The classification problem for compact computable metric spaces

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Abstract. We adjust methods of computable model theory to effective analysis. We use index sets and infinitary logic to obtain classification-type results for compact computable metric spaces. We show that every compact computable metric space can be uniquely described, up to isometry, by a computable Π_3 formula, and that orbits of elements are uniformly given by computable Π_2 formulas. We show that deciding if two compact computable metric spaces are isometric is a Π^0_2 complete problem *within* the class of compact computable spaces, which in itself is Π^0_3. On the other hand, if there is an isometry, then $\emptyset''$ can compute one. In fact, there is a set low relative to $\emptyset'$ which can compute an isometry. We show that the result can not be improved to $\emptyset'$. We also give further results for special classes of compact spaces, and for other related classes of Polish spaces.

1 Introduction

An equivalence relation on a standard Borel space is called *smooth* if it is Borel reducible to the equality relation on $\mathbb{R}$. By a result of Gromov (see [7, proof of 14.2.1]), the isometry relation on compact metric spaces is smooth. Thus, every compact metric space can be uniquely described, up to isometry, by a single real. In invariant descriptive set theory, a smooth equivalence relation E is considered trivial: by Silver's theorem, either E is Borel equivalent to equality on $\mathbb{R}$, or E has only countably many classes.

Recall that every compact metric space is separable and complete. Separable complete metric spaces occurring in mathematical practice are usually computable. For instance, $[0,1]^n$, the Hilbert cube, ℓ_2, $\mathcal{C}[0,1]$, and the Urysohn space are computable with any of the standard metrics [11,10]. In this paper, we adapt methods of computable model theory [2,6] to computable analysis [11,3,15] in order to study the classification problem for compact computable metric spaces. Although our paper is mostly restricted to the study of compact computable metric spaces up to isometry, we hope that our ideas and methods will find further applications to other topics of modern computable analysis, such as the study of computable Banach spaces and computable topological spaces.

In contrast to computable analysis, the main objects of computable algebra are countable algebraic structures. These are structures with domain $\mathbb{N}$ and in which the basic operations can be represented by computable functions on $\mathbb{N}$. In computable model theory and effective algebra there are several approaches to classification problems (see, e.g., [8,9,4,5]). We focus on two approaches which use index sets and infinitary computable logic, respectively.

Index sets and isomorphism problems. The first approach uses the fact that all partially computable functions can be effectively listed. As a consequence, there exists an effective listing of all partial computable algebraic structures $(\mathcal{A}_e)_{e\in\mathbb{N}}$ which includes all infinite computable algebras. For a class $\mathcal{K}$ of computable algebras, the difficulty of the classification problems is reflected in the following sets:

1. *the index set* $I_{\mathcal{K}} = \{e : \mathcal{A}_e \in \mathcal{K}\}$ of $\mathcal{K}$, and
2. *the isomorphism problem* $E_{\mathcal{K}} = \{(e,j) \in I_{\mathcal{K}}^2 : \mathcal{A}_e \cong \mathcal{A}_j\}$ for $\mathcal{K}$.

The complexity of the index sets is measured using the arithmetical, hyperarithmetical, and analytical hierarchies [2]. Recall that the arithmetical hierarchy is defined via iterating quantifiers over computable predicates, and the hyprarithmetical hierarchy extends the arithmetical hierarchy to computable ordinals. Deciding if two algebras from $\mathcal{K}$ are isomorphic might be simpler than detecting whether an algebra belongs to this class. In this case one usually considers the complexity of $E_{\mathcal{K}}$ *within* $I_{\mathcal{K}}$. For example, $E_{\mathcal{K}}$ is Π^0_2 *within* a Π^0_3 set $I_{\mathcal{K}}$ if there exists a Π^0_2 set $S \subset \mathbb{N}^2$ such that $E_{\mathcal{K}} = S \cap (I_{\mathcal{K}} \times I_{\mathcal{K}})$.

A collection of computable models $\mathcal{K}$ is called *classifiable* if both $I_{\mathcal{K}}$ and $E_{\mathcal{K}}$ are hyperarithmetical. (Usually $\mathcal{K}$ will be closed under isomorphism on computable models.) See [8,9,4,5] for further background and results in this direction.

Infinitary computable logic. Ash [1] introduced computable infinitary formulas in the context of computable algebras. An infinitary computable language extends a first-order language by allowing infinite conjunctions and disjunctions over computably enumerable families of formulas. The definition [1,2] uses a recursion scheme. Computable formulas have proved to be of a great importance in computable algebra; see the book of Ash and Knight [2]. We say that a class $\mathcal{K}$ of computable structures closed under isomorphism *admits a syntactic description*, if there exists a computable infinitary sentence Φ such that, for any computable M, we have $M \models \Phi$ if and only if $M \in \mathcal{K}$. Note that this condition implies that the index set is hyperarithmetical [8]. The converse is known without the restriction to indices for computable structures. Vanden Boom [14] has shown that every hyperarithmetical invariant class can be described by a computable sentence.

There is also a syntactic counterpart of requiring that $E_{\mathcal{K}}$ is hyperarithmetical.

Definition 1. We say that a class $\mathcal{K}$ of computable structures *admits a syntactic classification* if there is a hyperarithmetical bound on the complexity of infinitary formulas which describe the orbits of tuples of elements in $M \in \mathcal{K}$ under the action of the automorphism group of M.

To adjust the effective classification methods to computable analysis, we need some basic definitions. Following the tradition rooted in the works of Turing [12,13], we say that a real x is *computable* if for each k we can compute a rational within 2^{-k} of x.

Definition 2 ([3,11]). Let (M, d) be a complete separable metric space, and let $(q_i)_{i\in\mathbb{N}}$ be a dense sequence of points in M. The triple

$$\mathcal{M} = (M, d, (q_i)_{i\in\mathbb{N}})$$

is a *computable metric space* if $d(q_i, q_k)$ is a computable real uniformly in i, k. We say that $(q_i)_{i\in\mathbb{N}}$ is a *computable structure* on M, and that the q_i are the *special points* of $\mathcal{M}$. A *Cauchy name* for x is a sequence $(r_p)_{p\in\mathbb{N}}$ of special points converging rapidly to x in the sense that $d(r_p, r_{p+1}) < 2^{-p}$.

We introduce computable infinitary formulas in the context of computable metric spaces (see preliminaries). In Theorem 6 we prove that every computable compact metric space is uniquely described by a computable Π_3 infinitary sentence. Further, the orbits of special elements in a compact computable Polish space (under the action of its automorphism group) are given uniformly by computable infinitary Π_2 formulas. As a consequence, computable compact metric spaces admit a syntactic characterization. In Theorem 10 we will apply Theorem 6 to show that the index set of compact computable metric spaces is Π^0_3-complete, and the isomorphism problem for compact computable metric spaces is Π^0_2-complete within this index set. Thus, the collection of compact computable metric spaces is classifiable in the sense given above.

2 Preliminaries

We view a metric space (X, d) as a structure in the signature $\mathcal{S} = \{R_{<q}, R_{>q} \colon q \in \mathbb{Q}^+\}$, where $R_{<q}$ and $R_{>q}$ are binary relation symbols. The intended meaning of $R_{<q}(x, y)$ is that $d(x, y) < q$. The intended meaning of $R_{>q}(x,y)$ is that $d(x,y) > q$. We denote the first-order language of $\mathcal{S}$ by $\mathcal{L}$.

For a tuple $\overline{x} \in X^n$ consider the $n \times n$ distance matrix $D_n(\overline{x}) = d(x_i, x_j)_{i,j<n}$. We often view this matrix as a tuple in $\mathbb{R}^{n^2}$ with the max norm $\|.\|_{\max}$. Sometimes we suppress the subscript n. Note that for any matrix $A \in \mathbb{Q}^{n^2}$ and any positive rational p, there is a quantifier free positive first-order formula $\phi_{A,n,p}(\overline{x})$ in the signature above expressing that $\|D_n(\overline{x}) - A\|_{\max} < p$.

In this paper, the main objects are computable metric spaces. Notice that, in the notations of Definition 2, a separable space is computable if and only if $R_{<r}(q_i, q_k)$ and $R_{>r}(q_i, q_k)$ are uniformly Σ^0_1.

Definition 3. Since all partial functions can be effectively listed, we obtain a uniformly computable sequence of partial computable structures $(M_e)_{e\in\mathbb{N}}$ so that *some* of these M_e are computable structures on metric spaces: we view M_e as a partial computable function ψ such that $r_p = \psi(i,j,p)_{p\in\mathbb{N}}$ converges rapidly (in the sense above) to $d(i,j)$. It is a Π^0_2 property of ψ to be total and describe a metric space. We denote the completion of M_e, after modding out by equivalent points, by $\mathtt{cp}(M_e)$.

Fact 4 *For $(M, d, (p_i)_{i\in\mathbb{N}})$ a computable metric space, and W a c.e. set, $(p_i)_{i\in W}$ is a computable structure on the space $\mathtt{cp}((p_i)_{i\in W}), d)$.*

Proof. If W is infinite, we use a computable bijection $f : \omega \to W$ to define a computable structure $(r_i)_{i\in\mathbb{N}}$ on $\mathtt{cp}((p_i)_{i\in W}), d)$ by the rule $r_i = p_{f(i)}$.

Infinitary computable formulas. The language $\mathcal{L}^c_{\omega_1\omega}$ is a countable fragment of $\mathcal{L}_{\omega_1\omega}$. The atomic formulas are (syntactically) open finitary formulas in the language of metric spaces introduced above, with $\neg$ but without $=$. We allow computably enumerable conjunctions, computably enumerable disjunctions, and quantification over a variable.

In contrast to computable model theory, a computable structure on a space is not the whole space but a dense subset of it. Thus, for a computable metric structure M_e and ϕ a computable infinitary formula, $\mathtt{cp}(M_e) \models \phi$ and $M_e \models \phi$ have different interpretations.

The hierarchy of such formulas is defined similarly to the countable case; see the book of Ash and Knight [2]. In our specific case, the important modification is that $D_{<q}(x,y)$, for a rational q and special points x and y, should be understood as a Σ_1 formula, and similarly for $D_{>q}(x,y)$.

Informally, in the calculation of the complexity of a formula we also count alternations of infinitary conjunctions and disjunctions. When we count these alternations, we do not distinguish the infinitary conjunction from $\forall$, and disjunction from $\exists$. So, for example, a prefix of the form $\exists \bigwedge \forall \bigvee \exists$ will have only 3 alternations. More formally, the complexity of $\bigvee_i \psi_i$ is determined using $\inf\{\beta : \psi_i \in \Sigma_\beta\}$, and similarly for conjunctions. See [2] for formal definitions. We will omit the adjective "infinitary" when it is clear from the context.

Fact 5 *Let ψ be a computable formula of complexity Σ_n, where $n \in \omega$. Then the set $\{e : M_e \models \psi\}$ is Σ^0_n. (Similarly for Π_n.)*

Proof. By induction on the complexity of ψ we can show that, if M_e is a (partial) computable metric structure and $M_e \models \psi$, then $\emptyset^{(n-1)}$ will eventually recognize it.

3 Existential theories and infinitary formulas

Theorem 6.

(i) Within the class of computable Polish spaces, each compact member is uniquely described up to isometry by a computable Π_3 axiom.
(ii) The orbits of special elements in a compact computable metric space (under the action of its self-isometry group) are given uniformly by computable Π_2 formulas.

Proof. We will need a result due to Friedman, Fokina, Körwien and Nies (2012) which itself is based on Gromov's work (see [7, proof of 14.2.1]).

Proposition 7 *Let X, Y be compact metric spaces. Suppose that tuples $\widetilde{a} \in X^k, \widetilde{b} \in Y^k$ satisfy the same existential positive finitary formulas. Then there is an isometry from X to Y mapping $\widetilde{a}$ to $\widetilde{b}$.*

Proof. It is well-known that any isometric self-embedding of a compact metric space is onto (see [7, proof of 14.2.1]). Thus, by symmetry, it suffices to find an isometric embedding of X into Y mapping $\widetilde{a}$ to $\widetilde{b}$. The following lemma from [7, Exercise 14.2.3] slightly extends the above-mentioned result of Gromov.

Lemma 8. *Suppose that for every $\epsilon > 0$, for any n and tuple $\overline{x} \in X^n$ there is a tuple $\overline{y} \in Y^n$ such that $\left\| D(\widetilde{a}, \overline{x}) - D(\widetilde{b}, \overline{y}) \right\|_{\max} < \epsilon$. Then there is an an isometric embedding of X to Y mapping $\widetilde{a}$ to $\widetilde{b}$.*

It now suffices to show that if $\widetilde{a} \in X^n, \widetilde{b} \in Y^n$ satisfy the same existential positive formulas, the hypothesis of the lemma is satisfied. For every $n \times n$ rational matrix A, there is a formula $\phi_{A,n,\epsilon}(\overline{x})$ saying that

$\|D_n(\overline{x}) - A\|_{\max} < \epsilon/2$. Given $\overline{x} \in X^n$ choose a rational $(k+n) \times (k+n)$ matrix A such that

$$\|D(\widetilde{a}, \overline{x}) - A\|_{\max} < \epsilon/2.$$

Thus $\exists \overline{x}\, \phi_{A,n+k,\epsilon/2}(\widetilde{a}, \overline{x})$ holds in X. Hence there is $\overline{y} \in Y^n$ such that $\phi_{A,n+k,\epsilon/2}(\widetilde{b}, \overline{y})$ holds in Y. This implies $\left\|D(\widetilde{a}, \overline{x}) - D(\widetilde{b}, \overline{y})\right\|_{\max} < \epsilon$ as required.

We prove (i) of the theorem. Note that a complete metric space is compact iff it is totally bounded, namely, satisfies the computable sentence

$$\bigwedge_{q \in \mathbb{Q}^+} \bigvee_{n \in \mathbb{N}} \exists x_0 \dots x_{n-1} \forall y \bigvee_{i<n} d(x_i, y) < q. \tag{1}$$

We can replace each quantifier by a quantifier restricted to special points, and also replace $d(x_i, y) < q$ by $\neg(d(x_i, y) > q)$ with the meaning $d(x_i, y) \leq q$. Let θ be the resulting computable sentence. The quantifier $\bigvee_{i<n}$ is finitary and does not contribute any extra complexity to the formula. Thus, θ is computable Π_3. Clearly, $M_e \models \theta$ if and only if $\mathtt{cp}(M_e) \models \theta$.

We take M_e a computable structure on a Polish space. For the tuple $\widetilde{a} = \emptyset$ of special points we let ψ be a conjunction of all formulas $\exists \overline{x}\, \phi_{B,k,\epsilon}(\overline{x})$ (with quantification over special points, B a rational $k \times k$ matrix, ϵ a positive rational) which are true on M_e. Note that $\mathtt{cp}(M_e) \models \exists \overline{x}\, \phi_{B,k,\epsilon}(\overline{x})$ if and only if the corresponding restricted formula holds on M_e. Thus, the conjunction is in fact c.e. since we can enumerate all such sentences which are true on M_e. Therefore, ψ is computable Π_2. The desired computable axiom is $\mathcal{F} = \theta \wedge \psi$ which is of complexity Π_3.

We prove (ii). The orbit of a tuple $\widetilde{a}$ of special points in a compact computable Polish space is definable by the conjunction of $\exists \overline{x}\, \phi_{A,n+k,\epsilon/2}(\widetilde{a}, \overline{x})$ which hold on M_e. Given $\widetilde{a}$ we can effectively list all formulas $\phi_{A,n+k,\epsilon/2}(\widetilde{a}, \overline{x})$ such that $M_e \models \exists \overline{x}\, \phi_{A,n+k,\epsilon/2}(\widetilde{a}, \overline{x})$. Thus, the conjunction of all such formulas, with $\widetilde{a}$ replaced by a tuple of variables $\widetilde{y}$, is effective. Similarly to the proof of (1) above, we have $M_e \models \exists \overline{x}\, \phi_{A,n+k,\epsilon/2}(\widetilde{a}, \overline{x}) \Leftrightarrow \mathtt{cp}(M_e) \models \exists \overline{x}\, \phi_{A,n+k,\epsilon/2}(\widetilde{a}, \overline{x})$, for every $\widetilde{a} \in M_e$ and every parameters A, n, k and ϵ.

4 Descriptive complexity of index sets

Recall from Definition 3 that $\mathtt{cp}(M_e)_{e \in \omega}$ is an effective listing which includes all computable metric spaces. Recall also that a metric space X is connected iff for each nonempty open sets U, V, we have $C = X - (U \cup V) = \emptyset \Rightarrow U \cap V \neq \emptyset$.

Proposition 9. *(i) The set* $\{e : \mathtt{cp}(M_e)$ *is locally compact*$\}$ *is* Π^1_1*–complete. (ii) The set* $\{e : \mathtt{cp}(M_e)$ *is connected*$\}$ *is* Π^1_1*–hard.*

Theorem 10. *(i) The index set* CSp *of compact computable metric spaces is* Π^0_3*-complete. (ii) The isomorphism problem for compact computable metric spaces is* Π^0_2*-complete within* Π^0_3*.*

For the proof see the appendix. Next we study the complexity of whether a computable metric space is a continuum.

Proposition 11. *The index set* CCSp *of compact and connected computable metric spaces is* Π^0_3*-complete.*

Proof. Suppose now we are given a compact computable metric space $X = \mathtt{cp}(M_e)$. For connectedness, we need to check if for each non-empty open U and V, we have $C = X - (U \cup V) = \emptyset \Rightarrow U \cap V \neq \emptyset$. We may restrict U and V to finite unions of basic open sets of the form $B_\epsilon(p)$ where $\epsilon \in \mathbb{Q}^+$ and p is a special point. We may effectively in e obtain a $\emptyset'$-computable map g from 2^ω onto X. Thus $C = \emptyset$ is equivalent to $g^{-1}(C) = \emptyset$. Since the latter is a $\Pi^0_1(\emptyset')$ class, this condition is Σ^0_2. The condition $U \cap V \neq \emptyset$ is Σ^0_1 since this set contains a special point unless empty. Thus being connected is in fact Π^0_2 within the Π^0_3 set CSp.

Let S be any Π^0_3–complete set, and choose a uniformly c.e. double sequence $(V_{i,n})$ of initial segments of ω such that $i \in S \leftrightarrow \forall n V_{i,n} \neq \omega$. Let $a_k = 1 - 2^{-k}$. Given i, we can compute an index e for the computable metric space the Cartesian product $\Pi_{n\in\omega}[0, a_{|V_{i,n}|}]$ with the canonical computable structure obtained from the enumerations of the $V_{i,n}$, and the metric inherited from the standard metric on the Hilbert cube $[0, 1]^\omega$. Clearly M_e is connected, and M_e is compact iff $i \in S$.

5 Δ^0_3 categoricity

Definition 12. Let $S \subseteq \omega$ be an oracle. An isometry Φ from a computable metric space $(X, d, (q_i)_{i\in\mathbb{N}})$ to a computable metric space $(Y, d, (p_i)_{i\in\mathbb{N}})$ is *computable in* S if there is a Turing machine with oracle S which, on inputs i, k, outputs the k-th term of a Cauchy name for $\Phi(q_i)$.

We say that a computable metric space is Δ^0_n*–categorical* if between each of its computable presentations, there is an isometry computable relative to $\emptyset^{(n-1)}$.

Theorem 13 *Each compact computable metric space is* Δ^0_3 *categorical.*

Proof. Let $\mathcal{X} = (X, d, (p_i)_{i\in\mathbb{N}})$ and $\mathcal{Y} = (Y, d, (q_j)_{j\in\mathbb{N}})$ be compact computable metric spaces. Suppose that $\mathcal{X}$ can be isometrically embedded into $\mathcal{Y}$. We show that then there is a Δ^0_3 embedding; this is sufficient by symmetry.

Recall distance matrices D_n from Section 2. Let $\epsilon_n = 2^{-n}$. There is a computable triangular array of Y-special points $\langle y^n_i \rangle_{i<n}$ such that, where $\overline{y}^n = \langle y^n_0, \ldots, y^n_{n-1}\rangle$, we have

$$\|D_n(\langle p_0, \ldots, p_{n-1}\rangle) - D_n(\overline{y}^n)\|_{\max} < \epsilon_n.$$

We define a $\emptyset''$ computable triangular array of Y-special points $\langle w^n_i \rangle_{i\le n, 0<n}$ such that for each n, where $\overline{w}^n = \langle w^n_0, \ldots, w^n_{n-1}\rangle$, we have

$$|\{k > n\colon\ d(\overline{y}^k \restriction_n, \overline{w}^n) < \epsilon_n\}| = \infty. \tag{2}$$

We use compactness of Y and its finite powers Y^n throughout. Let $w^1_0 \in Y$ be a special point such that $A_1 = \{k\colon\ d(y^k_0, w^1_0) < \epsilon_1\}$ is infinite. Then (2) holds for $n = 1$.

(a) Increasing the dimension. Let w^1_1 be a special point in Y such that $C_1 = \{k \in A_1\colon\ d(y^k_1, w^1_1) < \epsilon_1\}$ is infinite.

(b) Refining the sequence. Let $\overline{w}^2 \in Y^2$ be a special point in the ball $B_{\epsilon_1}(\langle w^1_0, w^1_1\rangle)$ such that $A_2 = \{k \in C_1\colon\ d(\overline{y}^k \restriction_2, \overline{w}^2) < \epsilon_2\}$ is infinite.

We continue this process. Suppose $\overline{w}^n$ (and hence A_n) has been defined

(a) Let w^n_n be a special point in Y such that

$$C_n = \{k \in A_n\colon\ k > n \wedge d(y^k_n, w^n_n) < \epsilon_n\}$$

is infinite.

(b) Let $\overline{w}^{n+1} \in Y^{n+1}$ be a special point in $B_{\epsilon_n}((\overline{w}^n)\hat{\ } w^n_n)$ such that

$$A_{n+1} = \{k \in C_n\colon\ d(\overline{y}^k \restriction_{n+1}, \overline{w}^{n+1}) < \epsilon_{n+1}\}$$

is infinite. Then (2) holds for $n + 1$.

Note that the sequence $\langle w^n_i \rangle_{i\le n, 0<n}$ is indeed $\emptyset''$-computable because we uniformly in the previously defined special points obtain indices for the potential c.e. sets C_n, A_{n+1}. It takes $\emptyset''$ as an oracle to pick the next special points in such a way that the relevant set is infinite. Also note that $d(w^n_r, w^{n+1}_r) < \epsilon_n$ for each $n > r$. Thus, the sequence of points in Y $z_r = \lim_{n>r} w^n_r$ is computable in $\emptyset''$. It now suffices to show that the map $x_i \mapsto z_i$ preserves distances. Let $i < j$. Given n, by (2) pick $k > n$ such that $d(y^k \restriction_n, \overline{w}^n) < \epsilon_n$. Then, by the definitions,

$$|d(z_i, z_j) - d(w^n_i, w^n_j)| \le 2\epsilon_n$$
$$|d(w^n_i, w^n_j) - d(y^k_i, y^k_j)| \le \epsilon_n$$
$$|d(y^k_i, y^k_j) - d(x_i, x_j)| \le \epsilon_n.$$

Therefore, $|d(z_i, z_j) - d(x_i, x_j)| \leq 4\epsilon_n$.

The bound on the complexity of an isomorphism we obtained in Theorem 13 is not optimal. We can prove the following strengthening saying that some isomorphism is low relative to $\emptyset'$.

Theorem 14. *Let $\mathcal{X} = (X, d, (p_i)_{i\in\mathbb{N}})$ and $\mathcal{Y} = (Y, d, (q_j)_{j\in\mathbb{N}})$ be isometric compact computable metric spaces. Then there is a set S with $S' \leq_T \emptyset''$ which computes an isometry.*

The proof is an extension of the previous argument in that we build a nonempty $\Pi^0_1(\emptyset')$ class of isometries. Since the space is compact, the level size of the corresponding tree is bounded by a $\emptyset'$–computable function. Then, by the low basis theorem relative to $\emptyset'$, we obtain an isometry as required. We have also shown that the bound in Theorem 13 can not be improved to Δ^0_2, by building a metric space with two computable presentations and no Δ^0_2 isometry between them. Proofs of these results will appear in a journal paper.

6 Appendix: Proof of Theorem 10

(i) Recall that, for a sentence $\phi \in \mathcal{L}^c_{\omega_1\omega}$, the expressions $M_e \models \phi$ and $\mathtt{cp}(M_e) \models \phi$ have different interpretations: In the former we treat ϕ as a computable formula with quantifiers ranging over special points. In the latter ϕ is understood as an formula from $\mathcal{L}_{\omega_1\omega}$ with quantifiers ranging over the completion. We use notation from the proof of Theorem 6 (1). The sentence $\mathcal{F}$ has the following property. For each e, if M_e is a structure on a Polish space, then

$$M_e \models \mathcal{F} \Leftrightarrow \mathtt{cp}(M_e) \models \mathcal{F}.$$

Thus, we have $\mathtt{CSp} = \{e : \mathtt{cp}(M_e) \models \theta\} = \{e : M_e \models \theta\}$. Now, by Fact 5, we have that $\mathtt{CSp}$ is Π^0_3.

We now prove Π^0_3-completeness of $\mathtt{CSp}$. The standard computable structure on Baire space ω^ω is given by the collection of finite strings of natural numbers. We fix a Π^0_3-complete set S and a computable predicate P such that $x \in S \Leftrightarrow \forall y \exists^{<\infty} z P(x, y, z)$. By Fact 4, it is sufficient to construct a uniformly c.e. family $(C_x)_{x\in\mathbb{N}}$ of substructures of the standard structure on ω^ω which satisfies $x \in S \Leftrightarrow \mathtt{cp}(C_x)$ is compact. By uniformity, there will exist a total computable f such that $C_x = M_{f(x)}$ witnessing the desired reduction.

Construction. At stage -1, enumerate 01^y into the structure C_x for every y. At stage $s \geq 0$, we enumerate $01^y z$ with $z \leq s$ into C_x if $P(x, y, z)$ holds.

If $x \in S$ then each of the 01^y will have only finitely many extending strings, and the space $\mathtt{cp}(C_x)$ is compact. If $x \notin S$, then there is at least one string 01^y witnessing that $\mathtt{cp}(C_x)$ is not compact.

Remark 15. It follows from the Π^0_3-completeness of CSp and Fact 5 that the complexity of the sentence $\mathcal{F}$ from Theorem 6 can not be reduced.

(ii) Given $e, j \in$ CSp, we can effectively produce a computable Π_2 formula ψ in the notation of Theorem 6(1) which completely describes the isomorphism type of M_j. To see if $\mathtt{cp}(M_e) \cong \mathtt{cp}(M_j)$ it suffices to check if $M_e \models \psi$. By Fact 5, $\Psi_j = \{i : M_i \models \psi\}$ is Π^0_2, and it is actually uniformly Π_2 in the index of the formula ψ. Thus, the condition $e \in \Psi_j$ is Π^0_2 uniformly in e and j.

For the completeness, fix a Π^0_2-complete set S and a computable binary predicate R such that $x \in S \Leftrightarrow \exists^\infty y R(x, y)$. Let j be any computable index of the standard structure on Cantor space. For every x, we construct a c.e. closed subspace C_x of the standard structure on Cantor space. By Fact 4, we will get a computable structure on a compact space.

In the construction, if we see another y for which $R(x, y)$ holds, we enumerate all finite strings of length $\leq y$ from the standard structure into C_x. As a result, we will have C_x isomorphic to the whole Cantor space if, and only if, $x \in S$. Let f be a total computable function such that $C_x = M_{f(x)}$. We have $M_j \cong M_{f(x)}$ if and only if $x \in S$, as desired.

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