set Y that is Demuth random relative to A. To show that A is strongly jump traceable, for each order function h, we will build a uniformly c.e. sequence of sets $(T_x)_{x\in\mathbb{N}}$, $\#T_x \leq h(x)$, such that $J^A(x) \in T_x$ for almost all x such that $J^A(x)$ is defined.

For $m \in \mathbb{N}$ let

$$I_m = \{x\colon\ 2^m \leq h(x) < 2^{m+1}\}.$$

Construction. To help with the definition of the c.e. trace, we build an oracle Demuth test $(\mathcal{G}^A_m)_{m\in\mathbb{N}}$. At stage t, let u be the maximum use of the computations $J^A(x)$ for $x \in I_m$ that exist. We enumerate into the current version $\mathcal{G}^A_m[t]$ all oracles Z such that $\Phi^Z_t \succeq A\restriction_u$, as long as the measure stays below 2^{-m}. Whenever a new computation $J^A(x)$ for $x \in I_m$ converges, we start a new version of $\mathcal{G}^A_m$. Clearly, there will be at most $\#I_m$ versions.

More formally, let $\mathcal{U}_e = [W_e]^\prec$ be the c.e. open set given by W_e. There is a Turing functional Γ such that for each string α of length t,

$$\mathcal{U}_{\Gamma^\alpha(m,t)} = \{Z\colon\ \forall x \in I_m\ [J^\alpha_t(x)\downarrow \text{with use}\, u\ \Rightarrow\ \alpha\restriction_u \preceq \Phi^Z_t]\}.$$

Let $\mathcal{G}^A_m[t] = \mathcal{U}^{(\leq 2^{-m})}_{\Gamma^A(m,t)}$. By the uniformity of the Cut-off Lemma 2.1, from m, t with the help of oracle A we can compute an index for this effectively open class. Thus, the versions $\mathcal{G}^A_m[t]$ define a Demuth test $(\mathcal{G}^A_m)_{m\in\mathbb{N}}$ relative to A.

The c.e. trace $(T_x)_{x\in\mathbb{N}}$ is defined as follows. At stage t, for each string α of length t such that $y = J^\alpha_t(x)$ is defined and the measure of the current approximation to the c.e. open set $\mathcal{U}_{\Gamma^\alpha(m,t)}$ exceeds 2^{-m}, put y into T_x. The idea is that, if $y = J^A(x)$, then this must happen for some $\alpha \prec A$, otherwise the given Demuth random set Y can be put into $\mathcal{G}^A_m$ because there is no cut-off.

Claim 1. *$(T_x)_{x\in\mathbb{N}}$ is a c.e. trace such that for each x we have $\#T_x \le h(x)$.* Clearly the sequence is uniformly computably enumerable. Suppose now that distinct numbers y_i, $0 \le i < N$, enter T_x at stages t_i. Then $y_i = J^{\alpha_i}_{t_i}(x)$ for a string α_i. For $0 \le i < k < N$, the open sets $\mathcal{U}_{\Gamma^{\alpha_i}(m,t_i)}$ and $\mathcal{U}_{\Gamma^{\alpha_k}(m,t_k)}$ are disjoint because they consist of oracles computing incompatible initial segments of strings α_i and α_k. Since the measure of each such open set exceeds 2^{-m} at stage t_i, this shows that $N \le 2^m \le h(x)$ as required.

Claim 2. *For almost every x, if $y = J^A(x)$ is defined, then $y \in T_x$.*
Since Y is Demuth random in A, there is m_0 such that $Y \notin \mathcal{G}^A_m$ for each $m \ge m_0$. Given such an m, let t^* be a stage by which all $J^A(z)$ for $z \in I_m$ that are defined converge and, furthermore, $A\restriction_u \preceq \Phi^Y_{t^*}$ where u is the maximum use of these computations. Then the final version of $\mathcal{U}_{\Gamma^A(m,t)}$, and hence of $\mathcal{G}^A_m$, has been reached at stage t^*.

We claim that from some $t \ge t^*$ on, the measure of the approximation to $\mathcal{U}_{\Gamma^A(m,t)} = \mathcal{U}_{\Gamma^A(m,t^*)}$ exceeds 2^{-m}, whence at some stage we put $y = J^A(x)$ into T_x. Assume otherwise, that is, $\lambda\mathcal{U}_{\Gamma^A(m,t^*)} \le 2^{-m}$. Then by stage t^* the set Y has entered $\mathcal{U}_{\Gamma^A(m,t^*)} = \mathcal{G}^A_m$, contradiction. This completes the verification of Claim 2 and the proof. □

Note that the Demuth test $(\mathcal{G}^A_m)_{m\in\mathbb{N}}$ is not necessarily monotonic. Even if we modify the definition of the c.e. open classes $\mathcal{U}_{\Gamma^A(m,t)}$ by replacing "$x \in I_m$" with "$x \le \max I_m$", we cannot achieve monotonicity, because we cut off the enumeration of the $\mathcal{U}_{\Gamma^A(m,t)}$ when their measure attempts to exceed 2^{-m}. This cut-off might remove oracles Z that are needed for $\mathcal{U}_{\Gamma^A(m+1,t)}$. Thus, the proof of Theorem 3.2 cannot be extended to show that each c.e. base for weak Demuth randomness is strongly jump traceable. This is currently an open question.

We discuss a spin-off of the proof of Theorem 3.2 above, which will yield simpler proofs of Theorem 1.5 and also [10, Thm. 1.2].

Lemma 3.3. *For a given order function h and a superlow c.e. set A, we can build a Demuth test $(\mathcal{H}_m)_{m\in\mathbb{N}}$ such that, if $A \le_T Y$ for some Y passing this test, then A is jump traceable with bound h.*

Proof. Let Φ be the "universal" Turing functional given by $\Phi(0^e 1 Y) = \Phi_e(Y)$ for each e, Y. We actually will use the method in the proof of Theorem 3.2, with this Turing functional Φ, to show the following.

Claim 3.4. *We can build a Demuth test $(\mathcal{S}_m)_{m\in\mathbb{N}}$ such that, if $A = \Phi^Y$ for some Y passing this test, then A is jump traceable with bound h.*

This claim suffices to obtain Lemma 3.3: let $(\mathcal{H}_m)_{m\in\mathbb{N}}$ be the Demuth test obtained via Lemma 2.6 applied to the test $(\mathcal{S}_m)_{m\in\mathbb{N}}$. Thus, if a set Y passes $(\mathcal{H}_m)_{m\in\mathbb{N}}$, then 0^e1Y passes $(\mathcal{S}_m)_{m\in\mathbb{N}}$ for each e. By hypothesis of the lemma, $A \le_T Y$ for some Y passing $(\mathcal{H}_m)_{m\in\mathbb{N}}$, so we have $A = \Phi_e^Y$ for some e, and hence $A = \Phi(0^e1Y)$. Since 0^e1Y passes $(\mathcal{S}_m)_{m\in\mathbb{N}}$, we can conclude from the claim that A is jump traceable with bound h.

It remains to prove Claim 3.4. Based on the Turing functional Φ introduced above, we define the Turing functional Γ, the Demuth test relative to A $(\mathcal{G}_m^A)_{m\in\mathbb{N}}$, and the c.e. trace $(T_x)_{x\in\mathbb{N}}$ with bound h as in the proof of Theorem 3.2.

The idea is that $(\mathcal{G}_m^A)_{m\in\mathbb{N}}$ makes very limited use of the oracle A: each version $\mathcal{G}_m^A[t]$ of the m-th component is a c.e. open class, i.e. a Σ_1^0 class (while in full generality it could be $\Sigma_1^0(A)$ class). The oracle A is only needed to compute its index. Furthermore, the number of changes of versions is computably bounded. Since A is a superlow c.e. set, this allows us to cover $(\mathcal{G}_m^A)_{m\in\mathbb{N}}$ by a plain Demuth test $(\mathcal{S}_m)_{m\in\mathbb{N}}$.

Let Θ be a Turing functional such that $\Theta^X(m,i)$ is an index of the c.e. open set which is the i-th version of $\mathcal{G}_m^X$. The maximum value of i such that $\Theta^X(m,i)$ is defined is bounded by $\#I_m$.

Since A is superlow, by Lemma 2.7 there is a computable enumeration $(A_s)_{s\in\mathbb{N}}$ of A and a computable function f such that for at most $f(m,i)$ times a computation $\Theta^{A_s}(m,i)$ is destroyed. At stage t, let

$$\mathcal{S}_m[t] = \mathcal{U}_{\Theta^{A_t}(m,i)}$$

where i is maximal such that the expression on the right is defined. Clearly the number of times a version $\mathcal{S}_m[t]$ changes is bounded by $\sum_{i=0}^{\#I_m} f(m,i)$. Thus, $(\mathcal{S}_m)_{m\in\mathbb{N}}$ is a Demuth test. If an oracle Z is in $\mathcal{G}_m^A$ then Z is in the final version $\mathcal{S}_m[t]$.

By our hypothesis in Claim 3.4, $A = \Phi^Y$ for some Y passing $(\mathcal{S}_m)_{m\in\mathbb{N}}$. Thus Y is out of almost all $\mathcal{G}_m^A$. As before, this shows that $(T_x)_{x\in\mathbb{N}}$ is a trace for J^A with bound h. □

We next obtain the promised alternative proof of Theorem 1.5 above which is due to Kučera and Nies [15].

Corollary 3.5 ([15]). *Suppose the c.e. set A is Turing below a Demuth random set Y. Then A is strongly jump traceable.*

Proof. Every Demuth random set is generalized low$_1$. Hence by Theorem 1.2 A is K-trivial and therefore superlow. Now Lemma 3.3 shows that for each order function h, the set A is jump-traceable with bound h. □

4. Demuth test–compatible classes and strong jump traceability

The following concept forms part of a uniform method for showing that diamond classes are contained in the class $\mathsf{SJT}_{c.e.}$ of c.e., strongly jump traceable sets.

Definition 4.1. A class $\mathcal{C} \subseteq 2^\omega$ is called *Demuth test–compatible* if each Demuth test is passed by a member of $\mathcal{C}$.

Fact 4.2. *If $\mathcal{C}$ is Demuth test compatible, then so is $\mathcal{C} \cap \mathsf{MLR}$.*

Proof. Apply Fact 2.4 to a given Demuth test together with a universal ML-test. □

The core of the uniform method is the following lemma. Its proof is a "local" version of the proof of Cor. 3.5. That is, we argue in terms of single Demuth tests and correspondingly, single trace bounds. Think of $\mathcal{C}$ as a class that does not admit a Demuth random, such as the class of superlow sets.

Lemma 4.3. *Suppose the class $\mathcal{C}$ is Demuth test–compatible. Then, for each order function h, there is a ML-random set Y in $\mathcal{C}$ such that every c.e. set $A \leq_T Y$ is jump-traceable for bound h.*

Proof. By Theorem 2.3 and Fact 2.4, there is a single Demuth test such that any set passing the test is ML-random and generalized low_1, and hence Turing incomplete. Thus, by Theorem 1.2, the c.e. set A is K-trivial, and hence superlow.

Given an order function h, let $(\mathcal{H}_m)_{m\in\mathbb{N}}$ be the Demuth test from Lemma 3.3. By Fact 4.2, the class $\mathcal{C} \cap \mathsf{MLR}$ is Demuth test compatible as well. Thus, some ML-random set Y in $\mathcal{C}$ passes the test. Then Y is the required set. □

The following enables us to apply the core lemma to interesting classes. Note that a (weakly) Demuth randomn is not ω-c.a., and hence not superlow, as mentioned in Subsection 2.2. It is also not superhigh by [15, Cor. 3.6]. Nonetheless, these classes are Demuth test compatible.

Theorem 4.4. *Superlowness and superhighness are Demuth test–compatible. In other words, each Demuth test is passed by a superlow, and by a superhigh set.*

Before proving the theorem, we discuss two corollaries. We re-obtain the implications from right to left in Theorems 1.3 and 1.4: these results, obtained first in [11] by different means, are now immediate from Lemma 4.3, by the definitions of $\mathcal{C}^\diamond$ and strong jump traceability.

Corollary 4.5 ([11]). *The diamond classes of superlowness and superhighness are both contained in the strongly jump traceable sets.*

Remark 4.6. The inclusions of Corollary 4.5 were proved in [11] not just for c.e. sets, but in more generality for sets that are both superlow and jump traceable. Thus, as observed in the introduction, each set in $(\mathit{superlow})^{\Box}$ is K-trivial, hence superlow and jump traceable, hence strongly jump traceable. (In contrast, it is currently unknown whether $(\mathit{superhigh})^{\Box}$ is contained in the strongly jump traceables.)

On the other hand, the proofs of the results for c.e. sets given here are considerably simpler than the ones in [11].

As a further corollary we strengthen a result of Greenberg [10, Thm. 1.2]. Note that in [10] the set Y was merely ω-c.a.

Corollary 4.7. *For each order function h, there is a superlow ML-random set Y such that every c.e. set $A \leq_T Y$ is jump-traceable for bound h.*

To see this, it suffices apply to Lemma 3.3 when the class $\mathcal{C}$ is superlowness.

Proof of Theorem 4.4. By Lemma 2.5, it suffices to find a superlow [superhigh] set Z that passes $(\mathcal{G}_m)_{m\in\mathbb{N}}$, where $(\mathcal{G}_m)_{m\in\mathbb{N}}$ is a Demuth test such that $\lambda\mathcal{G}_m \leq 4^{-m}$ for each m. Let h be an order function such that the version $\mathcal{G}_m[s]$ changes at most $h(m)$ times.

By [19, p. 55], from an index $\mathcal{P}$ for a Π^0_1 class in 2^ω we can obtain a computable sequence $(\mathcal{P}_s)_{s\in\mathbb{N}}$ of clopen classes such that $\mathcal{P}_s \supseteq \mathcal{P}_{s+1}$ and $\mathcal{P} = \bigcap_s \mathcal{P}_s$.

Superlowness. We define a computable double sequence of (indices for) Π^0_1 classes $(\mathcal{P}^s_m)$, $m \leq s$, at stages s, where $\mathcal{P}^s_m \supseteq \mathcal{P}^s_{m+1}$. By convention, if these Π^0_1 classes are not explicitly changed at a stage $s > 0$, they retain their values at the previous stage. The number of stages s at which the index $\mathcal{P}^s_m$ changes will be computably bounded in m. The desired set Z will be in the intersection of the final versions $\lim_s \mathcal{P}^s_m$.

To define the double sequence $(\mathcal{P}^s_m)$, $m \leq s$, we extend the usual proof of the (super)low basis theorem. As in that proof, we meet conditions to determine the jump $J^Z(m)$ at arguments m. But now we interleave these conditions with conditions to avoid the open classes $\mathcal{G}_m$. In order to do this, we must have enough measure left at each step: we ensure that

$$\lambda(\mathcal{P}^s_m)_s \geq 4^{-m}$$

for each m, s. (Note that $(\mathcal{P}^s_m)_s$ denotes the clopen class approximating $\mathcal{P}^s_m$ at stage s.)

Suppose at a stage s, for an $m < s$ the class $\mathcal{P}^s_m$ has been defined. To define the next class $\mathcal{P}^s_{m+1}$, firstly, we remove the current version $\mathcal{G}_{m+1}[s]$.

This leaves a measure of at least $3 \cdot 4^{-m-1}$. Initially the guess was that $J^Z(m) \uparrow$. We change this guess to $J^Z(m) \downarrow$ if at this stage the measure of the class of oracles Y such that $J^Y(m) \downarrow$ exceeds $2 \cdot 4^{-m-1}$. Once a measure of 4^{-m-1} has left this current Π^0_1 class $\mathcal{P}^s_{m+1}$, we revert the guess back to $J^Y(m) \uparrow$. This will ensure that that there is always enough measure left in $\mathcal{P}^s_{m+1}$.

We now give the formal details. Each stage $s > 0$ of the construction has substages m for $m < s$.

Module m.

(a) Let $\widetilde{\mathcal{P}}^s_m = \mathcal{P}^s_m \setminus (\mathcal{G}_{m+1}[s])$. (Note that $\lambda(\widetilde{\mathcal{P}}^s_m)_s \geq 3 \cdot 4^{-m-1}$.) Whenever $\mathcal{G}_{m+1}[s]$ has a new value, the module returns to (a) (this overrides any instructions given in (b), (c) below).

(b) Let $\mathcal{D}$ be the clopen set $\{X\colon\ J^X_s(m)\downarrow\}$.

IF $\lambda(\widetilde{\mathcal{P}}^s_m)_s \cap \mathcal{D} \leq 2 \cdot 4^{-m-1}$ let $\mathcal{P}^s_{m+1} = \widetilde{\mathcal{P}}^s_m - \mathcal{D}$, and GOTO (b). (The current guess is that $J^Z(m) \uparrow$.)

ELSE let $\mathcal{P}^s_{m+1} = \widetilde{\mathcal{P}}^s_m \cap \mathcal{D}$ and GOTO (c). (The guess is now $J^Z(m) \downarrow$.)

(c) IF $\lambda(\mathcal{P}^s_{m+1})_s \geq 4^{-m-1}$ STAY AT (c). ELSE GOTO (b) and continue there at this same substage m. ($\mathcal{D}$ will be updated now.)

Construction. We say that a class $\mathcal{P}^s_m$ $(m < s)$ is *redefined* at stage s if the value of its index is different from the value at stage $s - 1$.
Stage $s > 0$.
Substage 0*:* let $\mathcal{P}^s_0 = 2^\omega$.
Substage m*,* $s > m > 0$*:* If $s = 1$, or $\mathcal{P}^s_m$ was redefined at s, go to the beginning of Module m. Otherwise continue where Module m was left at stage $s - 1$. This (re)defines $\mathcal{P}^s_{m+1}$.

Verification.

Claim 4.8. *At each stage s, for each $m \leq s$ we have $\lambda(\mathcal{P}^s_m)_s \geq 4^{-m}$.*

This is true for $m = 0$ because $(\mathcal{P}^s_0)_s = 2^\omega$. Suppose it holds for $m < s$. Since $\lambda \mathcal{G}_{m+1}[s] \leq 4^{-m-1}$ we have $\lambda(\widetilde{\mathcal{P}}^s_m)_s \geq 3 \cdot 4^{-m-1}$ in (a). Now $\lambda(\mathcal{P}^s_{m+1})_s \geq 4^{-m-1}$ follows from the inductive hypothesis and the case hypotheses in (b) and (c).

Claim 4.9. *There is a computable function r such that for each m*

$$r(m) \geq \#\{s > 0\colon\ \mathcal{P}^s_m \text{ is redefined at stage } s\}.$$

Let $r(0) = 0$. Now suppose inductively $r(m)$ has been defined correctly. Then $r(m) + h(m)$ bounds the number of stages s at which $\widetilde{\mathcal{P}}^s_m$ is redefined. As long as this class is not redefined, with stable value $\widetilde{\mathcal{P}}$, whenever Module m goes from (b) to (c) and, later on, back to (b), then $\lambda(\widetilde{\mathcal{P}} \cap \mathcal{D})$

has dropped by at least 4^{-m-1}. Since $\mathcal{D}$ is fixed while the module is at (c), this means that $\lambda\widetilde{\mathcal{P}}$ has dropped by at least 4^{-m-1}. The number of times this can happen is therefore bounded by 4^{m+1}. Thus, if we let $r(m+1) = r(m) + 2 \cdot 4^{m+1}(r(m) + h(m))$, then $r(m+1)$ is a correct bound for the number of times $\mathcal{P}^s_{m+1}$ is redefined. This proves the claim.

Now let $\mathcal{P}_m = \lim_s \mathcal{P}^s_m$. By the compactness of Cantor space there is a set $Z \in \bigcap_m \mathcal{P}_m$. Then Z is superlow because

$$J^Z(m)\downarrow \leftrightarrow \text{eventually Module } m \text{ is at (c)},$$

and the number of times the module goes from (c) to (b) is computably bounded as this means to redefine $\mathcal{P}^s_m$. Clearly Z passes the given Demuth test. This shows that superlowness is Demuth test–compatible.

Superhighness. Similar to [11], we actually show that for an arbitrary set $R \subseteq \mathbb{N}$, the class $\{Z \colon R \le_{tt} Z'\}$ is Demuth test–compatible. Then, for the case of superhighness, we let $R = \emptyset''$.

We now work with a full binary tree of Π^0_1 classes. We define a computable double sequence of indices for Π^0_1 classes $(\mathcal{P}^s_\alpha)$, $\alpha \in 2^{<\omega}$, $|\alpha| \le s$, at stages s. We ensure that $\mathcal{P}^s_{\alpha 0} \cap \mathcal{P}^s_{\alpha 1} = \emptyset$, and $\beta \prec \alpha$ implies $\mathcal{P}^s_\beta \supseteq \mathcal{P}^s_\alpha$. As before, if we don't explicitly change the indices, they retain their values from the previous stage. The number of stages s at which the index $\mathcal{P}^s_\alpha$ changes will be computably bounded in $|\alpha|$.

Let $\mathcal{P}_\alpha$ denote the final version $\lim_s \mathcal{P}^s_\alpha$. In the end we let $Z \in \bigcap\{\mathcal{P}_\alpha \colon \alpha \prec R\}$, and show that $R \le_{tt} Z'$ as required. For this, it is sufficient to define a Z-computable approximation $f(x,s)$ to $R(x)$ such that the number of stages $s > 0$ with $f(x,s) \ne f(x,s-1)$ is computably bounded in x (see [19, 1.4.4]). We will let $f(x,s)$ be the bit $R(x)$ assessed with the relevant information available at stage s, namely, with the classes $\mathcal{P}^s_\beta$ for $|\beta| \le s$.

We ensure that

$$\lambda(\mathcal{P}^s_\alpha)_s \ge 4^{-|\alpha|}$$

for each α, s such that $|\alpha| \le s$.

Module α. Let $m = |\alpha|$.

(a) Let $\widetilde{\mathcal{P}}^s_\alpha = \mathcal{P}^s_\alpha \setminus (\mathcal{G}_{m+1}[s])$. (Note that $\lambda(\widetilde{\mathcal{P}}^s_\alpha)_s \ge 3 \cdot 4^{-m-1}$.) Whenever $\mathcal{G}_{m+1}[s]$ has a new value, the module returns to (a).

(b) Let σ be the leftmost string of length s such that, where

$$\mathcal{D} = \{Z \colon Z \le_L \sigma\},$$

we have $\lambda((\widetilde{\mathcal{P}}^s_\alpha)_s \cap \mathcal{D}) \ge \frac{3}{8}4^{-m}$. Let

$$\mathcal{P}^s_{\alpha 0} = \widetilde{\mathcal{P}}^s_\alpha \cap \mathcal{D}, \text{ and } \mathcal{P}^s_{\alpha 1} = \widetilde{\mathcal{P}}^s_\alpha \setminus \mathcal{D}.$$

GOTO (c).

(c) IF $\lambda(\mathcal{P}^s_{\alpha 0})_s \geq 4^{-m-1}$ and $\lambda(\mathcal{P}^s_{\alpha 1})_s \geq 4^{-m-1}$ STAY AT (c). ELSE GOTO (b) and continue there at the same substage m.

Construction. Stage $s > 0$.
Substage 0*:* let $\mathcal{P}^s_{\emptyset} = 2^\omega$.
Substage m*,* $s > m > 0$*:* For each α of length m do the following. If $s = 1$, or $\mathcal{P}^s_\alpha$ was redefined at s, go to the beginning of Module α. Otherwise continue where Module α was left at stage $s-1$. This (re)defines $\mathcal{P}^s_{\alpha i}$ for $i \in \{0,1\}$.

Verification. By an argument similar to Claim 4.8 above, one checks that indeed $\lambda(\mathcal{P}^s_\alpha)_s \geq 4^{-|\alpha|}$ for each α, s such that $|\alpha| \leq s$.

Claim 4.10. *There is a computable function* r *such that for* $m = |\alpha|$*,*

$$r(m) \geq \#\{s > 0\colon\ \mathcal{P}^s_\alpha \text{ is redefined at stage } s\}.$$

Let $r(0) = 0$. Now suppose inductively that $r(m)$ has been defined correctly. Then $r(m) + h(m)$ bounds the number of stages s at which $\widetilde{\mathcal{P}}^s_\alpha$ can change. As long as this class is unchanged, with value $\widetilde{\mathcal{P}}$, each time Module α goes from (c) to (b), at least one of the classes $\widetilde{\mathcal{P}} \cap \mathcal{D}$, $\widetilde{\mathcal{P}} \setminus \mathcal{D}$ has lost a measure of $4^{-m}/8$. Since $\mathcal{D}$ is fixed while the module is at (c), this means that $\lambda\widetilde{\mathcal{P}}$ has dropped by at least $4^{-m}/8$. The number of times this can happen is therefore bounded by $8 \cdot 4^m$. Thus, if we let $r(m+1) = r(m) + 2 \cdot 8 \cdot 4^m(r(m) + h(m))$ then $r(m+1)$ is a correct bound for the number of times $\mathcal{P}^s_{\alpha i}$ can change ($i \in \{0,1\}$). This proves the claim.

Let $\mathcal{P}_\alpha = \lim_s \mathcal{P}^s_\alpha$. By the compactness of Cantor space there is a set $Z \in \bigcap\{\mathcal{P}_\alpha\colon\ \alpha \prec R\}$.

Claim 4.11. $R \leq_{\mathrm{tt}} Z'$.

To define a function $f \leq_T Z$ such that $R(x) = \lim_s f(x,s)$, given x, s, search for a string α of length $x+1$ and a $t \geq s$ such that

$$\forall \beta \forall i \in \{0,1\}\, [\beta\widehat{\ }i \preceq \alpha \rightarrow Z \notin (\mathcal{P}^t_{\beta\widehat{\ }(1-i)})_t].$$

Let $f_s(x)$ be the last bit of α.

Note that α, t exist by the hypothesis on Z. If $f(x,s) \neq f(x,s-1)$ then some $\mathcal{P}^s_\beta$ has been redefined for $|\beta| \leq x+1$. Thus the number of such stages s is computably bounded in x. If s is large enough such that $\mathcal{P}^s_\beta$ has stabilized for all β such that $|\beta| \leq x+1$, then $f_s(x) = R(x)$. ☐

5. Stronger variants of Demuth randomness

Recall that for a class $\mathcal{C} \subseteq 2^\omega$, we let

$$\mathcal{C}^\Box = \{A \colon \forall Y \in \mathcal{C} \cap \mathsf{MLR}[A \leq_T Y]\}.$$

In Section 3 we proved that all bases for Demuth randomness are strongly jump traceable. In this section we show that the class $(\omega^2\text{-c.a.})^\Box$ is properly contained in the bases for Demuth randomness. The proof is via sets passing a stronger variant of Demuth-tests: the number of version changes is merely limited by counting down along the canonical well-order of type ω^2.

5.1. ω^2-approximations

In the following we write $\mathbb{N}$ when we mean the natural numbers as a set, and ω when we mean their order type. We let $R = (\mathbb{N}, <_R)$ be the canonical computable well-order of type ω^2. Thus, $\langle x, y\rangle <_R \langle z, w\rangle$ if $x < y$, or $x = y$ and $z < w$. Recall that a *computable approximation* of a function $f \colon \mathbb{N} \to \mathbb{N}$ is a computable function $g_0 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $\lim_s g_0(x, s) = f(x)$. The following is a special case of a definition in [11], related to the Ershov hierarchy. To such a g_0 we adjoin a computable function g_1 with values in $\mathbb{N}$ which produces a descending sequence in the well-order R to count the number of times $g_0(x, s)$ changes.

Definition 5.1. An ω^2-*approximation* is a computable function

$$g = \langle g_0, g_1\rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$$

such that for each x and each $s > 0$,

$$g(x, s) \neq g(x, s-1) \to g_1(x, s) <_R g_1(x, s-1). \qquad (1)$$

In this case, g_0 is a computable approximation of a total Δ^0_2 function f. We say that g is an ω^2-*approximation of* f.

The following intuitive approach to ω^2-approximations is useful. Let $g = \langle g_0, g_1\rangle$ be an ω^2-approximation. Since we use the lexicographical ordering on $\mathbb{N} \times \mathbb{N}$, we may view g_1 itself as a pair of computable functions $\langle g_{10}, g_{11}\rangle$ where $g_{1i} \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ for $i = 0, 1$. Suppose $g_0(x, s) \neq g_0(x, s-1)$.

If $g_{10}(x, s) < g_{10}(x, s-1)$ we say there is a *top level change* at s. By assigning the value $g_{10}(x, 0)$, we state in advance how many such changes there will be.

If $g_{10}(x, s) = g_{10}(x, s-1)$ and hence $g_{11}(x, s) < g_{11}(x, s-1)$ we say this is a *bottom level change*. Whenever there has been a top level change at stage s, the new value $g_{11}(x, s)$ will bound the number of bottom level changes till the next top level change.

We show a robustness property of functions with an ω^2-approximation: it makes no difference whether given x, we count the changes at x, or at all the numbers less than x.

Fix some effective encoding of $\mathbb{N}^{<\omega}$ by natural numbers. For a function $f\colon \mathbb{N} \to \mathbb{N}$, let $\widehat{f}$ be the function $\lambda x. f\upharpoonright_x$ mapping x to the tuple of the first x values of f, encoded by a natural number.

Lemma 5.2. *Let $f : \mathbb{N} \to \mathbb{N}$. Then*

$$f \text{ has an } \omega^2\text{-approximation} \Leftrightarrow \widehat{f} \text{ has an } \omega^2\text{-approximation.}$$

Proof. $\Leftarrow$: Let $\langle h_0, h_1\rangle$ be an ω^2-approximation of $\widehat{f}$. Let $g_0(x,s)$ be the last entry of $h_0(x,s)$ if $h_0(x,s)$ encodes a tuple of length $x+1$; otherwise let $g_0(x,s) = 0$. Let $g_1(x,s) = h_1(x{+}1,s)$. Then $\langle g_0, g_1\rangle$ is an ω^2-approximation of f.
$\Rightarrow$: Let $\langle g_0, g_1\rangle$ be an ω^2-approximation of f. As before, we view g_1 as a pair of computable functions $\langle g_{10}, g_{11}\rangle$ where $g_{1i}\colon \omega\times\omega \to \omega$. Now let for $i = 0, 1$

$$h_{1i}(x,s) = \textstyle\sum_{y<x} g_{1i}(y,s).$$

Clearly $\langle h_0, \langle h_{10}, h_{11}\rangle\rangle$ is an ω^2-approximation, because $h_{10}(x,s)$ is nonincreasing in s, and $h_{11}(x,s) > h_{11}(x,s-1)$ is only possible if $h_{10}(x,s) < h_{10}(x,s-1)$.

If $f_s\upharpoonright_x \neq f_{s-1}\upharpoonright_x$ then $g_1(y,s) <_R g_1(y,s-1)$ for some $y < x$, whence $h_1(x,s) <_R h_1(x,s-1)$. ☐

5.2. ω^2-Demuth randomness

Definition 5.3. An ω^2-*Demuth test* is a sequence of c.e. open classes $(\mathcal{S}_m)_{m\in\mathbb{N}}$ such that $\forall m\, \lambda\mathcal{S}_m \le 2^{-m}$, and there is a function f with an ω^2-approximation such that $\forall m\, \mathcal{S}_m = [W_{f(m)}]^{\prec}$. A set Z *passes* the test if $Z \notin \mathcal{S}_m$ for almost every m. We say that Z is ω^2–*Demuth random* if Z passes each ω^2–Demuth test.

Since every Demuth test is an ω^2-Demuth test, every ω^2-Demuth random set is Demuth random.

We say that a set Y is called ω^2-*low* if Y' is ω^2-c.a. In this case, Y is clearly ω^2-c.a. itself. Adapting the proof of the first part of Theorem 4.4 yields the fact that the class of ω^2-low sets is ω^2-Demuth test-compatible:

Proposition 5.4. *Every ω^2-Demuth test is passed by an ω^2-low set Y.*

Proof. If $(\mathcal{S}_k)_{k\in\mathbb{N}}$ is an ω^2-Demuth test, then the test $(\mathcal{G}_m)_{m\in\mathbb{N}}$ such that $\lambda\mathcal{G}_m \le 4^{-m}$ obtained in the proof of Lemma 2.5 is an ω^2-Demuth test because of Lemma 5.2. Thus, it suffices to find an ω^2-low set Z that passes $(\mathcal{G}_m)_{m\in\mathbb{N}}$, where $(\mathcal{G}_m)_{m\in\mathbb{N}}$ is an ω^2-Demuth test such that $\lambda\mathcal{G}_m \le 4^{-m}$ for each m. We have versions $\mathcal{G}_m[s]$ at stage s, such that $(\mathcal{G}_m[s])_{m,s\in\mathbb{N}}$ can be extended to an ω^2-approximation of $(\mathcal{G}_m)_{m\in\mathbb{N}}$.

The Modules m and the construction are as before. In the verification we now show a variant of Claim 4.9, that $(\mathcal{P}^s_m)_{m,s\in\mathbb{N}}$ can be extended to an ω^2-approximation in the sense of Definition 5.1. As one would expect, the counting is a bit different now. By Lemma 5.2, the sequence of tuples of versions $(\mathcal{G}_i[s])_{i\le m+1}$ can be extended to an ω^2-approximation. As long as such a tuple is stable, each time a Module r goes from (b) to (c) and then back to (b), where $r \le m$ is least, the relevant class $\widetilde{\mathcal{P}}$ in that module has lost a measure of 4^{-r-1}; so there are at most $\Pi_{r=0}^m 4^{r+1}$ such transitions. This yields an ω^2-approximation extending $(\mathcal{P}^s_m)_{m,s\in\mathbb{N}}$, as required. Hence the set $Z \in \bigcap_m \mathcal{P}_m$ is ω^2-low by the same argument as before. □

It is not hard to build an ω^2-Demuth test that covers all Demuth tests. Thus, there is an ω^2-low Demuth random set. We will strengthen this shortly in Lemma 5.6.

5.3. A class properly contained in the bases for Demuth randomness

Theorem 5.5. *The class $(\omega^2\text{-c.a.})^\Box$ is a proper subclass of the bases for Demuth randomness.*

Proof. The main, technical part of the proof is the following:

Lemma 5.6. *There is an ω^2-Demuth test $(\mathcal{S}_m)_{m\in\mathbb{N}}$ such that every set Z passing this test is Demuth random in every K-trivial set A.*

Given the lemma, we obtain the theorem as follows. By Proposition 5.4, let Y be an ω^2-c.a. set Y passing the test $(\mathcal{S}_m)_{m\in\mathbb{N}}$ from the lemma.

For the inclusion, suppose $A \in (\omega^2\text{-c.a.})^\Box$. Then $A \in (\omega\text{-c.a.})^\Box$, so A is K-trivial as observed in Proposition 1.6. Note that Y is Demuth random in A by the lemma, and $A \le_T Y$ by the definition of the class $(\omega^2\text{-c.a.})^\Box$. Hence A is a base for Demuth randomness.

For the properness, we apply a result of Nies [21]. First we give some background. For cost functions, see [19, Section 5.3]. For the definition an ω^2-benign cost functions, see the last section of [11]. For any Δ^0_2 set Y one can define a cost function c_Y such that any c.e. set A obeying c_Y is Turing below Y; see [19, Fact 5.3.13]. If Y is ω^2-c.a. then c_Y is ω^2-benign.

In [16] it is shown that for each ω^2-benign cost function c, there is a c.e. set A obeying c and an ω^2-c.a. ML-random set Z such that $A \not\le_T Z$. We apply this to the ω^2-benign cost function c_Y, where Y is the ω^2-c.a. set obtained above. This yields a c.e. set $A \le_T Y$ and an ω^2-c.a. ML-random set Z such that $A \not\le_T Z$. Hence $A \notin (\omega^2\text{-c.a.})^\Box$.

The proof of Lemma 5.6 is in three steps. In Step 1 we use a golden run technique in the style of Nies [18] to cover a $\Sigma^0_1(A)$ class $\mathcal{U}$ for K-trivial A by a plain Σ^0_1 class $\mathcal{V}$ with only a slight excess in measure. While we can't obtain $\mathcal{V}$ effectively, the number of attempts at building $\mathcal{V}$ is computably bounded in an index for $\mathcal{U}$.

In Step 2, we apply this covering procedure to all the components of a Demuth test $(\mathcal{G}_m)_{m\in\mathbb{N}}$ relative to a K-trivial oracle A, thereby obtaining an ω^2-Demuth test $(\mathcal{R}_m)_{m\in\mathbb{N}}$. Kučera proved that there is a single ω-c.a. function h that dominates any function computed by a K-trivial set A. The function we apply this to is the A-computable bound on the number of version changes in $(\mathcal{G}_m)_{m\in\mathbb{N}}$. Using this we can argue that each of these ω^2-Demuth tests $(\mathcal{R}_m)_{m\in\mathbb{N}}$ has $p(m)$ top level version changes for a fixed computable function p, which bounds the number of times an approximation $h_s(m)$ to $h(m)$ increases.

In Step 3, we use this fact to engineer a single ω^2-Demuth test $(\mathcal{S}_m)_{m\in\mathbb{N}}$ that covers all the ω^2-Demuth test $(\mathcal{R}_m)_{m\in\mathbb{N}}$ obtained in the second step.

Step 1. Let $(W_e^X)_{e\in\mathbb{N}}$ be an effective listing of sets that are c.e. in an oracle X. We view these sets as subsets of $2^{<\omega}$. For each index e and oracle X, let $\mathcal{U}_e^X = [W_e^X]^{\prec}$.

Claim 5.7. *Let A be K-trivial. Then there is an effective double sequence $(\mathcal{V}_e^t)$ $(e, t \in \mathbb{N})$ of indices for Σ_1^0 classes in 2^ω with the following properties:*

(i) $\#\{r > 0 \colon \mathcal{V}_e^r \neq \mathcal{V}_e^{r-1}\} = O(4^e)$.

(ii) $\mathcal{U}_e^A \subseteq \mathcal{V}_e$ *and* $\lambda\mathcal{V}_e \leq \lambda\mathcal{U}_e^A + 2^{-e}$, *where* $\mathcal{V}_e = \lim_r \mathcal{V}_e^r$.

To prove the claim, let M be a prefix-free oracle machine such that $[\{\tau \colon M^A(0^e1\tau)\downarrow\}]^{\prec} = \mathcal{U}_e^A$. The Main Lemma [19, 5.5.1], which embodies the essence of the golden run method of Nies [18], states that given such a machine M, there is a computable sequence of stages $0 = q(0) < q(1) < \dots$ such that

$$\widehat{S} = \sum_r \widehat{c}_{M,A}(x,r) \, [\![x \text{ is least s.t. } A_{q(r+1)}(x) \neq A_{q(r+2)}(x)]\!] < \infty, \quad (2)$$

where

$$\widehat{c}_{M,A}(x,r) = \sum_\sigma 2^{-|\sigma|} \left[\!\!\left[\begin{array}{l} M^A(\sigma)[q(r+1)]\downarrow \ \& \\ x < \mathsf{use}\, M^A(\sigma)[q(r+1)] \leq q(r) \end{array} \right]\!\!\right].$$

We write $M^A(\sigma) \downarrow\downarrow [r]$ if $M^A(\sigma)[q(r+1)]\downarrow$ & $\mathsf{use}\, M^A(\sigma)[q(r+1)] \leq q(r)$. Let

$$U_{e,r} = \{\tau \colon M^A(0^e1\tau) \downarrow\downarrow [r]\}.$$

Construction of the double sequence $(\mathcal{V}_e^r)_{r,e\in\mathbb{N}}$. Fix e. At the end of each stage r we update a parameter $r^* \leq r$. Initially let $r^* = 0$. Whenever we see at stage r that our current attempt to cover $\mathcal{U}_e^A$ by a Σ_1^0 class $[W]^{\prec}$ is in danger because $[W]^{\prec}$ contains too much not in $[U_{e,r}]^{\prec}$, we start a new attempt to cover $\mathcal{U}_e^A$ by setting $r^* = r$. More formally:

Stage r. At the beginning of the stage let $W = \bigcup\{U_{e,t} \colon r^* \leq t \leq r\}$. If $\lambda([W]^{\prec} - [U_{e,r}]^{\prec}) > 2^{-e}$ let $r^* = r$. In any case, define

$$\mathcal{V}_e^r = [\bigcup\{U_{e,t}\colon\ r^* \le t\}]^{\prec}.$$

Verification. An *e-active stage* is a stage at which we redefine r^*. Clearly $[C]^{\prec} - [D]^{\prec} \subseteq [(C-D)]^{\prec}$ for any sets $C, D \subseteq 2^{<\omega}$. Thus, at an e-active stage r we have $\sum_\tau 2^{-|\tau|}[\![\tau \in V - U_{e,r}]\!] > 2^{-e}$. Any string $\tau \in V - U_{e,r}$ contributes $2^{-|\tau|-e-1}$ to the sum $\widehat{S}$ in (2), because for some t, $r^* \le t < r$, we had $M^A(0^e 1\tau) \downarrow\downarrow [t]$, and then A changed below the use of this computation. In total, this makes a contribution of $O(4^{-e})$ to the sum (2) restricted strings σ with prefix 0^e1. Thus, since $\widehat{S} < \infty$, there are only $O(4^e)$ many e-active stages.

Let k be the last e-active stage. For $r \ge k$ let $W^r = \bigcup_{k \le t \le r} U_{e,t}$. Then $\mathcal{V}_e = \lim_r \mathcal{V}_e^r = \bigcup_{k \le r} [W^r]^{\prec}$.

It remains to show that $\lambda(\mathcal{V}_e - \mathcal{U}_e^A) \le 2^{-e}$. By the definition of k we have $\lambda([W^r]^{\prec} - [U_{e,r}]^{\prec}) \le 2^{-e}$ for each e and each $r \ge k$. Let $\epsilon > 0$ be arbitrary. Pick $t \ge k$ so large such that $\lambda(\mathcal{V}_e - [W^t]^{\prec}) \le \epsilon$, and the tail sum in (2) for $\widehat{S}$ at t is at most $2^{-e-1}\epsilon$. Then

$$\begin{aligned}\lambda(\mathcal{V}_e - \mathcal{U}_e^A) &\le \lambda(\mathcal{V}_e - [W^t]^{\prec}) + \lambda([W^t]^{\prec} - [U_{e,t}]^{\prec}) + \lambda([U_{e,t}]^{\prec} - \mathcal{U}_e^A) \\ &\le \epsilon + 2^{-e} + \epsilon.\end{aligned}$$

This concludes the proof of Claim 5.7.

Step 2. Since every K-trivial set is Turing below a c.e. K-trivial [18], for Lemma 5.6 it suffices to consider K-trivial sets A that are computably enumerable. Fix such an A.

Claim 5.8. *There is a computable enumeration $(A_s)_{s\in\mathbb{N}}$ of A as follows: for every Turing functional Γ, for almost every x, the number of s such that $\Gamma^{A_{s-1}}(x)\downarrow$ but $\Gamma^{A_s}(x)\uparrow$ is bounded by x^2.*

Since A is K-trivial, it is jump traceable with trace bound $x\log^3 x + 1$ by [19, Exercise 8.4.8]. In fact, in that exercise we can trace instead of the jump any Turing functional Γ; so there is a c.e. trace $(T_x)_{x\in\mathbb{N}}$ such that $\#T_x \le x\log^3 x + 1$ and $\Gamma^A(x) \in T_x$ for all x such that $\Gamma^A(x)\downarrow$. There is a universal trace $(U_x)_{x\in\mathbb{N}}$, namely $U_x = \bigcup_{i \le \log x} T_x^i$, where $(T_x^i)_{i,x\in\mathbb{N}}$ is an effective listing of all the c.e. traces for the bound $x\log^3 x$.

Note that $\#U_x \le x^2$, and $\Gamma^A(x) \in U_x$ for almost all x such that $\Gamma^A(x)\downarrow$. We now argue as in the proof of Lemma 2.7 (which is [15, Lemma 4.1]) to establish the claim.

We now discuss how build an ω^2-Demuth test with a fixed bound on the number of top level changes that covers a given Demuth test relative to A. By a result of Kučera (see [19, Exercise 5.1.6]), let h be an increasing ω-c.a. function that dominates every function computable from some K-trivial set (for almost every input). Since h is ω-c.a., after increasing h if necessary, we may assume that there is a computable approximation $(h_s(x))_{x,s\in\mathbb{N}}$ and a computable function p such that $h(x) = \lim_s h_s(x)$, $h_{s-1}(x) \leq h_s(x)$ for each x, s, and for at most $p(x)$ many s is the inequality proper.

Suppose we are given a Demuth test $(\mathcal{G}_m)_{m\in\mathbb{N}}$ relative to A. Leaving out the first component we may assume that $\lambda\mathcal{G}_m \leq 2^{-m-1}$ for each m. We have $\mathcal{G}_m[t] = \mathcal{U}^A_{g(m,t)}$ for some $g \leq_T A$ such that $g(m,t) > m$ for each m. We may also assume that $g(m,t) \neq g(m,t-1)$ for at most $h(m)$ many t, because h dominates the A-computable bound on the number of such changes.

Claim 5.9. *There is an ω^2-Demuth test $(\mathcal{R}_m)$ such that the version of the m-th component has at most $p(m)$ top level changes, and $\mathcal{G}_m \subseteq \mathcal{R}_m$ for all m.*

To prove this, we cover $\mathcal{G}_m$, a component of a Demuth test relative to A, by the m-th component of the ω^2-Demuth test $(\mathcal{R}_m)_{m\in\mathbb{N}}$ that doesn't rely on the oracle A. Recall the computable enumeration $(A_s)_{s\in\mathbb{N}}$ of A from the Claim 5.7. The top level changes of $(\mathcal{R}_m)_{m\in\mathbb{N}}$ will be due to the changes of $h_t(m)$; the bottom level changes are due to the changes of $(A_s)_{s\in\mathbb{N}}$.

Let Γ be a Turing functional such that $\Gamma^A(\langle m,i\rangle)$ is the i-th value of the form $g(m,t)$. We may assume that $\lambda[\mathcal{U}_{\Gamma^X(\langle m,i\rangle)}]^{\prec} \leq 2^{-m-1}$ for each oracle X.

Let $d(m,i,k)$ be a computable function, nondecreasing in i,k, as follows: $e = d(m,i,k)$ is an index greater than m such that $W_e = \emptyset$ until (if ever) there is a k-th value of the form $v = \Gamma^{A_s}(m,i)$; once the value is there we let W_e copy W_v.

Now at stage t, given m, let $i \leq h_t(m)$ be maximal such that $\Gamma^{A_t}(\langle m,i\rangle)\downarrow$. Let this be the k-th value of this form. Let $e = d(m,i,k)$. Let

$$\mathcal{R}_m[t] = \mathcal{V}^t_e,$$

where $\mathcal{V}^t_e$ is defined in Claim 5.7.

Clearly $\lambda\mathcal{R}_m[t] \leq 2^{-m}$ because of our assumption on Γ and because $e > m$. Next we we show that the construction provides an ω^2-approximation where version m has at most $p(m)$ top level changes (recall that $p(m)$ bounds the number of times $h_s(m)$ changes).

Each change of $h_t(m)$ corresponds to a top level change; their number is bounded by $p(m)$. For the bottom level changes, suppose $[s_0, s_1]$ is a maximal interval such that $h_t(m) = r$ is stable for $s_0 \leq t \leq s_1$. Then in the definition of $\mathcal{R}_m[t]$ we have $i \leq r$, and hence $k \leq \langle m,r\rangle^2$ by Claim 5.8. Thus $e \leq q := d(m,r,\langle m,r\rangle^2)$. By Claim 5.7 the number of changes of versions $\mathcal{R}_m[t]$ for $s_0 \leq t \leq s_1$, i.e. of bottom level changes, is bounded by $\langle m,r\rangle^2 q4^q$ for almost all m. This quantity has been computed from m and r, as required for an ω^2-approximation.

Clearly $\mathcal{G}_m \subseteq \mathcal{R}_m$. This establishes the claim.

Step 3. No matter which K-trivial c.e. set A and Demuth test $(\mathcal{G}_m)_{m\in\mathbb{N}}$ relative to A is given, the ω^2-Demuth test $(\mathcal{R}_m)_{m\in\mathbb{N}}$ obtained in Claim 5.9

has $p(m)$ top level changes for a fixed computable function p. This enables us to define a single ω^2-Demuth test $(\mathcal{S}_m)_{m\in\mathbb{N}}$ that is stronger than all these ω^2-Demuth tests, which will conclude the proof of Lemma 5.6.

The construction of $(\mathcal{S}_m)_{m\in\mathbb{N}}$ is a refinement of the construction of a so-called special test in [11, Lemma 7.8] which is stronger than all Demuth tests. There, it was sufficient to build a test where the versions change finitely often. Here, we need an ω^2-approximation for the versions.

Let us say that a function $g\colon \mathbb{N} \to \mathbb{N}$ is $(\omega \cdot p)$-c.a. if it has an ω^2-approximation with $p(m)$ top level changes for each m. We first need to show that a single $(\omega \cdot p)$-c.a. function exists that can simulate all others. This is like [11, Lemma 7.4].

Claim 5.10. *There is an $(\omega\cdot p)$-c.a. function f as follows: for each $(\omega\cdot p)$-c.a. function g, there is e such that $g(m) = f(\langle e, m\rangle)$ for each m.*

To show this, recall that R is the computable well order which is induced on $\mathbb{N}$ by ω^2 with the lexicographical ordering via the canonical pairing function $\langle\, ,\, \rangle$. Define a *partial $(\omega \cdot p)$-approximation* to be a partial computable function $\psi = \langle \psi_0, \psi_1\rangle\colon \mathbb{N}^2 \to \mathbb{N}^2$ such that $\mathrm{dom}(\psi)$ is closed downward in both variables, and such that for all n and $s > 0$, if $(n, s) \in \mathrm{dom}(\psi)$ and $\psi(n, s) \neq \psi(n, s-1)$ then $\psi_1(n, s) <_R \psi_1(n, s-1)$; furthermore, the first component of $\psi_1(n, 0)$ viewed as a pair is $p(n)$ (i.e., we allow at most $p(n)$ top level changes of $\psi_0(n, s)$). There is an effective listing $(\psi^e)_{e\in\mathbb{N}}$ of all partial $(\omega \cdot p)$-approximations.

Write $\psi^e(n, t)\downarrow [s]$ to denote that $\langle n, t\rangle \in \mathrm{dom}(\psi^e)$ and that this fact is discovered after s steps of computation of a fixed universal machine (so $t \leq s$). We may assume that each $\mathrm{dom}(\psi^e)\,[s]$ is closed downward in both variables n, t. Given e, n and s, let $t \leq s$ be greatest such that $\psi^e(n, t)\downarrow[s]$, and let $g^e(n, s) = \psi^e_0(n, t)$. If there is no such t, then let $g^e(n, s) = 0$.

Now $(g^e)_{e\in\mathbb{N}}$ is a uniformly computable sequence of functions, and the function f defined by letting $f(\langle e, n\rangle) = \lim_s g^e(n, s)$ is total for all e. Clearly f is $(\omega \cdot p)$-c.a. If ψ^e is total then $f(\langle e, n\rangle) = \lim_s \psi^e_0(n, s)$ for all n. This proves Claim 5.10.

Recall the notation $\mathcal{W}^{(\leq\epsilon)}$ introduced in the Cut-off Lemma 2.1, where $\mathcal{W}$ is a c.e. open class and $\epsilon > 0$ a rational. Also recall that $\mathcal{U}_e = [W_e]^{\prec}$. Now let q be an $(\omega \cdot p)$-c.a. function such that

$$\mathcal{U}_{q(\langle e,n\rangle)} = \mathcal{U}^{(\leq 2^{-n})}_{f(\langle e,n\rangle)}$$

(where f is the function from Claim 5.10). Let

$$\mathcal{S}_m = \bigcup\nolimits_{e<m} \mathcal{U}_{q(e,e+m+1)},$$

then $\lambda\mathcal{S}_m \leq \sum_{e<m} 2^{-(e+m+1)} \leq 2^{-m}$. Furthermore, $(\mathcal{S}_m)_{m\in\mathbb{N}}$ is an ω^2-Demuth test because q is an ω^2-c.a. function and by Lemma 5.2. If Z passes this test, then Z passes every test $(\mathcal{R}_m)_{m\in\mathbb{N}}$ obtained in Claim 5.9, and

hence Z is Demuth random in every K-trivial set. This completes the proof of Lemma 5.6 and of Theorem 5.5. □

6. Towards a traceability characterization of $(\omega^2\text{-c.a.})^\diamond$

In the following, the binary function $\overline{p}$ will always be a computable approximation from above to an unbounded nondecreasing function p. Thus $\overline{p}(x,s) \geq \overline{p}(x,s+1)$ for each x,s, and $p(x) = \lim_s \overline{p}(x,s)$ for each x. We say that $\overline{p}$ is an *inverse ω-c.a. approximation* if for the function

$$r_s(n) = \max\{x\colon\ \overline{p}(x,s) \leq n\},$$

the number of stages s such that $r_s(n) \neq r_{s-1}(n)$ is computably bounded in n.

Example 6.1. Consider $\overline{p}(x,s) = \min\{C_s(y)\colon\ y \geq x\}$, where $C_s(y)$ is the usual plain Kolmogorov complexity of y at stage s. Then $p(x) = \lim_s \overline{p}(x,s)$ is dominated by each order function (see [19, 2.1.22]). The approximation $r_s(n)$ increases at most 2^{n+1} times. Thus, $\overline{p}$ is inverse ω-c.a.

A $\overline{p}$*-trace* is a computable double sequence $(T_{x,s})_{x,s\in\mathbb{N}}$ of strong indices for finite sets such that $T_{x,s} \subseteq T_{x,s+1}$ for each x,s, and where $T_x = \bigcup_s T_{x,s}$,

$$\#(T_x \setminus T_{x,s}) \leq \overline{p}(x,s)$$

for each x,s. The idea is that at any stage s we can adjust downwards the bound on the number of elements that are allowed to go into T_x after stage s.

By Example 6.1, the following notion implies strong jump traceability.

Definition 6.2. We say that A is ω-jump traceable if J^A has a $\overline{p}$ trace for each inverse ω-c.a. computable approximation $\overline{p}$ such that p is unbounded.

Theorem 6.3. *Every set A in $(\omega^2\text{-low})^\diamond$ is ω-jump traceable. That is, if a c.e. set A is Turing below each ω^2-low ML-random set, then it is ω-jump traceable.*

Proof. We adapt our argument leading to the first statement of Corollary 4.5, that the diamond class of superlowness is contained in the strongly jump traceable sets. First we adapt Theorem 3.2 and its proof in order to show that each base for ω^2-Demuth randomness is ω-jump traceable. For $m,s \in \mathbb{N}$, we let

$$I_{m,s} = \{x\colon\ 2^m \leq \overline{p}(x,s) < 2^{m+1}\}.$$

To define the Turing functional $\Gamma^X(m,t)$, we use the interval $I_{m,t}$: for each string α of length t,

$$\mathcal{U}_{\Gamma^\alpha(m,t)} = \{Z\colon \forall x \in I_{m,t}\ [J^\alpha_t(x)\downarrow \text{with use}\, u \ \Rightarrow\ \alpha\restriction_u \preceq \Phi^Z_t]\}.$$

Now define versions $\mathcal{G}^A_m[t]$ as before. They determine an ω^2-Demuth test relative to A by the hypothesis that $\overline{p}$ is inverse ω-c.a.

The computable double sequence of finite sets $(T_{x,s})_{x,s\in\mathbb{N}}$ is defined as expected. At stage t, for each string α of length t such that $y = J^\alpha_t(x)$ is defined and the measure of the current approximation to the c.e. open set $\mathcal{U}_{\Gamma^\alpha(m,t)}$ exceeds 2^{-m}, put y into $T_{x,s}$. It is not hard to verify that $(T_{x,s})_{x,s\in\mathbb{N}}$ is a $\overline{p}$ trace for J^A.

Claim 3.4 is adapted as follows: given $\overline{p}$ as above, a Turing functional Φ and a superlow c.e. set A, there is an ω^2–Demuth test $(\mathcal{S}_m)_{m\in\mathbb{N}}$ such that, if $A = \Phi^Y$ for some Y passing this test, then J^A has a $\overline{p}$-trace.

An analog of Lemma 4.3 now shows that, if a class $\mathcal{C}$ is ω^2-Demuth test–compatible, then each set $A \in \mathcal{C}^\diamond$ is ω-jump traceable. It remains to adapt the first statement of Theorem 4.4, which now says that ω^2-lowness is ω^2-Demuth test–compatible. This was already done in Proposition 5.4. $\square$

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