2015 30th Annual ACM/IEEE Symposium on Logic in Computer Science

A unifying approach to the Gamma question

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***Abstract*—The Gamma question was formulated by Andrews et al. in "Asymptotic density, computable traceability and 1-randomness" (2013, available at http://www.math.wisc.edu/~lempp/papers/traceable.pdf). It is related to the recent notion of coarse computability which stems from complexity theory. The Gamma value of an oracle set measures to what extent each set computable with the oracle is approximable, in the sense of density, by a computable set. The closer to 1 this value is, the closer the oracle is to being computable. The Gamma question asks whether this value can be strictly in between 0 and $1/2$.**

We say that an oracle is weakly Schnorr engulfing if it computes a Schnorr test that succeeds on all computable reals. We show that each non weakly Schnorr engulfing oracle has a Gamma value of at least $1/2$. Together with a recent result of Kjos-Hanssen, Stephan, and Terwijn, this establishes new examples of such oracles. We also give a unifying approach to oracles with Gamma value 0. We say that an oracle is infinitely often equal with bound h if it computes a function that agrees infinitely often with each computable function bounded by h. We show that every oracle which is infinitely equal with bound 2^{d^n} for $d > 1$ has a Gamma value of 0. This provides new examples of such oracles as well.

We present a combinatorial characterization of being weakly Schnorr engulfing via traces, which is inspired by the study of cardinal characteristics in set theory.

I. Introduction

Generic-case complexity is a subfield of computational complexity. It started with the observation that some problems that are difficult to solve in full are easy to solve on "most inputs", namely on a set of inputs of density 1. This notion was introduced by Kapovich et al. [11]. They showed among other things that for a large class of finitely generated groups, the generic case complexity of the word problem is linear.

This notion has recently been extended by Jockusch and Schupp [10]. The authors identify two notions that can be proved to be incomparable. The first is generic computability, where one must always give the right answer, without having to provide an answer for a small set of inputs. The second is coarse computability, where one always has to provide an answer, with the possibility of being wrong on a small set of inputs. In both cases, a set of inputs is considered small if it is of upper density 0; this will be made precise in the paper.

Then Andrew et al. [1] assign a value γ to each set of natural numbers. They use this to assign a value Γ to each Turing degree. They prove that the Gamma values of 0, $1/2$ and 1 can be realized by a Turing degree. If a degree has a Gamma value strictly larger than $1/2$, then it is computable and its Gamma value, in fact, equals 1. They ask whether a Turing degree can have a Gamma value strictly in between 0 and $1/2$.

We provide a unifying approach to this question. Among the non-computable degrees, the members of two kinds of degree classes are known to have a Gamma value of $1/2$: the computably traceable degrees, and the computably dominated random degrees. The proofs suggest that in the two cases this holds for very different reasons. We show here that, on the contrary, they are both contained in a common class that implies a Gamma value of $1/2$: being not weakly Schnorr engulfing. Together with recent, yet unpublished work of Kjos-Hanssen, Stephan, and Terwijn [14], this establishes new examples of such degrees.

We also unify the examples of degrees with a Gamma value of 0, by relating them with a property of degrees we call H-*infinitely often equal* for an appropriate computable bound H. We end the paper by giving a combinatorial characterization of being weakly Schnorr engulfing, close to the notion of being H-infinitely often equal.

II. Preliminaries and notation

In this paper, we work in the space of infinite sequences of 0's and 1's, the Cantor space, denoted by 2^ω. We call finite sequences of 0's and 1's *strings*, elements of the Cantor space *sequences* and sets of sequences *sets*. We sometimes also use the word sequence to denote sequences of various objects (typically integers), and when we do so we will always specify it to avoid any ambiguity. We also sometimes view a sequence X as the subset of ω containing n iff $X(n) = 1$ without necessarily specify it. For a string σ, we denote the set of sequences extending σ by $[\sigma]$ and we call those sets *cylinders*. We denote by λ the unique probability measure on 2^ω such that $\lambda([\sigma]) = 2^{-|\sigma|}$ for any string σ, where $|\sigma|$ denotes the length of σ.

The Cantor space is endowed with the product topology, for which a set is clopen iff it is a finite union of cylinders, and open iff it is a countable union of cylinders. A set $\mathcal{U}$ is *effectively open*, or Σ^0_1, if there exists a computably enumerable sequence of strings $\{\sigma_i\}_{i\in\omega}$ such that $\mathcal{U} = \bigcup_i [\sigma_i]$. A set $\mathcal{U}$ is *effectively closed*, or Π^0_1, if it is the complement of a Σ^0_1 set. Finally we denote by $\langle .,. \rangle$ a fixed computable bijection from $\omega \times \omega$ to ω.

We now introduce notation which is less standard than the one of the previous paragraphs. It will be convenient to expose the work of this paper, especially for results of Section V. For

1043-6871/15 $31.00 © 2015 IEEE
DOI 10.1109/LICS.2015.60

a sequence $X \in 2^\omega$ and a finite interval $I \subset \omega$, we denote by $X \upharpoonright_I$ the string $X(I(0))^\frown \dots {}^\frown X(I(m-1))$, where m is the length of I, and $I(k)$ denotes the k-th element of I. For a string σ and a finite interval I, we write $[\sigma]_I$ to denote the set of sequences extending σ, that is, $\{X \in 2^\omega \ : \ X \upharpoonright_I = \sigma\}$. For a finite interval $I \subset \omega$ and a clopen set $\mathcal{J} \subseteq 2^\omega$, we write $\mathcal{J} \subseteq 2^I$ if there exists a set of strings $\sigma_0, \dots, \sigma_k$ of length $|I|$ such that $\mathcal{J} = \bigcup_{i \leq k} [\sigma_i]_I$.

In this paper we will be interested in having a canonical coding between sequences and functions $f : \omega \to \omega$ which are strictly bounded by some $H : \omega \to \omega$. Such a function H will generally be an *order function*, that is, a computable function H such that $H(n) \leq H(n+1)$ and $\lim_n H(n) = +\infty$. To make the coding work nicely, we will always assume that H is of the form $2^{\widetilde{H}(n)}$. Given a sequence X and such a bound $H(n) = 2^{\widetilde{H}(n)}$, we denote by f_X the function corresponding to X. Formally we define $H'(n) = \sum_{m<n} \widetilde{H}(m)$ (with $H'(0) = 0$), and $f_X(n)$ to be the integer less than $2^{\widetilde{H}(n)}$ which is represented in binary by the string $X \upharpoonright_{[H'(n), H'(n+1))}$. Conversely, given f with $f(n) < H(n) = 2^{\widetilde{H}(n)}$, we write X_f to denotes the sequence X such that $f_X = f$.

A. Preliminaries on coarse computability

The notion of coarse computability has received a lot of recent attention; see for instance [1] and [9].

Definition II.1. A sequence A is *coarsely computable* if there is a computable sequence X such that the lim inf of the frequency of positions n on which $A(n) = X(n)$, equals 1. More formally, let us introduce the function:

$$\underline{\rho}(Z) = \liminf_n \frac{|Z \cap [0,n]|}{n}$$

The sequence A is coarsely computable if for some computable sequence X we have $\underline{\rho}(A \leftrightarrow X) = 1$, where $A \leftrightarrow X$ denotes the sequence which, seen as a subset of ω, contains n iff $A(n) = X(n)$.

A real number can naturally be assigned to sequences. This number can be seen as an indication of how far the sequence is from being coarsely computable. It is defined by:

$$\gamma(A) = \sup_{X \text{ computable}} \underline{\rho}(A \leftrightarrow X)$$

We will refer to this as the *gamma value* of A. Andrews, Cai, Diamondstone, Jockusch and Lempp [1] had the interesting idea to define a similar value for Turing degrees, which indicates how far a degree is from being computable:

$$\Gamma(d) = \inf\{\gamma(A) \colon \ A \in 2^\omega \text{ is of degree } d\}$$

This will be referred to in this paper as the *Gamma value* of d (with a capital 'G'). In practice we will often write $\Gamma(A)$ for a set $A \in 2^\omega$ to mean $\Gamma(d)$ where d is the Turing degree of A. It is easy to see that one can equivalently consider $\Gamma(A)$ to be the infinium over the values $\gamma(B)$ for every $B \leq_T A$, rather than just every $B \equiv_T A$. The reason is that given any $B <_T A$, we can add to the sequence B all the information about A at some very sparse computable set of positions, giving a new set Turing equivalent to A, with the same gamma value as B has.

The Gamma question is: which real numbers can be realized by the Gamma value of a degree? In [1] Andrews et al. proved that $\Gamma(A) > 1/2$ iff $\Gamma(A) = 1$ iff A is computable. They also gave examples of sequences A with $\Gamma(A) = 1/2$ and examples of sequences A with $\Gamma(A) = 0$. These examples will be detailed in Section III. It is unkown whether some sequence has a Gamma value strictly between 0 and $1/2$.

B. Preliminaries on algorithmic randomness

Algorithmic randomness uses the tools of computability theory to give a formal definition of the notion of a random infinite binary sequence, the type of sequence we would expect as the result of independent coin tosses. The reader can refer to [19] or [8] for a background on algorithmic randomness. We briefly discuss the notions important for this paper.

1) Martin-Löf randomness: The first satisfactory definition of randomness was given by Martin-Löf in [17]:

Definition II.2. A *Martin-Löf test* is given by a uniform intersection $\bigcap_n \mathcal{U}_n$ of effectively open sets such that the function $n \mapsto \lambda(\mathcal{U}_n)$ is bounded by a decreasing computable function with limit 0. We say that a sequence X is *Martin-Löf random* if it belongs to no Martin-Löf test.

Note that one can analogously define sequences which are Martin-Löf random for any computable probability measure μ, by simply replacing λ by μ in the above definition. Such notions will be discussed at the end of Section III-A.

2) Schnorr randomness: The notion of Schnorr randomness was introduced by Schnorr [23], who was aiming at a concept that is more constructive than the one of Martin-Löf.

Definition II.3. A *Schnorr test* is given by a uniform intersection $\bigcap_n \mathcal{U}_n$ of effectively open sets such that the function $n \mapsto \lambda(\mathcal{U}_n)$ is computable and decreasing with limit 0. We say that a sequence X is *Schnorr random* if it belongs to no Schnorr test. For a sequence A we denote by A-Schnorr tests and A-Schnorr random sequences the relativized notions.

It is well known that Schnorr randomness is strictly weaker than Martin-Löf randomness. Downey and Griffiths [7] gave another characterization of Schnorr randomness. It can be seen as an effective version of the Borel Cantelli lemma. We review their result in terms of the following test notion. Our term is derived from the corresponding term "Solovay test"; recall Solovay tests characterize Martin-Löf randomness.

Definition II.4. A *Schnorr-Solovay test* is a computable sequence $\{\mathcal{C}_n\}_{n\in\omega}$ of clopen sets such that $\sum_n \lambda(\mathcal{C}_n)$ is finite and computable. A sequence X is *captured* by a Schnorr-Solovay test $\{\mathcal{C}_n\}_{n\in\omega}$ if X is in infinitely many sets $\mathcal{C}_n$, that is, $X \in \bigcap_n \bigcup_{k\geq n} \mathcal{C}_k$.

We say that Test 2 *covers* Test 1 if every sequence failing Test 1 also fails Test 2. Downey and Griffiths [7] proved that Schnorr-Solovay tests characterize Schnorr randomness. Actually, their notion of a *total Solovay tests* is slightly more general than ours in that their components $\mathcal{C}_n$ in Definition II.4 are effectively open sets uniformly in n, rather than clopen sets. They proved that each Schnorr test can be covered by a total Solovay test, and vice-versa. On the other hand, it is clear that every total Solovay test can be covered by a Schnorr-Solovay test, by replacing each open set C_n by an effective sequence of clopen sets with union C_n. So our notion used here is indeed equivalent to theirs.

Proposition II.5 ([7])**.** *A sequence X is Schnorr random iff it is not captured by any Schnorr-Solovay test.*

Definition II.6. A Schnorr-Solovay test $\{\mathcal{C}_n\}_{n\in\omega}$ is called *independent* if the sequence $\{\mathcal{C}_n\}_{n\in\omega}$ is independent in the usual sense of probability theory: $\lambda(\bigcap_{r\in F}\mathcal{C}_r) = \prod_{r\in F}\lambda(\mathcal{C}_r)$ for each finite set F.

As we will see in Section V, each non-Schnorr random is captured by an independent Schnorr-Solovay test of a very special kind.

Definition II.7. An *interval test* is given by a uniformly computable sequence of pairs $\{I_n, \mathcal{J}_n\}_{n\in\omega}$ where the I_n are pairwise disjoint increasing finite intervals (i.e., $\max(I_n) < \min(I_{n+1})$), each $\mathcal{J}_n \subseteq 2^{I_n}$ is a clopen set uniformly computable in n, and $\sum_n \lambda(\mathcal{J}_n)$ is finite and computable.

3) Higher randomness: Randomness has recently been studied from the viewpoint of higher computability. The reader may refer to [22] or [18] for background on higher computability, and to [19] or [18] for background on higher randomness. Here we summarise the main notions used in the paper.

A set of sequences or of integers is Π^1_1 if it can be defined by a second order formula of arithmetic with arbitrary quantifiers over the integers, but only universal quantifiers over infinite objects (sequences or functions). We also forbid negations in a Π^1_1 formula, to avoid having a 'hidden' existential quantifier over infinite objects. A set of sequences or of integers is Σ^1_1 if it can be defined by a formula of arithmetic with only existential quantifiers over infinite objects. Finally a set sequences or of integers is Δ^1_1 if it is both Π^1_1 and Σ^1_1.

Definition II.8. A sequence X is Δ^1_1*-random* if it belongs to no Δ^1_1 set $\mathcal{A} \subseteq 2^\omega$ with $\lambda(\mathcal{A}) = 0$. A sequence X is Π^1_1-*random* if it belongs to no Π^1_1 set $\mathcal{A} \subseteq 2^\omega$ with $\lambda(\mathcal{A}) = 0$.

The Gandy-Spector theorem gives a powerful analogy between the notions of computable/recursively enumerable sets of integers and Δ^1_1/Π^1_1 sets of integers. Informally, one can picture a Π^1_1 set of integers as being effectively enumerable along some "ordinal stages of computation" $\alpha < \omega_1^{ck}$, rather than just finite stages of computation, where ω_1^{ck} is the least non-computable ordinal, that is, the least ordinal α such that there is no c.e. relation $R \subseteq \omega\times\omega$ which is a well-order of order type α. Similarly, one can picture a Δ^1_1 set of integers as being effectively enumerable along some "ordinal stages of computation" which are bounded by some $\alpha < \omega_1^{ck}$.

Of particular interest in higher randomness is the set $\{X \in 2^\omega : \omega_1^X > \omega_1^{ck}\}$ of sequences X such that the least non-computable ordinal relatively to X, namely ω_1^X, is larger than ω_1^{ck}. We have the following theorem:

Theorem II.9 (Chong, Nies, Yu [4])**.** *A sequence X is Π^1_1-random iff it is Δ^1_1-random and $\omega_1^X = \omega_1^{ck}$.*

In higher computability there is a version of the halting problem, called Kleene's $\mathcal{O}$, which is defined as the set of all e such that the binary c.e. relation coded by e is a total well-order of the integers[1]. Every Π^1_1 set is many-one reducible to Kleene's $\mathcal{O}$.

Theorem II.10. *For a sequence X we have $\omega_1^X > \omega_1^{ck}$ iff Klenne's $\mathcal{O}$ is $\Delta^1_1(X)$.*

Finally we state here what is known as the van Lambalgen theorem for Π^1_1-randomness. The sequence $X \oplus Y$ is defined by $(X\oplus Y)(2n) = X(n)$ and $(X\oplus Y)(2n+1) = Y(n)$.

Theorem II.11. *For sequences X, Y we have that $X \oplus Y$ is Π^1_1-random iff X is Π^1_1-random and Y is $\Pi^1_1(X)$-random.*

In particular, if a Π^1_1-random sequence X is in a $\Pi^1_1(Y)$ nullset then Y is in a $\Pi^1_1(X)$ nullset.

C. Preliminaries on weakly Schnorr engulfing sequences, and traces

It is well known that unlike the case of Martin-Löf randomness, there exists no universal Schnorr test, that is, no Schnorr test covering all the others. One can prove this by showing that every Π^0_1 set of computable positive measure contains a computable sequence, so that no Schnorr test contains every computable sequence (see for instance Fact 3.5.9 in [19]). We are interested in oracles that strengthen the power of Schnorr tests, in that some Schnorr test relative to the oracle captures all the computable sequences.

Definition II.12. A sequence A is *weakly Schnorr engulfing* if there exists an A-Schnorr test containing all the computable sequences.

A weaker property of oracles has already been proved equivalent to a tracing property: some Schnorr test relative to the oracle is not covered by any plain Schnorr test. We first define the tracing property.

Definition II.13 (Terwijn and Zambella [24])**.** A *computable trace* is a computable sequence $\{T_n\}_{n\in\omega}$ of finite sets of integers given by strong indices. Formally $T_n = D_{p(n)}$ where $p : \omega \to \omega$ is a computable function and D_n is the set containing m iff there is a 1 at position m of the binary expansion of n. An oracle A is *computably traceable* if there

[1]The definition of Kleene is actually more complex in order to ease effective definitions by induction over elements of this set. For our purposes the present definition suffices.

is a computable bound $H : \omega \to \omega$ such that: for every function $f \leq_T A$, there is a computable trace $\{T_n\}_{n\in\omega}$ with $|T_n| < H(n)$ and $f(n) \in T_n$ for each n.

Intuitively, an oracle is computably traceable if every function f it computes is 'close to computable', in that one can compute a small set of values $f(n)$ belongs to. By a result of Terwijn and Zambella [24], together with Kjos-Hanssen, Nies and Stephan [13], we have:

Theorem II.14. *For an oracle A the following are equivalent:*

- *A is not computably traceable.*
- *There is an A-Schnorr test covered by no Schnorr test.*
- *There is an A-Schnorr test containing a Schnorr random.*

Oracles failing any of these properties are said to be *low for Schnorr randomness*. We will show the connection between being weakly Schnorr engulfing and the Gamma question. We shall also give in Section V a combinatorial characterization of being weakly Schnorr engulfing, by showing its equivalence with another tracing property.

D. Preliminaries on hyperimmune sequences

The following is a central notion of computability theory.

Definition II.15. An oracle X is *computably dominated* if for every function $f \leq_T X$, there exists a computable function g such that $f \leq g$, that is, $f(n) \leq g(n)$ for every n. An oracle X that is not computably dominated is also said to be of *hyperimmune degree*.

Kurtz proved in [16] that a sequence is hyperimmune iff it Turing computes a weakly-1-generic sequence, that is, a sequence which is in every dense Σ^0_1 subset of 2^ω. We state here a less famous, though interesting, third equivalent notion:

Definition II.16. A function $f : \omega \mapsto \omega$ is *infinitely often equal* (i.o.e. for short) if it coincides infinitely often with every computable function. An sequence A is of *i.o.e. degree* if it Turing computes an i.o.e. function.

For the sake of completeness we include a proof of a well-known equivalence.

Proposition II.17. *The following are equivalent for a sequence A.*

- *A is of hyperimmune degree.*
- *A is of i.o.e degree.*

Proof. If $f : \omega \mapsto \omega$ is an i.o.e. function, then $f + 1$ is clearly dominated by no computable function. Conversely suppose that a function $g : \omega \mapsto \omega$ is bounded by no computable function. Let $\{\Phi_e\}_{e\in\omega}$ be an effective list of the partial computable functions, and $\{R_e\}_{e\in\omega}$ be a list of Boolean values initialized to "false". At stage n, we define $f(n)$ to be $\Phi_k(n)[g(n)]$, where k is the least integer less than n such that $\Phi_k(n)[g(n)]$ halts and R_k is false. Then we set R_k to "true". If no such k exists we let $f(n) = 0$.

Note first that a function f is i.o.e. iff it is coincides once with every computable function (using that the class of computable functions is invariant under finite changes). If Φ_e is total, then the function $n \mapsto \min\{t : \Phi_e(n)[t] \text{ halts}\}$ is total and computable. Thus it is dominated by g infinitely often. Now it is easy to check that f coincides at least once with every computable function. □

In this paper, we will be interested in sequences of hyperimmune degree through their i.o.e. characterization. We will also be interested in computably dominated sequences that are random. It is well-known that, while the set of computably dominated sequences has measure 0, some are still Schnorr random [19]; further, each computably dominated Schnorr random is already Martin-Löf random. We will refer to these sequences simply as the computably dominated random sequences.

III. The Gamma values

A. Gamma value of 1/2 and being weakly Schnorr engulfing

The only two known examples of sequences with a Gamma value of $1/2$ have been the non-computable, computably traceable sequences and the computably dominated random sequences [1]. They are quite different from each other: while sequences of the first kind are close to computable, sequences of the second kind are far from computable (as they are random).

We identify a third property of an oracle implied by both properties above, which suffices to get a Gamma value of $1/2$: being not weakly Schnorr engulfing. As computably traceable sequences are low for Schnorr randomness, they are not weakly Schnorr engulfing (see Theorem II.14). Rupprecht [21] proved that computably dominated random sequences are not weakly Schnorr engulfing. It remains to prove that any non-weakly Schnorr engulfing sequences has a Gamma value of $1/2$ (or of 1 in case it is computable).

Theorem III.1. *Let A be not weakly Schnorr engulfing. Then $\Gamma(A) \geq 1/2$.*

Proof. Let $B \leq_{\mathrm{T}} A$. For each $d \in \mathbb{N}$ we will define a Schnorr-Solovay test $(G_k)_{k\in\mathbb{N}}$ relative to A such that $\underline{\rho}(B \leftrightarrow R) \geq 1/2 - 1/d$ for each sequence R not captured by this test. For each d some computable sequence R passes the test for d, so this will show that $\gamma(B) \geq 1/2$.

Let I_k be defined inductively as the set of k consecutive integers following I_{k-1}: $I_0 = \emptyset, I_1 = \{1\}, I_2 = \{2, 3\}, \dots$. Given k let

$$G_k = \{Z \colon Z(i) \neq B(i) \text{ for a ratio of bits in } I_k \text{ of at least } 1/2 + 1/d\}$$

which is a clopen set computed uniformly in k from A. By the usual Chernoff bounds we have $\lambda G_k \leq e^{-2k/d^2}$. Clearly $\int_r^\infty e^{-2x/d^2}dx \geq \sum_{k=r+1}^\infty e^{-2k/d^2}$. Since $\int_r^\infty e^{-2x/d^2}dx = d^2 e^{-2r/d^2}$ effectively converges to 0 as $r \to \infty$, the real $\sum_k \lambda G_k$ is computable in A. Thus $(G_k)_{k\in\mathbb{N}}$ is a Schnorr-Solovay test relative to A as required. □

In [1] it is proved that any sequence which is computably dominated and random with respect to a computable measure has a Gamma value of 1/2 (unless it is computable, which can happen if it is an atom of the measure). Rupprecht's proof in [21] can be modified to show that no sequence which is computably dominated and random for a computable measure is weakly Schnorr engulfing. It is natural to wonder whether besides these and the non-computable, computably traceable sequences, there are other sequences with a Gamma value of 1/2. Recently Kjos-Hanssen, Stephan, and Terwijn [14] constructed a sequence which is not weakly Schnorr engulfing (and hence computably dominated), not computably traceable, and not DNC (see e.g. [19, Ch. 4] for the definition of DNC, which abbreviates "diagonally non-computable"). A non-DNC sequence cannot be random for any atomless computable measure. (For instance, use Demuth's result [6] that any Martin-Löf random with respect to an atomless computable measure Turing computes a Martin-Löf random with respect to the uniform measure λ, together with the fact that every Martin-Löf random is DNC.) Thus, together with the result of Kjos-Hanssen et al., we have obtained new examples of sequences with a Gamma value of $1/2$.

B. Gamma value of 0 *and infinite equality*

The two known examples of sequences with a Gamma value of 0 are the sequences of hyperimmune degree, and the sequences of PA degree. The latter are the sequences which Turing compute a complete extension of Peano arithmetic. It is well-known that these conicide with the sequences that Turing compute a member of any non-empty Π^0_1 set, uniformly in a code for this set. It is also well known that some of them are computably dominated. These two types of sequences seem to have a Gamma value of 0 for quite different reasons, when one looks at the respective proofs in [1]. We identify here a third notion implied by both, which already suffices to get a Gamma value of 0. This notion is a weakening of being i.o.e., where one introduces a bound on the functions.

Definition III.2. Given a computable bound $H : \omega \mapsto \omega$, we say that $f : \omega \mapsto \omega$ is H-*infinitely often equal* (or H-i.o.e.) if f coincides infinitely often with every computable function strictly bounded by H. A sequence A is of H-i.o.e. degree if A Turing computes an H-i.o.e. function.

Recall that by Proposition II.17, sequences of hyperimmune degree are of i.o.e. degree (with no bound).

Theorem III.3. *Any PA degree is* H*-i.o.e. for any computable bound* H *of the form* $2^{\widetilde{H}}$.

Proof. Let A be of PA degree. There exists an A-computable list $\{X_e\}_{e\in\omega}$ of sequences that contains all the computable ones. To see this, note that uniformly in a partial computable function Φ_e, A computes a member of the Π^0_1 set

$$\{X \ : \ \forall n \ \forall t \ [\Phi_e(n)[t]\uparrow \ \text{or} \ \Phi_e(n)[t] = X(n)]\}.$$

Given any total computable function $\widetilde{H}$, define $H'(n) = \sum_{m<n} \widetilde{H}(m)$ (with $H'(0) = 0$). We let the A-computable function f map n to the natual number corresponding to the string $X_n\restriction_{[H'(n),H'(n+1))}$. If is clear that f is H-i.o.e. for $H(n) = 2^{\widetilde{H}(n)}$. □

We now prove that sequences of H-i.o.e. degree for a sufficiently fast growing function H have a Gamma value of 0.

Theorem III.4. *Let* $d > 1$ *be a real. If* A *is of* $2^{(d^n)}$*-i.o.e. degree, then* $\Gamma(A) = 0$.

Proof. We first prove that if A is of $2^{(a^n)}$-i.o.e. degree for a natural number $a > 1$, then $\Gamma(A) \le 1/a$. Consider a $2^{(a^n)}$-i.o.e. function f that we can bound without loss of generality by $2^{(a^n)}$. Let $H(n) = \sum_{m<n} a^m$ (with $H(0) = 0$). We define the f-computable set B_f such that $B_f\restriction_{[H(n),H(n+1))}$ equals the string corresponding to the n-th value of f. Consider now any computable sequence X and its bitwise complement $\overline{X}$, together with the function $f_{\overline{X}}$ which to n associates the integer corresponding to the string $\overline{X}\restriction_{[H(n),H(n+1))}$. In particular as f infinitely often coincides with $f_{\overline{X}}$, there are infinitely many n such that from position $H(n)$ to $H(n+1)-1$, sequences X and B disagree all the time, which implies

$$\frac{|(B \leftrightarrow X) \cap [0, H(n+1))|}{H(n+1)} \le \frac{H(n)}{H(n+1)}$$

As $a^n = (a-1)H(n) - 1$, we have $H(n+1) = H(n) + (a-1)H(n) - 1$. Hence a ratio of at most $H(n)/(aH(n) - 1)$ bits is guessed correctly by X on the initial segment of length $H(n+1)$. As this happens infinitely often and for every computable sequence X, we conclude that $\gamma(B) \le 1/a$ and hence $\Gamma(A) \le 1/a$.

We shall now prove that if A is of $2^{(d^n)}$-i.o.e. degree for any real $d > 1$, then A is of $2^{(a^n)}$-degree for any natural number $a > 1$. To do so we first argue that if f is H-i.o.e. for some function H, then the function $n \mapsto f(2n)$ is $H(2n)$-i.o.e. or the function $n \mapsto f(2n+1)$ is $H(2n+1)$-i.o.e. Indeed suppose that some computable function g_1 bounded by $H(2n)$ is such that $g_1(n) \neq f(2n)$ for every n and that some computable function g_2 bounded by $H(2n+1)$ is such that $g_2(n) \neq f(2n+1)$ for every n. Then the computable function g such that $g(2n) = g_1(n)$ and $g(2n+1) = g_2(n)$ is never equal to f, which is a contradiction.

Given a $2^{(d^n)}$-i.o.e. function f and any integer $a > 1$, let k be such that $2^{(d^{k\times n})} \ge 2^{(a^n)}$ (a value k bigger than $\log_d(a)$ suffices). By repeating the operation described above sufficiently often, we easily see how to compute from f a function f' which is $2^{(d^{k\times n})}$-i.o.e. and hence $2^{(a^n)}$-i.o.e. It follows that $\Gamma(A) = 0$. □

Rupprecht [21], Thm. 19, constructed a sequence which is weakly Schnorr engulfing, computably dominated and not DNC. His proof can be slightly modified to construct a sequence which is for any given computable function $\widetilde{H}$, of $2^{\widetilde{H}(n)}$-i.o.e. degree, and both computably dominated and not DNC. As every PA degree is DNC, this provides new examples of sequences with a Gamma value of 0.

C. Gamma values with respect to bases other than 2

We consider here the Gamma value for real numbers expressed in different bases. For an integer $b \geq 2$ we denote the space of infinite sequences of elements in $\{0, \ldots, b-1\}$ by b^ω. For $A \in b^\omega$ we define the value $\gamma_b(A)$ as before, except we now consider a supremum over computable elements of b^ω. The definition of $\Gamma_b(A)$ is also as before, except we now consider an infimum over elements of b^ω which are Turing equivalent to A. Finally for a (non rational) real $r \in \mathbb{R}$ we define $\Gamma_b(r)$ to be $\Gamma_b(A)$ for $A \in b^\omega$ the canonical representation of r in base b.

Let us argue that for any base $b \geq 2$ and any real r we have $\Gamma_{b+1}(r) \leq \Gamma_b(r)$. Indeed for every sequence $A \in b^\omega$ we obviously have $\gamma_b(A) = \gamma_{b+1}(A)$. As every elements of b^ω is also an element of $(b+1)^\omega$, the infimum in the definition of $\Gamma_{b+1}(r)$ is done over more elements than in the definition of $\Gamma_b(r)$. In particular if $\gamma_2(r) = 0$ then $\gamma_b(r) = 0$ for any $b \geq 2$.

By straightforward modifications of the proof of Theorem III.1, for any sequence $A \in b^\omega$ which is not weakly Schnorr engulfing we have $\Gamma_b(A) = 1/b$. What is really of interest here is the proof that any sequence $A \in b^\omega$ such that $\Gamma_b(A) > 1/b$ is computable: The proof of this in [1] for the case $b = 2$ uses a "majority vote" technique, that cannot be used directly for larger bases. This will be made clear in what follows.

Definition III.5. For any sequence $A \in 2^\omega$, seen as a subset of ω, we denote by $\#_c^A : \omega^c \to \omega$ the function which on $x_1, \ldots, x_c$ returns $|A \cap \{x_1, \ldots, x_c\}|$.

Note that $\#_c^A$ can take at most $c+1$ distinct values. Kummer [15] proved that if A is not computable, one cannot trace $\#_c^A$ by a c.e. trace containing strictly less than $c+1$ values. The proof was later simplified by Owings [20]:

Theorem III.6. *[Kummer] Let $c \geq 2$. Suppose A is an oracle such that $\#_c^A$ is traceable via some trace $\{T_n\}_{n\in\omega}$, where each T_n is c.e. uniformly in n and $|T_n| \leq c$. Then A is computable.*

We will use this in the proof of the following theorem:

Theorem III.7. *Let $A \in b^\omega$. If $\Gamma_b(A) > 1/b$ then A is computable.*

Proof. Let $\widetilde{A} \in 2^\omega$ be some effective binary encoding of A. The sequence $\#_{b-1}^{\widetilde{A}}$ can be seen as an element of to b^ω. We perform a majority vote argument as in [1], except that we now do not need an absolute majority to win. We define a sequence $B \in b^\omega$ that encodes every bit of $\#_{b-1}^{\widetilde{A}}$ with many repetitions. Inductively we define intervals I_n by $I_0 = \{0\}$ and $I_{n+1} = \{k \ : \ a_n < k \leq (n+1) \times s_n\}$, where a_n is the last position in the interval I_n and s_n the sum of the length of the intervals I_0 to I_n. For any $k \in I_n$ we define $B(k) = \#_{b-1}^{\widetilde{A}}(n)$. As B is Turing computable from A, we must have $\gamma_b(B) > 1/b$. Hence there is a computable sequence C such that $\liminf_n |B \leftrightarrow C \cap [0,n]|/n > 1/b$.

We claim that for sufficiently large n, the ratio of positions k in I_n such that $C(k) = \#_{b-1}^{\widetilde{A}}(n)$ is strictly greater than $1/b$. Assume otherwise. Then there are arbitrarily large n such that at most $1/b$ positions in I_n are guessed correctly by C. Recall s_{n-1} is the number of positions before I_n. As there are n more positions in I_n than in all the previous intervals together, we have a ratio of at most

$$\frac{s_{n-1} + (ns_{n-1})/b}{s_{n-1} + ns_{n-1}} = \frac{1 + n/b}{1+n}$$

positions up to the maximum of I_n that are guessed correctly. This expression has limit $1/b$ as n goes to infinity. It then follows that

$$\liminf_k |B \leftrightarrow C \cap [0,k]|/k \leq 1/b,$$

which is a contradiction.

Therefore for n large enough, the sequence C must guess in the interval I_n strictly more than $1/b$ of the bits correctly. Also they can be at most $b-1$ values which are given by C with a ratio strictly bigger than $1/b$. By building the computable trace with all these values, we have a trace for $\#_{b-1}^{\widetilde{A}}$, which implies by Theorem III.6 that $\widetilde{A}$, and then A, is computable. ☐

Question III.8. *Let $r \in [0,1]$ be non-computable. Do we have for all integers $b, c \geq 2$ that $\Gamma_b(r) = 1/b$ iff $\Gamma_c(r) = 1/c$?*

D. The Gamma value in the higher setting

In this section we study a notion analogous to being weakly Schnorr engulfing in the setting of higher computability. Thereafter we discuss a higher version of the Gamma question.

Definition III.9. A sequence A is *weakly Δ_1^1 engulfing* if there is a $\Delta_1^1(A)$ nullset containing every Δ_1^1 sequence.

For an A-computable ordinal α, we write $A^{(\alpha)}$ to denote the α-th jump of A. There are several equivalent ways to concretely define this set, for example via H-sets, as initially done by Kleene and Spector (see [22]), or as the set of codes for A-c.e. binary relations coding total orders of order-type strictly smaller than some ordinal, as in [18]. The important point is that $A^{(\alpha)}$ should be a $\Sigma_\alpha^0(A)$-complete set.

It is well known (see for example [5] or Proposition 4.2.5 in [18]) that if A is Δ_1^1-random, then it is GL_α for every computable ordinal α, that is, $A^{(\alpha)} \equiv_T A \oplus \emptyset^{(\alpha)}$. Using this, we prove the following theorem about Π_1^1-random sequences:

Theorem III.10. *Suppose A is Π_1^1 random. Then A is not weakly Δ_1^1 engulfing.*

Proof. Consider a $\Delta_1^1(A)$ nullset $\mathcal{S}$. As $\omega_1^A = \omega_1^{ck}$ we have some computable ordinal α such that $\mathcal{S}$ is a $\Sigma_\alpha^0(A)$ set. By the effective regularity of Lebesgue measure relativized to A, we can approximate $\mathcal{S}$ from above by a uniform intersection of $\Delta_1^1(A)$ open sets $\bigcap_n \mathcal{U}_n$ with $\lambda(\mathcal{U}_n) \leq 2^{-n}$. Using again that $\omega_1^A = \omega_1^{ck}$, we have a computable ordinal α such that each $\mathcal{U}_n$ is $\Sigma_1^0(A^{(\alpha)})$ uniformly in n. Now because A is Δ_1^1 random it is GL_α and then each $\mathcal{U}_n$ is also $\Sigma_1^0(A \oplus \emptyset^{(\alpha)})$ uniformly in n. Consider an index for the $\Pi_2^0(A \oplus \emptyset^{(\alpha)})$ set $\bigcap_n \mathcal{U}_n$, namely an effective list of Σ_1^0 sets $\mathcal{U}_n$ relative to the oracle. We can now modify this index such that for any oracle X we have $\lambda(\mathcal{U}_n^X) \leq 2^{-n}$, without changing $\mathcal{U}_n$ for oracles for which we already have $\lambda(\mathcal{U}_n^X) \leq 2^{-n}$ (in particular the oracle $A \oplus \emptyset^{(\alpha)}$). For any X consider now the $\Delta_1^1(X)$ set:

$$\mathcal{S}_X = \{Z \ : \ X \in \bigcap_n \mathcal{U}_n^{Z\oplus\emptyset^{(\alpha)}}\}$$

and the Δ_1^1 set:

$$\mathcal{N} = \{X \ : \ \lambda(\mathcal{S}_X) > 0\}$$

We claim that $\mathcal{N}$ contains every Δ_1^1 sequence belonging to $\bigcap_n \mathcal{U}_n^{A\oplus\emptyset^{(\alpha)}}$ and that $\lambda(\mathcal{N}) = 0$. Since some Δ_1^1 sequence is not contained in $\mathcal{N}$, this will show that A is not weakly Δ_1^1 engulfing.

If X is Δ_1^1 and belongs to $\bigcap_n \mathcal{U}_n^{A\oplus\emptyset^{(\alpha)}}$ we clearly have $A \in \mathcal{S}_X$. Since $\mathcal{S}_X$ is Δ_1^1 and A is Δ_1^1 random, this implies $\lambda(\mathcal{S}_X) > 0$. This shows the claim.

To verify that $\lambda(\mathcal{N}) = 0$ it suffices to show that $\mathcal{N}$ contains no Π_1^1 random sequence. We apply the van Lambalgen theorem for Π_1^1-randomness Theorem II.11. Suppose that Y is Π_1^1-random in order to prove $Y \notin \mathcal{N}$. Suppose for some Z we have $Y \in \bigcap_n \mathcal{U}_n^{Z\oplus\emptyset^{(\alpha)}}$. In particular as Y is not $\Pi_1^1(Z)$-random Z is not $\Pi_1^1(Y)$-random. Then $\mathcal{N}_Y$ is included in the set of sequences which are not $\Pi_1^1(Y)$-random. As this is a set of measure 0 we have $\lambda(\mathcal{N}_Y) = 0$. Hence Y is not in $\mathcal{N}$. ☐

We now define counterparts of γ and Γ in the higher setting:

$$\gamma_h(A) = \sup_{X \in \Delta_1^1} \underline{\rho}(A \leftrightarrow X)$$

$$\Gamma_h(A) = \inf\{\gamma_h(B) \ : \ B \text{ is } \Delta_1^1(A)\}$$

As in the setting of computability, $\Gamma_h(A) > 1/2$ iff A is Δ_1^1. To show this, one follows the same proof as the one of Proposition 1.4. in [1], using a "majority vote" technique.

Still as in the setting of computability, and following the same proof as the one of Theorem III.1, if A is not weakly Δ_1^1 engulfing then $\Gamma_h(A) = 1/2$. In particular:

Theorem III.11. *Every Π_1^1-random sequence Z satisfies $\Gamma_h(Z) = 1/2$.*

Consequently, for Γ, the situation is quite different in the higher setting: The set of sequences with a Gamma value of $1/2$ has measure 1, whereas it has measure 0 in the lower setting.

A notion of being infinitely often equal also makes sense in the higher setting.

Definition III.12. Let $H : \omega \mapsto \omega$ be a Δ_1^1 bound. A function $f : \omega \mapsto \omega$ is *higher H-i.o.e.* if it is infinitely often equal to every Δ_1^1 function bounded by H. A sequence A is of *H-i.o.e. hyperdegree* if there is some higher H-i.o.e. function which is $\Delta_1^1(A)$.

It is clear that the proof of Theorem III.4 can be adapted to the higher setting to obtain for any real $d > 1$ that any A of 2^{d^n}-i.o.e. hyperdegree has a higher Gamma value of 0. In particular, if $\omega_1^X > \omega_1^{ck}$ then $\Gamma_h(X) = 0$.

The fact that every hyperimmune function computes an infinitely often equal function (see II.17) has no analogue in the higher setting. Kihara [12] has defined a sequence X such that some function f bounded by no Δ_1^1 function is $\Delta_1^1(X)$, but no infinitely often equal function is $\Delta_1^1(X)$.

IV. Infinitely often equality

We have seen that both being weakly Schnorr engulfing and the Gamma question are connected with the notion of infinite equality. We study this notion on its own right in some more detail.

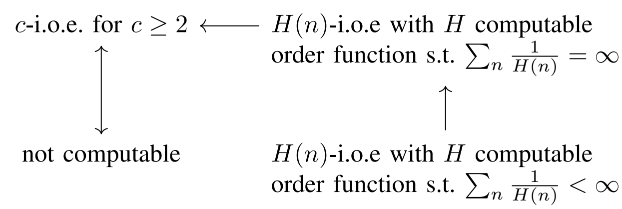


Fig. 1. Known implications for i.o.e. degrees

It is clear that the two implications of Fig. 1 hold. We also need to argue that they are strict. First let us prove the left part of Fig. 1. It is obvious that being of 2-i.o.e. degree is equivalent to being not computable, but for $c > 2$ some more complicated argument is required. Here again, we can use the function $\#_c^A$ of Definition III.5 and Theorem III.6 to deduce the following:

Theorem IV.1. *Suppose A is not c-i.o.e. for some integer $c \geq 2$. Then A is computable.*

Proof. The function $g(\langle x_1, \ldots, x_{c-1}\rangle) = \#_c^A(x_1, \ldots, x_{c-1})$ is clearly A-computable and strictly bounded by c. Suppose it never coincides with a computable function f that is strictly bounded by c. Then we can capture g (and hence $\#_c^A$) with the computable trace $\{T_n\}_{n\in\omega}$ such that T_n contains every number less than c except $f(n)$. It follows from Theorem III.6 that A is computable. ☐

By the following proposition, the first implication is strict: there exists a non-computable sequence which is not of H-i.o.e. degree for any order function H.

Proposition IV.2. *Suppose A is computably traceable. For any order function H and any function $f \leq_T A$, there exists a computable function $g < H$ such that f is always different from g.*

Proof. Suppose $f < H$ is Turing below some computably traceable set A. As proved in [24], for any computable order function H', as slowly growing as we want, the function f can be traced via a computable trace $\{T_n\}_{n\in\omega}$ bounded by H'. We simply take $H' < H$. To compute $g(n)$ we can then take any value smaller than $H(n)$ and not in T_n. ☐

Let us now prove that the second implication is strict: for any order function H there exists a sequence of $H(n)$-i.o.e. degree such that $\sum_n 1/H(n) = \infty$, but not of $H(n)$-i.o.e. degree for any order function H such that $\sum_n 1/H(n) < \infty$. This is done via a computably dominated random. For the next proposition, we use the weak notion of Kurtz randomness, which is defined by not being in any Π^0_1 null set.

Proposition IV.3. *Suppose A is Kurtz random. Then A is of $H(n)$-i.o.e. degree for any order function $H = 2^{\widetilde{H}}$ such that $\sum_n 1/H(n) = \infty$.*

Proof. Consider a computable function $f < H$. Let $H'(n) = \sum_{m<n} \widetilde{H}(m)$ (with $H'(0) = 0$). Let $I_n = [H'(n), H'(n+1))$, then we have by hypothesis that $\sum_n \lambda([f(n)]_{I_n}) = \infty$ (where $f(n)$ is seen as a string of length I_n). Also as the I_n are pariwise disjoint, by Borel-Cantelli we have

$$\lambda\left(\bigcap_n \bigcup_{m\geq n} [f(m)]_{I_m}\right) = 1$$

and then for each n we have $\lambda(\bigcup_{m\geq n}[f(m)]_{I_m}) = 1$. It follows that if X is Kurtz random there are infinitely many n such that $X\upharpoonright_{I_n} = f(n)$, where $X\upharpoonright_{I_n}$ is seen as an integer less than $2^{|I_n|}$. Also this is true for any computable function f, and it then follows that the X-computable function $g(n) = X\upharpoonright_{I_n}$ is $H(n)$-i.o.e. □

It is easy to see that if a sequence is of $H(n)$-i.o.e. degree for an order function H such that $\sum_n 1/H(n) < \infty$, then it is weakly Schnorr engulfing. A more general statement will actually be proven in Proposition V.3. It follows that any computably dominated random is of $H(n)$-i.o.e. degree for any order function H such that $\sum_n 1/H(n) = \infty$, but not of $H(n)$-i.o.e. degree for any computable order function H such that $\sum_n 1/H(n) < \infty$. For instance, such a sequence is of $2^{\log(n)}$-i.o.e. degree and not of $2^{2\log(n)}$-i.o.e. degree.

Question IV.4. *Is there some X of 2^n-i.o.e degree that is not of $f(n)$-i.o.e degree for some computable $f(n) >> 2^n$? If yes, can $f(n)$ be taken to be 2^{2^n}?*

V. Weakly Schnorr engulfing and tracing

The notions of being weakly Schnorr engulfing and being H-infinitely often equal for H sufficiently fast growing are both connected to the Gamma question stated in the abstract, via Theorem III.1 and Theorem III.4, respectively. At first sight, these notions appear to be unrelated. We give a combinatorial characterization of being weakly Schnorr engulfing, using traces whose members are strictly bounded by some computable function. This will bring the notion of being weakly Schnorr engulfing closer to the notion of being of H-i.o.e. degree for some H. Both the tracing property and being infinitely often equal were introduced in search of computability theoretic analogues of combinatorial notions from set theory used to analyse cardinal characteristics (see [3]). We will also apply analogues of the methods employed in that area.

Definition V.1. For a computable function $H : \omega \mapsto \omega$ of the form $2^{\widetilde{H}(n)}$, we say that a computable trace $\{T_n\}_{n\in\omega}$ is a *computable H-trace* if for each n, elements of T_n are strictly bounded by $H(n)$. If furthermore $\sum_n |T_n|/H(n)$ is finite and computable, the trace $\{T_n\}_{n\in\omega}$ is called a *small computable H-trace*. We say that a function $f < H$ is *captured* by a computable H-trace $\{T_n\}_{n\in\omega}$ if for infinitely many n we have $f(n) \in T_n$.

Note that a computable H-trace can be small only for functions H such that $\sum_n 1/H(n) < \infty$. The idea underlying small traces is to have $\sum_n \lambda([T_n]_{H(n)})$ finite and computable, where $[T_n]_{H(n)}$ is the set of strings of length $\widetilde{H}(n)$ corresponding to elements of T_n.

We shall see that a sequence is weakly Schnorr engulfing iff it computes some small H-trace capturing every computable function bounded by H. In order to conduct the proof, the notion of an interval test defined in II.7 plays a key role: still via some coding between strings of length $\widetilde{H}(n)$ and integers smaller than $H(n)$, we can view any small computable H-trace as an interval test, and vice-versa. The goal of this section if to prove the following theorem:

Theorem V.2. *A sequence A is weakly Schnorr engulfing $\Leftrightarrow$ for some computable function H, there is an A-computable small H-trace capturing every computable function bounded by H.*

We begin with the easier implication "$\Leftarrow$" of Theorem V.2.

Proposition V.3. *Suppose $\{T_n\}_{n\in\omega}$ is a small H-trace relative to A that captures every computable function bounded by $H = 2^{\widetilde{H}}$. Then A is weakly Schnorr engulfing.*

Proof. We define an interval test $\{I_n, \mathcal{J}_n\}_{n\in\omega}$ by viewing each member of T_n as a string σ of length $\widetilde{H}(n)$. Let $H'(n) = \sum_{m<n} \widetilde{H}(m)$ (with $H'(0) = 0$). For each n, let $I_n = [H'(n), H'(n+1))$, and let $\mathcal{J}_n$ be the union of all the sets $[\sigma]_{[H'(n),H'(n+1))}$ for every member of T_n encoding a string σ of length $\widetilde{H}(n)$.

It is clear that $\{I_n, \mathcal{J}_n\}_{n\in\omega}$ is an interval (and hence Schnorr) test relative to A that captures every computable sequence. □

We now prove the implication "$\Rightarrow$" of Theorem V.2. The idea is to try to cover any A-computable Schnorr test by an A-computable interval test. In order to do so we consider algorithmic versions of some results from set theory. An algorithmic version of a theorem of Bartoszyński [2] (see also Theorem 2.5.11 in [3]) implies that it is not possible in general to cover a Schnorr test by a single interval test [2]. We can, however, always cover a Schnorr test by two interval tests. The proof we give here is similar to the proof of Theorem 2.5.7 in [3].

Lemma V.4. *Given a Schnorr test and a computable sequence of positive rationals $\{\varepsilon_n\}_{n\in\omega}$, there exists a computable*

[2]We don't know if every Schnorr test can be covered by a single independent Schnorr-Solovay test

sequence of integers $n_0 < m_0 < n_1 < m_1 < \dots$ and computable sequences of clopen sets $\{\mathcal{J}_{1,n}\}_{n\in\omega}$ and $\{\mathcal{J}_{2,n}\}_{n\in\omega}$ such that:

- $\mathcal{J}_{1,k} \subseteq 2^{[n_k,n_{k+1})}$ *and* $\mathcal{J}_{2,k} \subseteq 2^{[m_k,m_{k+1})}$
- $\lambda(\mathcal{J}_{1,k}) \leq \varepsilon_k$ *and* $\lambda(\mathcal{J}_{2,k}) \leq \varepsilon_k$
- *Any sequence captured by the Schnorr test is in* $\bigcap_n \bigcup_{k\geq n} \mathcal{J}_{1,k}$ *or in* $\bigcap_n \bigcup_{k\geq n} \mathcal{J}_{2,k}$.

The proposition can be relativized in the usual way to any oracle A.

Proof. By the discussion after Definition II.4 there is a Schnorr-Solovay test covering the given Schnorr test. We can suppose without loss of generality that each clopen set of the Schnorr-Solovay test is of the form $[\sigma]$ for a string σ. Thus, the test is given by a computable sequence of strings $\{\sigma_n\}_{n\in\omega}$. We define an auxiliary computable sequence of integers $p_0 < q_0 < p_1 < q_1 < \dots$ such that $\mathcal{J}_{1,k}$ depends on the strings σ_i for $q_k \leq i < p_{k+1}$. and $\mathcal{J}_{2,k}$ depends on the strings σ_i for $p_{k+1} \leq i < q_{k+1}$. The idea is the following: once we have put the first p strings σ_i into the first component of our first interval test, we remember the maximal length n of those strings. We then put the next q strings into the first component of the second interval test, for sufficiently large q so that the sum of the measure of each remaining string is smaller than $\varepsilon \times 2^{-n}$. In particular we then know that the sum of the measures of each remaining string of which we remove the n first bits, is still small enough. We repeat the operation, now making sure that the measure of what remains to be put in the second trace is small enough. We proceed in this fashion, alternating between the two traces. Fig. 2 illustrates the choice of $p_k < q_k < p_{k+1} < q_{k+1} < \dots$. Fig. 3 illustrates the choice of $n_k < m_k < n_{k+1} < m_{k+1} < \dots$.

We now give the formal construction. Let $n_0 = 0$ and $p_0 = 0$. Let $q_0 > 0$ be the least integer such that $\sum_{n\geq q_0} 2^{-|\sigma_n|} \leq \varepsilon_0$. Let m_0 be the maximal value between 1 and the length of the longest string σ_i for $i < q_0$. Suppose p_k, n_k and q_k, m_k have been defined. Let us define p_{k+1} and n_{k+1}. Let $p_{k+1} > q_k$ be the least integer such that

$$\sum_{n\geq p_{k+1}} 2^{-|\sigma_n|} \leq 2^{-m_k}\varepsilon_k$$

and n_{k+1} the maximal value between $m_k + 1$ and the length of the longest string σ_i for $q_k \leq i < p_{k+1}$. Finally let $\mathcal{J}_{1,k}$ be the clopen set equal to the union of $[\sigma_i \restriction_{[n_k,n_{k+1})}]_{[n_k,n_{k+1})}$ for any $q_k \leq i < p_{k+1}$.

Suppose q_k, m_k and p_{k+1}, n_{k+1} have been defined. Let us define q_{k+1} and m_{k+1}. Let $q_{k+1} > p_{k+1}$ be the least integer such that

$$\sum_{n\geq q_{k+1}} 2^{-|\sigma_n|} \leq 2^{-n_{k+1}}\varepsilon_{k+1}$$

and let m_{k+1} be the maximal value between $n_{k+1} + 1$ and the length of the longest string σ_i for $p_{k+1} \leq i < q_{k+1}$. Finally let $\mathcal{J}_{2,k}$ be the clopen set equal to the union of $[\sigma_i \restriction_{[m_k,m_{k+1})}]_{[m_k,m_{k+1})}$ (i.e., the sequences agreeing with σ_i on $[m_k, m_{k+1})$) for any $p_{k+1} \leq i < q_{k+1}$.

We verify that the constructed objects satisfy the required conditions. Firstly, since $n_0 < m_0 < n_1 < m_1 < \dots$ is a computable sequence of integers, $\mathcal{J}_{1,k} \subseteq 2^{[n_k,n_{k+1})}$ and $\mathcal{J}_{2,k} \subseteq 2^{[m_k,m_{k+1})}$ are effective sequences of clopen sets.

Secondly, for any k, by the choice of q_k,

$$\sum_{i\geq q_k} 2^{-|\sigma_i|} < 2^{-n_k}\varepsilon_k.$$

Note that by the definition of $\mathcal{J}_{1,k}$ we have

$$\lambda(\mathcal{J}_{1,k}) \leq 2^{n_k} \sum_{i\geq q_k} 2^{-|\sigma_i|}.$$

It follows that $\lambda(\mathcal{J}_{1,k}) \leq \varepsilon_k$. The argument for $\lambda(\mathcal{J}_{2,k}) \leq \varepsilon_k$ is similar.

Finally, if $\{\sigma_n\}_{n\in\omega}$ contains infinitely many prefixes of X, then X is in $\bigcap_n \bigcup_{m\geq n} \mathcal{J}_{1,m}$ or in $\bigcap_n \bigcup_{m\geq n} \mathcal{J}_{2,m}$. ☐

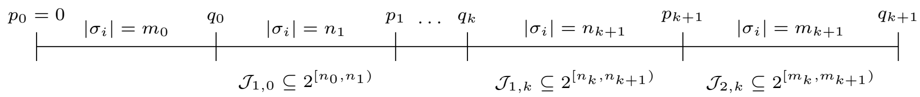
In order to obtain Theorem V.2 we would need to merge the two interval tests covering a Schnorr test into a single interval test. As we mentioned already, this is not possible by the algorithmic result of Bartoszyński's result. It would be enough, given an A-Schnorr test $\bigcap_n \mathcal{U}_n$ covering every computable sequence, to mix the two interval A-Schnorr-Solovay tests obtained via Lemma V.4 into a single interval A-Schnorr-Solovay test that also captures every computable sequence, without necessarily covering $\bigcap_n \mathcal{U}_n$. This is what we achieve now, adapting the proof of Theorem 2.5.12 of [3]. To do so, we need to slightly modify Lemma V.4, so the sequence $n_0 < m_0 < n_1 < m_1 < \dots$ is not just A-computable (once we relativize the lemma to A) but computable. Also in order to achieve this, we need to restrict ourselves to computably dominated oracles A. Fortunately, this is not an obstacle for the harder implication $\Rightarrow$ of Theorem V.2. We argued already in Proposition II.17 that if A is hyperimmune, A is of i.o.e. degree, and therefore certainly computes a small H-trace capturing every computable function.

Lemma V.5. *Suppose A is computably dominated. Given an A-Schnorr test and a computable sequence of positive rationals $\{\varepsilon_n\}_{n\in\omega}$, there exists a computable sequence of integers $n_0 < m_0 < n_1 < m_1 < \dots$ and A-computable sequences of clopen sets $\{\mathcal{J}_{1,n}\}_{n\in\omega}$ and $\{\mathcal{J}_{2,n}\}_{n\in\omega}$ such that:*

- $\mathcal{J}_{1,k} \subseteq 2^{[n_k,n_{k+1})}$ *and* $\mathcal{J}_{2,k} \subseteq 2^{[m_k,m_{k+1})}$,
- $\lambda(\mathcal{J}_{1,k}) \leq \varepsilon_k$ *and* $\lambda(\mathcal{J}_{2,k}) \leq \varepsilon_k$
- *Any sequence captured by the A-Schnorr test is in* $\bigcap_n \bigcup_{k\geq n} \mathcal{J}_{1,k}$ *or in* $\bigcap_n \bigcup_{k\geq n} \mathcal{J}_{2,k}$.

Proof. The proof is similar to the one of Lemma V.4. Let $\{\sigma_n\}_{n\in\omega}$ be an A-Schnorr-Solovay test. Similarly we define the sequence of integers $p_0 < q_0 < p_1 < q_1 < \dots$ such that $\mathcal{J}_{1,k}$ will depend on the strings σ_i for $q_k \leq i < p_{k+1}$, whereas $\mathcal{J}_{2,k}$ will depend on the strings σ_i for $p_{k+1} \leq i < q_{k+1}$. The only change from the proof of Lemma V.4 is that the sequence $p_0 < q_0 < p_1 < q_1 < \dots$ now is computable, not merely A-computable.

To make these sequences computable, we define $F_A : \omega \times \mathbb{Q} \to \omega$ to be the A-computable function which to (p, ε)



The strings σ_i for $q_k \leq i < p_{k+1}$ have length at most n_{k+1}

The strings σ_i for $p_{k+1} \leq i < q_{k+1}$ have length at most m_{k+1}

Fig. 2. The choice of $p_k < q_k < p_{k+1} < q_{k+1} < \ldots$ in the proof of Lemma V.4

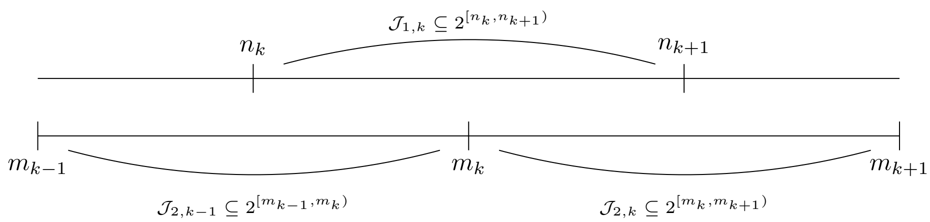


Fig. 3. The choice of $n_k < m_k < n_{k+1} < m_{k+1} < \ldots$ in the proof of Lemma V.4

associates the least integer $q > p$ such that $\sum_{n \geq q} 2^{-|\sigma_n|} \leq \varepsilon$. Note that F_A is A-computable because $\sum_n 2^{-|\sigma_n|}$ is A-computable. We also define $G_A : \omega \times \omega \times \omega \to \omega$ be the A-computable function which to (p, q, n) associates the maximal value between $n + 1$ and the length of the longest string σ_i for $p \leq i < q$. As A is computably dominated, both F_A and G_A are bounded by some computable functions F and G.

Let $n_0 = 0$ and $p_0 = 0$. Let q_0 be the result of computing $F(0, \varepsilon_0)$. Let m_0 be the result of computing $G(0, q_0, 0)$. Suppose p_k, n_k and q_k, m_k have been defined. Let us define p_{k+1} and n_{k+1}. Let p_{k+1} be the result of computing $F(q_k, 2^{-m_k}\varepsilon_k)$ and n_{k+1} the result of computing $G(q_k, p_{k+1}, m_k)$. Finally let $\mathcal{J}_{1,k}$ be the clopen set equal to the union of $[\sigma_i \upharpoonright_{[n_k,n_{k+1})}]_{[n_k,n_{k+1})}$ for any $q_k \leq i < p_{k+1}$.

Suppose q_k, m_k and p_{k+1}, n_{k+1} have been defined. Let us define q_{k+1} and m_{k+1}. Let q_{k+1} be the result of computing $F(p_{k+1}, 2^{-n_{k+1}}\varepsilon_{k+1})$ and let m_{k+1} be the result of computing $G(p_{k+1}, q_{k+1}, n_{k+1})$. Finally let $\mathcal{J}_{2,k}$ be the clopen set equal to the union of $[\sigma_i \upharpoonright_{[m_k,m_{k+1})}]_{[m_k,m_{k+1})}$ for any $p_{k+1} \leq i < q_{k+1}$.

It is clear that $n_0 < m_0 < n_1 < m_1 < \ldots$ is a computable sequence of integers. It follows that $\mathcal{J}_{1,k} \subseteq 2^{[n_k,n_{k+1})}$ and $\mathcal{J}_{2,k} \subseteq 2^{[m_k,m_{k+1})}$ is an A-computable sequence of clopen sets. The rest of the verification is as in Lemma V.4. ☐

We are now ready to mix the two interval A-Schnorr-Solovay tests into one interval A-Schnorr-Solovay test capturing every computable sequence.

Theorem V.6. *Let A be weakly Schnorr engulfing. There is an interval test $\{I_n, \mathcal{J}_n\}_{n\in\omega}$ relative to A that captures every computable sequence. Moreover, the sequence $\{I_n\}_{n\in\omega}$ is computable.*

Proof. We already discussed the case that A is hyperimmune. So let us suppose that A is computably dominated. Fix a decreasing computable sequence of positive rationals $\{\varepsilon_n\}_{n\in\omega}$ such that $\sum_n \varepsilon_n \times 2^{n+1} < \infty$. By the previous lemma, we can assume that we have a computable sequence of integers $n_0 < m_0 < n_1 < m_1 < \ldots$, together with A-computable sequences of clopen sets $\{\mathcal{J}_{1,n}\}_{n\in\omega}$ and $\{\mathcal{J}_{2,n}\}_{n\in\omega}$ such that:

- $\mathcal{J}_{1,k} \subseteq 2^{[n_k,n_{k+1})}$ and $\mathcal{J}_{2,k} \subseteq 2^{[m_k,m_{k+1})}$
- $\lambda(\mathcal{J}_{1,k}) \leq \varepsilon_k$ and $\lambda(\mathcal{J}_{2,k}) \leq \varepsilon_k$
- Any computable sequence is in $\bigcap_n \bigcup_{m\geq n} \mathcal{J}_{1,m}$ or in $\bigcap_n \bigcup_{m\geq n} \mathcal{J}_{2,m}$

We are going to create relative to A an interval test $\{[n_{k+1}, n_{k+2}), \mathcal{R}_k\}_{k\in\omega}$ by mixing $\{\mathcal{J}_{1,k}\}_{k\in\ \omega}$ and $\{\mathcal{J}_{2,k}\}_{k\in\ \omega}$. Then, assuming that a computable sequence X is not in $\bigcap_n \bigcup_{m\geq n} \mathcal{R}_m$ we are going to create, using X, another interval A-Schnorr-Solovay test $\{[m_{k+1}, n_{k+2}), \mathcal{T}_k\}_{k\in\omega}$ such that every computable sequence is necessarily in $\bigcap_n \bigcup_{m\geq n} \mathcal{T}_m$. Fig. 4 illustrates the construction.

For $k > 0$, Let $\mathcal{S}_{1,k}$ be the union of $[\sigma]_{[n_k,m_k)}$ for every strings σ of length $m_k - n_k$ such that there are at least $2^{n_{k+1}-m_k-k}$ many strings τ with $[\sigma\,\hat{}\,\tau]_{[n_k,n_{k+1})} \subseteq \mathcal{J}_{1,k}$. We have $\lambda([S_{1,k}]^{\prec})2^{-k} \leq \lambda(\mathcal{J}_{1,k})$ and then $\lambda([S_{1,k}]^{\prec}) \leq 2^k\varepsilon_k \leq 2^k\varepsilon_{k-1}$.

For $k > 0$, let $\mathcal{S}_{2,k}$ be the union of $[\sigma]_{[n_k,m_k)}$ for every string σ of length $m_k - n_k$ such that there are at least $2^{n_k-m_{k-1}-k}$ many strings τ with $[\tau\,\hat{}\,\sigma]_{[m_{k-1},m_k)} \subseteq \mathcal{J}_{2,k-1}$. We have $\lambda([S_{2,k}]^{\prec})2^{-k} \leq \lambda(\mathcal{J}_{2,k-1})$ and then $\lambda([S_{2,k}]^{\prec}) \leq 2^k\varepsilon_{k-1}$.

Let $\mathcal{R}_k = [S_{1,k+1}]^{\prec} \cup [S_{2,k+1}]^{\prec}$. By the choice of the ε_k, clearly $\{[n_{k+1}, n_{k+2}), \mathcal{R}_k\}_{k\in\omega}$ is an interval A-Schnorr-Solovay test. If every computable sequence belongs to

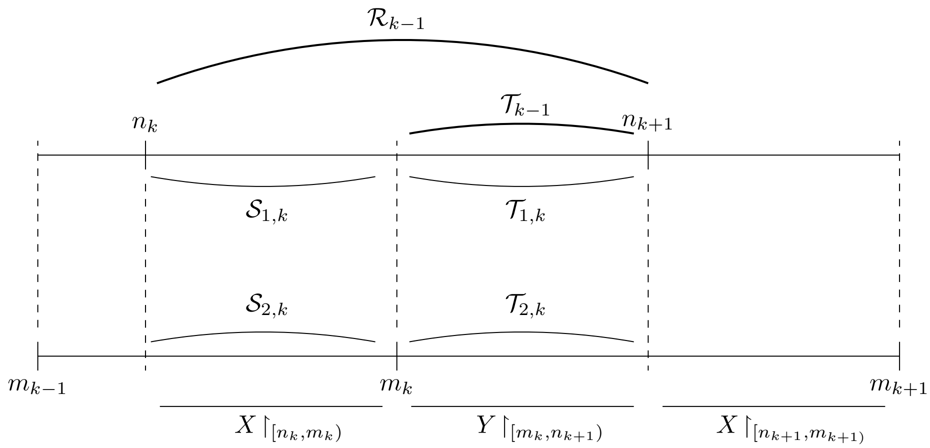


Fig. 4. Construction for Theorem V.6

$\bigcap_n \bigcup_{m\geq n} \mathcal{R}_m$ we are done. Otherwise there is a computable sequence X such that $X\restriction_{[n_{k+1},n_{k+2})}$ does not belong to $\mathcal{R}_k$ for any k.

For $k>0$ let $\mathcal{T}_{1,k}$ be the union of $[\tau]_{[m_k,n_{k+1})}$ for any string τ of length $n_{k+1}-m_k$ such that $[X\restriction_{[n_k,m_k)}\,\hat{}\,\tau]_{[n_k,n_{k+1})}\subseteq \mathcal{J}_{1,k}$. Let $\mathcal{T}_{2,k}$ be the union of $[\tau]_{[m_k,n_{k+1})}$ for any string τ of length $n_{k+1}-m_k$ such that $[\tau\,\hat{}\,X\restriction_{[n_{k+1},m_{k+1})}]_{[m_k,m_{k+1})}\subseteq \mathcal{J}_{2,k}$.

As X must be in $\bigcap_n \bigcup_{m\geq n} \mathcal{J}_{1,m}$ or in $\bigcap_n \bigcup_{m\geq n} \mathcal{J}_{2,m}$ and as no prefix of X belongs to $\mathcal{R}_k$ for any k, it implies by definition of $\mathcal{S}_{1,k}$ that there are at most $2^{n_{k+1}-m_k-k}$ many $[\tau]_{[m_k,n_{k+1})}$ in $T_{1,k}$ and by definition of $\mathcal{S}_{2,k+1}$ that there are at most $2^{n_{k+1}-m_k-k-1}$ many $[\tau]_{[m_k,n_{k+1})}$ in $T_{2,k}$.

In particular the clopen set $\mathcal{T}_k$ defined to be $\mathcal{T}_{1,k+1}\cup \mathcal{T}_{2,k+1}$, has measure less than 2^{-k+1}. Also, the sequence $\{[m_{k+1},n_{k+2}),\mathcal{T}_k\}_{k\in\omega}$ is an A -interval test.

We now use the fact that the sequence $n_0<m_0<n_1<m_1<\ldots$ is computable in order to prove that $\bigcap_n \bigcup_{m\geq n} \mathcal{T}_m$ contains every computable sequence. Suppose otherwise, that is, some computable sequence Y is in no set $\mathcal{T}_k$ for any k. Then the sequence $Z = X\restriction_{[n_0,m_0)}\,\hat{}\,Y\restriction_{[m_0,n_1)}\,\hat{}\,\ldots\,\hat{}\,X\restriction_{[n_k,m_k)}\,\hat{}\,Y\restriction_{[m_k,n_{k+1})}$ is computable. Also we easily see that Z is not in $\bigcap_n \bigcup_{m\geq n} \mathcal{J}_{1,m}$ as otherwise by definition of $\mathcal{T}_{1,k}$ the sequence Y would be captured by $\{[m_{k+1},n_{k+2}),\mathcal{T}_k\}_{k\in\omega}$. Similarly we see that Z is not in $\bigcap_n \bigcup_{m\geq n} \mathcal{J}_{2,m}$ as otherwise by definition of $\mathcal{T}_{2,k}$ the sequence Y would be captured by $\{[m_{k+1},n_{k+2}),\mathcal{T}_k\}_{k\in\omega}$. This contradicts the fact that both $\bigcap_n \bigcup_{m\geq n} \mathcal{J}_{1,m}$ and $\bigcap_n \bigcup_{m\geq n} \mathcal{J}_{2,m}$ contain every computable sequence. Thus $\bigcap_n \bigcup_{m\geq n} \mathcal{T}_m$ contains every computable sequence. ☐

Finally we can prove the harder direction of Theorem V.2:

Theorem V.2 "⇒". Given a sequence A that is weakly Schnorr engulfing, we have an interval A-Schnorr-Solovay test $\{I_n,\mathcal{J}_n\}_{n\in\omega}$ capturing every computable sequence such that, in addition, $\{I_n\}_{n\in\omega}$ is computable. We define a computable function H by $H(n)=2^{|I_n|}$, and a small A-computable H-trace $\{T_n\}_{n\in\omega}$ by letting T_n be the set of integers encoded by the strings σ such that $[\sigma]_{I_n}\subseteq \mathcal{J}_n$. Suppose that a computable function $f:\omega\to\omega$ bounded by H is not captured by $\{T_n\}_{n\in\omega}$. Then the computable sequence X such that $X\restriction_{I_n}$ equals the string of length $|I_n|$ corresponding to $f(n)$ and such that $X(i)=0$ for i not in any I_n is not captured by $\{I_n,\mathcal{J}_n\}_{n\in\omega}$ either, which is a contradiction. ☐

We end the paper with a curious fact: in a sense, among the weakly Schnorr engulfing sequences, the computably dominated ones have more capturing power than the ones which are not. Recall that a sequence is Δ^0_2 iff it is computable from the halting problem, $\emptyset'$, iff it is the pointwise limit of a uniform list of computable sequences.

Proposition V.7. *Suppose A is a computably dominated, weakly Schnorr engulfing oracle. Then the set of Δ^0_2 sequences is contained in an A-Schnorr null set.*

Proof. We shall prove that any A-Schnorr-Solovay test $\{I_n,\mathcal{J}_n\}_{n\in\omega}$, with each I_n computable uniformly in n, capturing every computable sequence, also captures every Δ^0_2 sequence. Consider a Δ^0_2 sequence X, given as the pointwise limit of a list $\{X_s\}_{s\in\omega}$ where each $X_s\in 2^\omega$ is computable uniformly in s.

Suppose X is not captured by our A-Schnorr-Solovay test, without loss of generality we can suppose that for every n we have $X\restriction_{I_n}\notin \mathcal{J}_n$. Let us define the A-computable function f which to the pair (t,n) associates the smallest $s\geq t$ such that $X_s\restriction_{I_n}\notin \mathcal{J}_n$. As A is computably dominated, the function f is bounded by some computable function $g:\omega\times\omega\mapsto\omega$.

We then define a computable sequence Y such that $Y\restriction_{I_n}\notin \mathcal{J}_n$ for each n. Fix some n and compute successive values of $g(s,n)$ for every s starting from 0, until every value $X_t\restriction_{I_n}$ is identical for $s\leq t\leq g(s,n)$. As the sequence $\{X_t\restriction_{I_n}\}_{t\in\omega}$ converges, we know that this will happen eventually. Define then $Y\restriction_{I_n}$ to be $X_{g(s,n)}\restriction_{I_n}$ and Y be 0 on positions which are in no I_n.

It is clear that $Y \upharpoonright_{I_n} \notin \mathcal{J}_n$ for every n, because $X_{f(s,n)} \upharpoonright_{I_n} \notin \mathcal{J}_n$, and as $s \leq f(s,n) \leq g(s,n)$, $X_{g(s,n)} \upharpoonright_{I_n} = X_{f(s,n)} \upharpoonright_{I_n}$ by the choice of s. $\square$

The conclusion of Proposition V.7 does not hold if A is hyperimmune. For if $A = \emptyset'$ then no A-Schnorr test contains every A-computable element. It is worth mentioning that the following analogue of Proposition V.7 is true with a similar proof: If f is computably dominated and H-i.o.e., then also f coincides infinitely often with every Δ^0_2 function bounded by H.

Acknowledgment

The authors would like to thank the Marsden fund of New Zealand, as well as the Science Faculty of the University of Auckland for supporting Monin's research stay in Auckland.

References

[1] U. Andrews, M. Cai, D. Diamondstone, C. Jockusch, and S. Lempp, "Asymptotic density, computable traceability, and 1-randomness," preprint, 2013.

[2] T. Bartoszyński, "On covering of real line by null sets," *Pacific Journal of Mathematics*, vol. 131, no. 1, pp. 1–12, 1988.

[3] T. Bartoszyński and H. Judah, *Set Theory. On the structure of the real line*. Wellesley, MA: A K Peters, 1995, 546 pages.

[4] C. T. Chong, A. Nies, and L. Yu, "Lowness of higher randomness notions," *Israel journal of mathematics*, vol. 66, no. 1, pp. 39–60, 2008.

[5] C. T. Chong and L. Yu, "Randomness in the higher setting."

[6] O. Demuth, "Remarks on the structure of tt-degrees based on constructive measure theory," *Comment. Math. Univ. Carolin.*, vol. 29, no. 2, pp. 233–247, 1988.

[7] R. Downey and E. Griffiths, "Schnorr randomness," *J. Symbolic Logic*, vol. 69, no. 2, pp. 533–554, 2004.

[8] R. Downey and D. Hirschfeldt, *Algorithmic randomness and complexity*. Berlin: Springer-Verlag, 2010, 855 pages.

[9] D. R. Hirschfeldt, C. G. Jockusch Jr, T. McNicholl, and P. E. Schupp, "Asymptotic density and the coarse computability bound," preprint, 2013.

[10] C. G. Jockusch and P. E. Schupp, "Generic computability, turing degrees, and asymptotic density," *Journal of the London Mathematical Society*, vol. 85, no. 2, pp. 472–490, 2012.

[11] I. Kapovich, A. Myasnikov, P. Schupp, and V. Shpilrain, "Generic-case complexity, decision problems in group theory, and random walks," *Journal of Algebra*, vol. 264, no. 2, pp. 665–694, 2003.

[12] T. Kihara, "Higher randomness and lim-sup forcing within and beyond hyperarithmetic," *In preparation*.

[13] B. Kjos-Hanssen, A. Nies, and F. Stephan, "Lowness for the class of Schnorr random sets," *SIAM J. Computing*, vol. 35, no. 3, pp. 647–657, 2005.

[14] B. Kjos-Hanssen, F. Stephan, and S. Terwijn, "Covering the recursive sets," preprint, 2015.

[15] M. Kummer, "A proof of Beigel's cardinality conjecture," *J. Symbolic Logic*, vol. 57, no. 2, pp. 677–681, 1992.

[16] S. Kurtz, "Randomness and genericity in the degrees of unsolvability," Ph.D. Dissertation, University of Illinois, Urbana, 1981.

[17] P. Martin-Löf, "The definition of random sequences," *Inform. and Control*, vol. 9, pp. 602–619, 1966.

[18] B. Monin, "Higher computability and randomness," Ph.D. dissertation, Université Paris Diderot, http://www.liafa.univ-paris-diderot.fr/~benoitm/ressources/misc/Thesis_report_benoit_monin_v1.7.pdf, 2014.

[19] A. Nies, *Computability and randomness*, ser. Oxford Logic Guides. Oxford: Oxford University Press, 2009, vol. 51, 444 pages. Paperback version 2011. [Online]. Available: http://dx.doi.org/10.1093/acprof:oso/9780199230761.001.0001

[20] J. C. Owings, "A cardinality version of Beigel's Nonspeedup Theorem," *J. Symbolic Logic*, vol. 54, no. 3, pp. 761–767, 1989.

[21] N. Rupprecht, "Relativized Schnorr tests with universal behavior," *Arch. Math. Logic*, vol. 49, no. 5, pp. 555–570, 2010. [Online]. Available: http://dx.doi.org.ezproxy.auckland.ac.nz/10.1007/s00153-010-0187-6

[22] G. Sacks, *Higher Recursion Theory*, ser. Perspectives in Mathematical Logic. Heidelberg: Springer–Verlag, 1990.

[23] C. Schnorr, *Zufälligkeit und Wahrscheinlichkeit. Eine algorithmische Begründung der Wahrscheinlichkeitstheorie*. Berlin: Springer-Verlag, 1971, lecture Notes in Mathematics, Vol. 218.

[24] S. Terwijn and D. Zambella, "Algorithmic randomness and lowness," *J. Symbolic Logic*, vol. 66, pp. 1199–1205, 2001.