HIGHER KURTZ RANDOMNESS

BJØRN KJOS-HANSSEN, ANDRÉ NIES, FRANK STEPHAN, AND LIANG YU

Abstract. A real x is Δ^1_1-Kurtz random (Π^1_1-Kurtz random) if it is in no closed null Δ^1_1 set (Π^1_1 set). We show that there is a cone of Π^1_1-Kurtz random hyperdegrees. We characterize lowness for Δ^1_1-Kurtz randomness as being Δ^1_1-dominated and Δ^1_1-semi-traceable.

1. Introduction

Traditionally one uses tools from recursion theory to obtain mathematical notions corresponding to our intuitive idea of randomness for reals. However, already Martin-Löf [11] suggested to use tools from higher recursion (or equivalently, effective descriptive set theory) when he introduced the notion of Δ^1_1-randomness. This approach was pursued to greater depths by Hjorth and Nies [8] and Chong, Nies and Yu [1]. Hjorth and Nies investigated a higher analog of the usual Martin-Löf randomness, and a new notion with no direct analog in (lower) recursion theory: a real is Π^1_1-random if it avoids each null Π^1_1 set. Chong, Nies and Yu [1] studied Δ^1_1-randomness in more detail, viewing it as a higher analog of both Schnorr and recursive randomness. By now a classical result is the characterization of lowness for Schnorr randomness by recursive traceability (see, for instance, Nies' textbook [13]). Chong, Nies and Yu [1] proved a higher analog of this result, characterizing lowness for Δ^1_1 randomness by Δ^1_1 traceability.

Our goal is to carry out similar investigations for higher analogs of Kurtz randomness [3]. A real x is Kurtz random if avoids each Π^0_1 null class. This is quite a weak notion of randomness: each weakly 1-generic set is Kurtz random, so for instance the law of large numbers can fail badly.

It is essential for Kurtz randomness that the tests are *closed* null sets. For higher analogs of Kurtz randomness one can require that these tests are closed and belong to a more permissive class such as Δ^1_1, Π^1_1, or Σ^1_1.

Restrictions on the computational complexity of a real have been used successfully to analyze randomness notions. For instance, a Martin-Löf random real is weakly 2-random iff it forms a minimal pair with $\emptyset'$ (see [13]). We prove a result of that kind in the present setting. Chong, Nies, and Yu [1] studied a property restricting the

2010 *Mathematics Subject Classification.* Primary 03D32, Secondary 03D30, 03E15, 03E35, 68Q30.

Kjos-Hanssen's research was partially supported by NSF (U.S.A.) grants DMS-0652669 and DMS-0901020. Nies is partially supported by the Marsden Fund of New Zealand, grant No. 08-UOA-184. Stephan is supported in part by NUS grants number R146-000-114-112 and R252-000-308-112. Yu is supported by NSF of China No. 10701041 and Research Fund for the Doctoral Program of Higher Education, No. 20070284043.

complexity of a real: being Δ^1_1-dominated. This is the higher analog of being recursively dominated (or of hyperimmune-free degree). We show that a Δ^1_1-Kurtz random Δ^1_1 dominated set is already Π^1_1-random. Thus Δ^1_1-Kurtz randomness is equivalent to a proper randomness notion on a conull set. We also study the distribution of higher Kurtz random reals in the hyperdegrees. For instance, there is a cone of Π^1_1-Kurtz random hyperdegrees. However, its base is very complex, having the largest hyperdegree among all Σ^1_2 reals.

Thereafter we turn to lowness for higher Kurtz randomness. Recursive traceability of a real x is easily seen to be equivalent to the condition that for each function $f \leq_T x$ there is a recursive function $\hat{f}$ that agrees with f on at least one input in each interval of the form $[2^n, 2^{n+1} - 1)$ (see [13, 8.2.21]). Following Kjos-Hanssen, Merkle, and Stephan [10] one says that x is recursively semi-traceable (or infinitely often traceable) if for each $f \leq_T x$ there is a recursive function $\hat{f}$ that agrees with f on infinitely many inputs. It is straightforward to define the higher analog of this notion, Δ^1_1-semi-traceability. Our main result is that lowness for Δ^1_1-Kurtz randomness is equivalent to being Δ^1_1-dominated and Δ^1_1-semi-traceable. We also show using forcing that being Δ^1_1-dominated and Δ^1_1-semi-traceable is strictly weaker than being Δ^1_1-traceable. Thus, lowness for Δ^1_1 Kurtz randomness is strictly weaker than lowness for Δ^1_1-randomness.

2. Preliminaries

We assume that the reader is familiar with elements of higher recursion theory, as presented, for instance, in Sacks [16]. See [13, Ch. 9] for a summary.

A real is an element in 2^ω. Sometimes we write $n \in x$ to mean $x(n) = 1$. Fix a standard Π^0_2 set $H \subseteq \omega \times 2^\omega \times 2^\omega$ so that for all x and $n \in \mathcal{O}$, there is a unique real y satisfying $H(n, x, y)$. Moreover, if $\omega_1^x = \omega_1^{\mathrm{CK}}$, then each real $z \leq_h x$ is Turing reducible to some y so that $H(n, x, y)$ holds for some $n \in \mathcal{O}$. Roughly speaking, y is the $|n|$-th Turing jump of x. These y's are called H^x sets and denoted by H^x_n. For each $n \in \mathcal{O}$, let $\mathcal{O}_n = \{m \in \mathcal{O} \mid |m| < |n|\}$. $\mathcal{O}_n$ is a Δ^1_1 set.

We use the Cantor pairing function, the bijection $p : \omega^2 \to \omega$ given by $p(n, s) = \frac{(n+s)^2+3n+s}{2}$, and write $\langle n, s\rangle = p(n, s)$. For a finite string σ, $[\sigma] = \{x \succ \sigma \mid x \in 2^\omega\}$. For an open set U, there is a presentation $\hat{U} \subseteq 2^{<\omega}$ so that $\sigma \in \hat{U}$ if and only if $[\sigma] \subseteq U$. We sometimes identify U with $\hat{U}$. For a recursive functional Φ, we use $\Phi^\sigma[s]$ to denote the computation state of Φ^σ at stage s. For a tree T, we use $[T]$ to denote the set of infinite paths in T. Some times we identify a finite string $\sigma \in \omega^{<\omega}$ with a natural number without confusion.

The following results will be used in later sections.

Theorem 2.1 (Gandy). *If $A \subseteq 2^\omega$ is a nonempty Σ^1_1 set, then there is a real $x \in A$ so that $\mathcal{O}^x \leq_h \mathcal{O}$.*

Theorem 2.2 (Spector [17] and Gandy [6]). *$A \subset 2^\omega$ is Π^1_1 if and only if there is an arithmetical predicate $P(x, y)$ such that*

$$y \in A \leftrightarrow \exists x \leq_h y P(x, y).$$

Theorem 2.3 (Sacks[14]). *If x is non-hyperarithmetical, then $\mu(\{y|y \geq_h x\}) = 0$.*

Theorem 2.4 (Sacks [16]). *The set $\{x|x \geq_h \mathcal{O}\}$ is Π^1_1. Moreover, $x \geq_h \mathcal{O}$ if and only if $\omega_1^x > \omega_1^{\mathrm{CK}}$.*

A consequence of the last two theorems above is that the set $\{x \mid \omega_1^x > \omega_1^{\mathrm{CK}}\}$ is a Π^1_1 null set.

Given a class $\mathbf{\Gamma}$, an element $x \in \omega^\omega$ is called a $\mathbf{\Gamma}$-*singleton* if $\{x\}$ is a $\mathbf{\Gamma}$ set. Note that if $x \in \omega^\omega$ is a Π^1_1-singleton, then too is $x_0 = \{\langle n, m\rangle \mid x(n) = m\} \equiv_T x$. Hence we do not distinguish Π^1_1-singletons between Baire space and Cantor space.

A subset of 2^ω is $\mathbf{\Pi}^0_0$ if it is clopen. We can define $\mathbf{\Pi}^0_\gamma$ sets by a transfinite induction for all countable γ. Every such set can be coded by a real (for more details see [16]). Given a class $\mathbf{\Gamma}$ (for example, $\mathbf{\Gamma} = \Delta^1_1$) of subsets of 2^ω, a set A is $\mathbf{\Pi}^0_\gamma(\mathbf{\Gamma})$ if A is $\mathbf{\Pi}^0_\gamma$ and can be coded by a real in $\mathbf{\Gamma}$.

In the case $\gamma = 1$, every hyperarithmetic closed subset of reals is $\mathbf{\Pi^0_1}(\Delta^1_1)$. We also have the following result with an easy proof.

Proposition 2.5. *If $A \subseteq 2^\omega$ is Σ^1_1 and $\mathbf{\Pi}^0_1$, then A is $\mathbf{\Pi}^0_1(\Sigma^1_1)$.*

Proof. Let $z = \{\sigma \mid \exists x(x \in A \wedge x \succ \sigma)\}$. Then $x \in A$ if and only if $\forall n(x \upharpoonright n \in z)$. So A is $\Pi^0_1(z)$. Obviously z is Σ^1_1. □

Note that Proposition 2.5 fails if we replace Σ^1_1 with Π^1_1 since $\mathcal{O}^{\mathcal{O}}$ is a Π^1_1 singleton of hyperdegree greater than $\mathcal{O}$.

The ramified analytical hierarchy was introduced by Kleene, and applied by Fefferman [4] and Cohen [2] to study forcing, a tool that turns out to be powerful in the investigation of higher randomness theory. We recall some basic facts from Sacks [16] whose notations we mostly follow:

The ramified analytic hierarchy language $\mathfrak{L}(\omega_1^{\mathrm{CK}}, \dot{x})$ contains the following symbols:

(1) Number variables: $j, k, m, n, \ldots$;
(2) Numerals: $0, 1, 2, \ldots$;
(3) Constant: $\dot{x}$;
(4) Ranked set variables: $x^\alpha, y^\alpha, \ldots$ where $\alpha < \omega_1^{\mathrm{CK}}$;
(5) Unranked set variables: $x, y, ldots$;
(6) Others symbols include: $+$, $\cdot$ (times), $'$ (successor) and $\in$.

Formulas are built in the usual way. A formula φ is *ranked* if all of its set variables are ranked. Due to its complexity, the language is not codable in a recursive set but rather in the countable admissible set $L_{\omega_1^{\mathrm{CK}}}$.

To code the language in a uniform way, we fix a Π^1_1 path $\mathcal{O}_1$ through $\mathcal{O}$ (by [5] such a path exists). Then a ranked set variable x^α is coded by the number $(2, n)$ where $n \in \mathcal{O}_1$ and $|n| = \alpha$. Other symbols and formulas are coded recursively. With such a coding, the set of Gödel number of formulas is Π^1_1. Moreover, the set of Gödel numbers of ranked formulas of rank less than α is r.e. uniformly in the unique notation for α in $\mathcal{O}_1$. Hence there is a recursive function f so that $W_{f(n)}$ is the set of Gödel numbers of the ranked formula of rank less than $|n|$ when $n \in \mathcal{O}_1$ ($\{W_e\}_e$ is, as usual, an effective enumeration of r.e. sets).

One now defines a structure $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x)$, where x is a real, analogous to the way Gödel's L is defined, by induction on the recursive ordinals. Only at successor stages are new sets defined in the structure. The reals constructed at a successor stage are arithmetically definable from the reals constructed at earlier stages. The details may be found in [16]. We define $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi$ for a formula φ of $\mathfrak{L}(\omega_1^{\mathrm{CK}}, \dot{x})$ by allowing the unranked set variables to range over $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x)$, while the symbol x^α will be interpreted as the reals built before stage α. In fact, the domain of $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x)$ is the set $\{y \mid y \leq_h x\}$ if and only if $\omega_1^x = \omega_1^{\mathrm{CK}}$ (see [16]).

A sentence φ of $\mathfrak{L}(\omega_1^{\mathrm{CK}}, \dot{x})$ is said to be Σ_1^1 if it is ranked, or of the form $\exists x_1, \ldots, \exists x_n \psi$ for some formula ψ with no unranked set variables bounded by a quantifier.

The following result is a model-theoretic version of the Gandy-Spector Theorem.

Theorem 2.6 (Sacks [16]). *The set $\{(n_\varphi, x) \mid \varphi \in \Sigma_1^1 \wedge \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi\}$ is Π_1^1, where n_φ is the Gödel number of φ. Moreover, for each Π_1^1 set $A \subseteq 2^\omega$, there is a formula $\varphi \in \Sigma_1^1$ so that*

(1) $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi \implies x \in A$;
(2) *if* $\omega_1^x = \omega_1^{\mathrm{CK}}$, *then* $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi \Longleftrightarrow x \in A$.

Note that if φ is ranked, then both the sets $\{x \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi\}$ (the Gödel number of φ is omitted) and $\{x \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \neg\varphi\}$ are Π_1^1. So both sets are Δ_1^1. Moreover, if $A \subseteq 2^\omega$ is Δ_1^1, then there is a ranked formula φ so that $x \in A \Leftrightarrow \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi$ (see Sacks [16]).

Theorem 2.7 (Sacks [14]). *The set*

$$\{(n_\varphi, p) \mid \mu(\{x \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi\}) > p \wedge \varphi \in \Sigma_1^1 \wedge p \text{ is a rational number}\}$$

is Π_1^1 where n_φ is the Gödel number of φ.

Theorem 2.8 (Sacks [14]). *There is a recursive function $f : \omega \times \omega \to \omega$ so that for all n which is Gödel number of a ranked formula:*

(1) *$f(n,p)$ is Gödel number of a ranked formula;*
(2) *the set $\{x \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi_{f(n,p)}\} \supseteq \{x \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi_n\}$ is open; and*
(3) $\mu(\{x \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi_{f(n,p)}\} - \{x \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi_n\}) < \frac{1}{p}$.

Theorem 2.9 (Sacks [14] and Tanaka [18]). *If A is a Π_1^1 set of positive measure, then A contains a hyperarithmetical real.*

We also remind the reader of the higher analog of ML-randomness first studied by [8].

Definition 2.10. *A* Π_1^1-ML-test *is a sequence $(G_m)_{m\in\omega}$ of open sets such that for each m, we have $\mu(G_m) \leq 2^{-m}$, and the relation $\{\langle m, \sigma\rangle \mid [\sigma] \subseteq G_m\}$ is Π_1^1. A real x is* Π_1^1-ML-random *if $x \notin \cap_m G_m$ for each Π_1^1-ML-test $(G_m)_{m\in\omega}$.*

3. Higher Kurtz random reals and their distribution

Definition 3.1. *Suppose we are given a point class $\mathbf{\Gamma}$ (i.e. a class of sets of reals). A real x is* $\mathbf{\Gamma}$-Kurtz random *if $x \notin A$ for every closed null set $A \in \mathbf{\Gamma}$. Further, x is said to be* Kurtz random *(y-Kurtz random) if $\mathbf{\Gamma} = \Pi_1^0$ ($\mathbf{\Gamma} = \Pi_1^0(y)$).*

We focus on Δ^1_1, Σ^1_1 and Π^1_1-Kurtz randomness. By the proof of Proposition 2.5, it is not difficult to see that a real x is Δ^1_1-Kurtz random if and only if x does not belong to any $\mathbf{\Pi}^0_1(\Delta^1_1)$ null set.

Theorem 3.2. Π^1_1*-Kurtz randomness* $\subset$ Σ^1_1*-Kurtz randomness* $=$ Δ^1_1*-Kurtz-randomness.*

Proof. It is obvious that Π^1_1-Kurtz randomness $\subseteq$ Δ^1_1-Kurtz randomness and Σ^1_1-Kurtz randomness $\subseteq$ Δ^1_1-Kurtz randomness. It suffices to prove that Σ^1_1-Kurtz randomness $=$ Δ^1_1-Kurtz-randomness and Π^1_1-Kurtz randomness $\subset$ Δ^1_1-Kurtz randomness.

Note that every Π^1_1-ML-random is Δ^1_1-Kurtz random and there is a Π^1_1-ML-random real $x \equiv_h \mathcal{O}$ (see [8] and [1]). But $\{x\}$ is a Π^1_1 closed set. So x is not Π^1_1-Kurtz random. Hence Π^1_1-Kurtz randomness $\subset$ Δ^1_1-Kurtz randomness.

Suppose we are given a Π^1_1 open set A of measure 1. Define

$$x = \{\sigma \in 2^{<\omega} \mid \forall y(y \succ \sigma \implies y \in A)\}.$$

Then x is a Π^1_1 real coding A (i.e. $y \in A$ if and only if there is a $\sigma \in x$ for which $y \succ \sigma$, or $y \in [\sigma]$). So there is a recursive function $f : 2^{<\omega} \to \omega$ so that $\sigma \in x$ if and only if $f(\sigma) \in \mathcal{O}$. Define a Π^1_1 relation $R \subseteq \omega \times \omega$ so that $(k, n) \in R$ if and only if $n \in \mathcal{O}$ and $\mu(\bigcup\{[\sigma] \mid \exists m \in \mathcal{O}_n(f(\sigma) = m)\}) > 1 - \frac{1}{k}$. Obviously R is a Π^1_1 relation which can be uniformized by a Π^1_1 function f^* (see [12]). Since $\mu(A) = 1$, f^* is a total function. So the range of f^* is bounded by a notation $n \in \mathcal{O}$. Define $B = \{y \mid \exists\sigma(y \succ \sigma \wedge f(\sigma) \in \mathcal{O}_n)\}$. Then $B \subseteq A$ is a Δ^1_1 open set with measure 1. So every Π^1_1 open conull set has a Δ^1_1 open conull subset. Hence Σ^1_1-Kurtz randomness equals Δ^1_1-Kurtz randomness. □

It should be pointed out that, by the proof of Theorem 3.2, not every Π^1_1-ML-random real is Π^1_1-Kurtz random.

The following result clarifies the relationship between Δ^1_1- and Π^1_1-Kurtz randomness.

Proposition 3.3. *If* $\omega^x_1 = \omega^{\mathrm{CK}}_1$*, then* x *is* Π^1_1*-Kurtz random if and only if* x *is* Δ^1_1*-Kurtz random.*

Proof. Suppose that $\omega^x_1 = \omega^{\mathrm{CK}}_1$ and x is Δ^1_1-Kurtz random. If A is a Π^1_1 closed null set so that $x \in A$, then by Theorem 2.6, there is a formula $\varphi(z, y)$ whose only unranked set variables are z and y so that the formula $\exists z\varphi(z, y)$ defines A. Since $\omega^x_1 = \omega^{\mathrm{CK}}_1$, $x \in B = \{y \mid \mathfrak{A}(\omega^{\mathrm{CK}}_1, y) \models \exists z^\alpha\varphi(z^\alpha, y)\} \subseteq A$ for some recursive ordinal α. Define $T = \{\sigma \in 2^{<\omega} \mid \exists y \in B(y \succ \sigma)\}$. Obviously $B \subseteq [T]$. Since B is Δ^1_1, $[T]$ is Σ^1_1. Since A is closed, $B \subseteq A$, and $[T]$ is the closure of B, we have $[T] \subseteq A$. Hence since A is null, so is $[T]$. By the proof of Theorem 3.2, there is a Δ^1_1 closed null set $C \supseteq [T]$. Hence $x \in C$, a contradiction. □

From the proof of Theorem 3.2, one sees that every hyperdegree above $\mathcal{O}$ contains a Δ^1_1-Kurtz random real. But this fails for Π^1_1-Kurtz randomness. We say that a hyperdegree $\mathbf{d}$ is a *base for a cone of* $\mathbf{\Gamma}$*-Kurtz randoms* if for every hyperarithmetic degree $\mathbf{h} \geq \mathbf{d}$, $\mathbf{h}$ contains a $\mathbf{\Gamma}$-Kurtz random real.

The hyperdegree of $\mathcal{O}$ is a base for a cone of Δ^1_1-Kurtz randoms as proved in Theorem 3.2. In Corollary 5.3 we will show that not every nonzero hyperdegree is a base of a cone of Δ^1_1-Kurtz randoms.

Is there a base for a cone of Π^1_1-Kurtz randoms? If such a base **b** exists, then **b** is not hyperarithmetically reducible to any Π^1_1 singleton. Intuitively, this means that such bases must be complex.

To obtain such a base we need a lemma.

Lemma 3.4. *For any reals x and $z \geq_T x'$, there is an x-Kurtz random real $y \equiv_T z$.*

Proof. Fix an enumeration of the x-r.e. open sets $\{U^x_n\}_{n\in\omega}$.

We inductively define an increasing sequence of binary strings $\{\sigma_s\}_{s<\omega}$.

Stage 0. Let σ_0 be the empty string.

Stage $s+1$. Let $l_0 = 0$, $l_1 = |\sigma_s|$, and $l_{n+1} = 2^{l_n}$ for all $n > 1$. For every $n > 1$, let

$$A_n = \{\sigma \in 2^{l_n - 1} \mid \exists m < n \forall i \forall j (l_m \leq i, j < l_{m+1} \implies \sigma(i) = \sigma(j))\}.$$

Then

$$|A_n| \leq 2 \cdot 2^{l_{n-1}}.$$

In other words,

$$\mu(\bigcup\{[\sigma] \mid \sigma \succeq \sigma_s \wedge \sigma \notin A_n\}) \geq 2^{-l_1} \cdot (1 - 2^{l_n + 1 - l_{n+1}}).$$

Case(1): There is some $m > l_1 + 1$ so that $|\{\sigma \succeq \sigma_s \mid \sigma \in 2^m \wedge [\sigma] \subseteq U^x_s\}| > 2^{m-l_1-1}$. Let $n = m+1$. Then $l_{n+1} - 1 - l_n > 2$ and $l_n > m$. So there must be some $\sigma \in 2^{l_n - 1} - A_n$ so that there is a $\tau \preceq \sigma$ for which $[\tau] \subseteq U^x_s$ and $\tau \in 2^m$.

Let $\sigma_{s+1} = \sigma^\frown(z(s))^{l_n - 1}$.

Case(2): Otherwise. Let $\sigma_{s+1} = \sigma_s^\frown(z(s))^{l_1 - 1}$.

This finishes the construction at stage $s+1$.

Let $y = \bigcup_s \sigma_s$.

Obviously the construction is recursive in z. So $y \leq_T z$. Moreover, if U^x_n is of measure 1, then Case (1) happens at the stage $n+1$. So y is x-Kurtz random.

Let $l_0 = 0, l_{n+1} = 2^{l_n}$ for all $n \in \omega$. To compute $z(n)$ from y, we y-recursively find the n-th l_m for which for all i, j with $l_m \leq i < j < l_{m+1}$, $y(i) = y(j)$. Then $z(n) = y(l_m)$. □

Let $\mathcal{Q} \subseteq \omega \times 2^\omega$ be a universal Π^1_1 set. In other words, $\mathcal{Q}$ is a Π^1_1 set so that every Π^1_1 set is some $\mathcal{Q}_n = \{x \mid (n, x) \in \mathcal{Q}\}$. By Theorem 2.2.3 in [9], the real $x_0 = \{n \mid \mu(\mathcal{Q}_n) = 0\}$ is Σ^1_1. Let

$$\mathfrak{c} = \{(n, \sigma) \mid n \in x_0 \wedge \exists x((n, x) \in \mathcal{Q} \wedge \sigma \prec x)\} \subseteq \omega \times 2^{<\omega}.$$

Then $\mathfrak{c}$ can be viewed as a Σ^1_2 real. Since every Π^1_1 null closed set is $\Pi^0_1(\mathfrak{c})$, every $\mathfrak{c}$-Kurtz random real is Π^1_1-Kurtz random.

Theorem 3.5. *$\mathfrak{c}$ is a base for a cone of Π^1_1-Kurtz randoms.*

Proof. For every real $y_0 \geq_h \mathfrak{c}$, there is a real $y_1 \equiv_h y_0$ so that $y_1 \geq_T \mathfrak{c}'$, the Turing jump of $\mathfrak{c}$. By Lemma 3.4, there is a real $z \equiv_T y_1$ for which z is $\mathfrak{c}$-Kurtz random and so Π^1_1-Kurtz random. □

Recall that every Σ^1_2 real is constructible (see e.g. the last chapter of Moschovakis [12]). In the following we will determine the position of $\mathfrak{c}$ within the constructible hierarchy. A real is called constructible if it belongs to some level L_α of Gödel's hierarchy of constructible sets

$$L = \bigcup\{L_\beta : \beta \text{ is an ordinal}\}.$$

More generally, for each real x we have the hierarchy

$$L[x] = \bigcup\{L_\beta[x] : \beta \text{ is an ordinal}\}$$

of sets constructible from x.

Let

$$\delta^1_2 = \sup\{\alpha : \alpha \text{ is an ordinal isomorphic to a } \Delta^1_2 \text{ wellordering of } \omega\},$$

and

$$\delta = \min\{\alpha \mid L \setminus L_\alpha \text{ contains no } \Pi^1_1 \text{ singleton}\}.$$

Proposition 3.6 (Forklore)**.** $\delta = \delta^1_2$.

Proof. If $\alpha < \delta$, then there is a Π^1_1 singleton $x \in L_\delta \setminus L_\alpha$. Since $x \in L_{\omega_1^x}$ and ω_1^x is a $\Pi^1_1(x)$ wellordering, it must be that $\alpha < \omega_1^x < \delta^1_2$. So $\delta \leq \delta^1_2$.

If $\alpha < \delta^1_2$, there is a Δ^1_2 wellordering relation $R \subseteq \omega \times \omega$ of order type α. So there are two recursive relations $S, T \subseteq (\omega^\omega)^2 \times \omega^3$ so that

$$R(n,m) \Leftrightarrow \exists f \forall g \exists k S(f,g,n,m,k), \text{ and}$$
$$\neg R(n,m) \Leftrightarrow \exists f \forall g \exists k T(f,g,n,m,k).$$

Define a Π^1_1 set $R_0 = \{(f,n,m) \mid \forall g \exists k S(f,g,n,m,k)\}$. By the Gandy-Spector Theorem 2.2, there is an arithmetical relation S' so that $R_0 = \{(f,n,m) \mid \exists g \leq_h f(S'(f,g,n,m))\}$. Recall that every nonempty Π^1_1 set contains a Π^1_1-singleton (Kondo-Addison [16]). Then

$$R(n,m) \Leftrightarrow \exists f \in L_\delta \exists g \in L_{\omega_1^f}[f](S'(f,g,n,m)).$$

In other words, R is Σ_1-definable over L_δ. By the same method, the complement of R is Σ_1-definable over L_δ too. So R is Δ_1-definable over L_δ. It is clear that L_δ is admissible. So $R \in L_\delta$. Hence $\alpha < \delta$. Thus $\delta^1_2 = \delta$. □

Note that if x is a Δ^1_2-real, then ω_1^x is isomorphic to a Δ^1_2 wellordering of ω. So

$$\sup\{\omega_1^x \mid x \text{ is a } \Pi^1_1\text{-singleton}\} \leq \delta^1_2.$$

Since $x \in L_{\omega_1^x}$ for every Π^1_1-singleton x,

$$\sup\{\omega_1^x \mid x \text{ is a } \Pi^1_1\text{-singleton}\} \geq \delta = \delta^1_2.$$

Thus

$$\sup\{\omega_1^x \mid x \text{ is a } \Pi^1_1\text{-singleton}\} = \delta = \delta^1_2.$$

Since every Π^1_1 singleton is recursive in $\mathfrak{c}$, we have $\mathfrak{c} \notin L_{\delta^1_2}$ and $\omega_1^{\mathfrak{c}} \geq \delta^1_2$.

By the same argument as in Proposition 3.6, the reals lying in $L_{\delta^1_2}$ are exactly the Δ^1_2 reals. So $\mathfrak{c}$ is not Δ^1_2. Moreover, since $\mathfrak{c}$ is Σ^1_2, it is Σ_1 definable over $L_{\delta^1_2}$. Hence $\mathfrak{c} \in L_{\delta^1_2+1}$. In other words, for any real z, if $\omega_1^z > \omega_1^{\mathfrak{c}}$, then $\mathfrak{c} \in L_{\omega_1^z}$ and so $\mathfrak{c} \leq_h z$. Then by [15], $\mathfrak{c} \in L_{\omega_1^{\mathfrak{c}}}$. Thus $\omega_1^{\mathfrak{c}} > \delta^1_2$. Since actually all Σ^1_2 reals lie in $L_{\delta^1_2+1}$. This means that

$$\mathfrak{c} \text{ has the largest hyperdegree among all } \Sigma^1_2 \text{ reals.}$$

4. Δ^1_1-traceability and dominability

We begin with the characterization of Π^1_1-randomness within Δ^1_1-Kurtz randomness.

Definition 4.1. *A real x is hyp-dominated if for all functions $f : \omega \to \omega$ with $f \leq_h x$, there is a hyperarithmetic function g so that $g(n) > f(n)$ for all n.*

Recall that a real is Π^1_1-random if it does not belong to any Π^1_1-null set. The following result is a higher analog of the result that Kurtz randomness coincides with weak 2-randomness for reals of hyperimmune-free degree.

Proposition 4.2. *A real x is Π^1_1-random if and only if x is hyp-dominated and Δ^1_1-Kurtz random.*

Proof. Every Π^1_1-random real is Δ^1_1-Kurtz random and also hyp-dominated (see [1]). We prove the other direction.

Suppose x is hyp-dominated and Δ^1_1-Kurtz random. We show that x is Π^1_1-Martin-Löf random. If not, then fix a universal Π^1_1-Martin-Löf test $\{U_n\}_{n\in\omega}$ (see [8]). Then there is a recursive function $f : \omega \times 2^{<\omega} \to \omega$ so that for any pair (n, σ), $\sigma \in U_n$ if and only if $f(n, \sigma) \in \mathcal{O}$. Since x is hyp-dominated, $\omega_1^x = \omega_1^{\mathrm{CK}}$ (see [1]). Then we define a $\Pi^1_1(x)$ relation $R \subseteq \omega \times \omega$ so that $R(n, m)$ if and only if there is a σ so that $m \in \mathcal{O}$, $f(n, \sigma) \in \mathcal{O}_m = \{i \in \mathcal{O} \mid |i| < |m|\}$ and $\sigma \prec x$. Then by the Π^1_1-uniformization relativized to x, there is a partial function p uniformizing R. Since $x \in \bigcap_n U_n$, p is a total function. Since $\omega_1^x = \omega_1^{\mathrm{CK}}$, there must be some $m_0 \in \mathcal{O}$ so that $p(n) \in \mathcal{O}_{m_0}$ for every n. Then define a Δ^1_1-Martin-Löf test $\{\hat{U}_n\}_{n\in\omega}$ so that $\sigma \in \hat{U}_n$ if and only if $f(n, \sigma) \in \mathcal{O}_{m_0}$. So $x \in \bigcap_n \hat{U}_n$. Let $\hat{f}(n) = \min\{l \mid \exists \sigma \in 2^l(\sigma \in \hat{U}_n \wedge x \in [\sigma])\}$ be a $\Delta^1_1(x)$ function. Then there is a Δ^1_1 function f dominating $\hat{f}$. Define $V_n = \{\sigma \mid \sigma \in 2^{\leq f(n)} \wedge \sigma \in \hat{U}_n\}$ for every n. Then $P = \bigcap_n V_n$ is a Δ^1_1 closed set and $x \in P$. So x is not Δ^1_1-Kurtz random, a contradiction.

Since is Π^1_1-Martin-Löf random and $\omega_1^x = \omega_1^{\mathrm{CK}}$, x is already Π^1_1-random (see [1]). ☐

Next we proceed to traceability.

Definition 4.3. (i) *Let $h : \omega \to \omega$ be a nondecreasing unbounded function that is hyperarithmetical. A Δ^1_1 trace with bound h is a uniformly Δ^1_1 sequence $(T_e)_{e\in\omega}$ such that $|T_e| \leq h(e)$ for each e.*

(ii) *$x \in 2^\omega$ is Δ^1_1-traceable* [1] *if there is $h \in \Delta^1_1$ such that, for each $f \leq_h x$, there is a Δ^1_1 trace with bound h such that, for each e, $f(e) \in T_e$.*

(iii) *$x \in 2^\omega$ is Δ^1_1-semi-traceable if for each $f \leq_h x$, there is a Δ^1_1 function g so that, for infinitely many n, $f(n) = g(n)$. We say that g semi-traces f.*

(iv) *$x \in 2^\omega$ is Π^1_1-semi-traceable if for each $f \leq_h x$, there is a partial Π^1_1 function p so that, for infinitely many n we have $f(n) = p(n)$.*

Note that, if $(T_e)_{e\in\omega}$ is a uniformly Δ^1_1 sequence of finite sets, then there is $g \in \Delta^1_1$ such that for each e, $D_{g(e)} = T_e$ (where D_n is the nth finite set according to some recursive ordering). Thus

$$g(e) = \mu n\, \forall u\, [u \in D_n \leftrightarrow u \in T_e].$$

In this formulation, the definition of Δ^1_1 traceability is very close to that of recursive traceability.

Also notice that the choice of a bound as a witness for traceability is immaterial:

Proposition 4.4 (As in Terwijn and Zambella [19])**.** *Let A be a real that is Δ^1_1 traceable with bound h. Then A is Δ^1_1 traceable with bound h' for any monotone and unbounded Δ^1_1 function h'.*

Lemma 4.5. *x is Π^1_1-semi-traceable if and only if x is Δ^1_1-semi-traceable.*

Proof. It is not difficult to see that if x is Π^1_1-semi-traceable, then $\omega_1^x = \omega_1^{\mathrm{CK}}$. For otherwise, $x \geq_h \mathcal{O}$. So it suffices to show that $\mathcal{O}$ is not Π^1_1-semi-traceable. Let $\{\phi_i\}_{i\in\omega}$ be an effective enumeration of partial recursive functions. Define a function $g \leq_T \mathcal{O}'$ so that $g(i) = \sum_{j\leq i} m^i_j + 1$ where m^i_j is the least number k so that $p_j(i,k) \in \mathcal{O}$; if there is no such k, then $m^i_j = 0$. Note that for any Π^1_1 partial function p, there must be some partial recursive function p_j so that for every pair n, m, $p(n) = m$ if and only if $p_j(n,m) \in \mathcal{O}$. Then by the definition of g, for any $i > j$, $g(k) \neq p(i)$. So g cannot be traced by p.

Suppose that x is Π^1_1-semi-traceable, $\omega_1^x = \omega_1^{\mathrm{CK}}$, and $f \leq_h x$. Fix a Π^1_1 partial function p for f. Since p is a Π^1_1 function, there must be some recursive injection h so that $p(n) = m \Leftrightarrow h(n,m) \in \mathcal{O}$.

Let $R(n,m)$ be a $\Pi^1_1(x)$ relation so that $R(n,m)$ iff there exists $m > k \geq n$ for which $f(k) = p(k)$. Then some total function g uniformizes R such that g is $\Pi^1_1(x)$, and so $\Delta^1_1(x)$. Thus, for every n, there is some $m \in [g(n), g(g(n)))$ so that $f(m) = p(m)$. Let $g'(0) = g(0)$, and $g'(n+1) = g(g'(n))$ for all $n \in \omega$. Define a $\Pi^1_1(x)$ relation $S(n,m)$ so that $S(n,m)$ if and only if $m \in [g'(n), g'(n+1))$ and $p(m) = f(m)$. Uniformizing S we obtain a $\Delta^1_1(x)$ function g''.

Define a $\Delta^1_1(x)$ set by $H = \{h(m,k) \mid \exists n (g''(n) = m \wedge f(m) = k)\}$. Since $\omega_1^x = \omega_1^{\mathrm{CK}}$, $H \subseteq \mathcal{O}_n$ for some $n \in \mathcal{O}$. Since $\mathcal{O}_n$ is a Δ^1_1 set, we can define a Δ^1_1 function $\hat{f}$ by: $\hat{f}(i) = j$ if $h(i,j) \in \mathcal{O}_n$; $\hat{f}(i) = 1$, otherwise. Then there are infinitely many i so that $f(i) = \hat{f}(i)$. □

Note that the Δ^1_1-dominated reals form a measure 1 set [1] but the set of Δ^1_1-semi-traceable reals is null. Chong, Nies and Yu [1] constructed a non-hyperarithmetic Δ^1_1-traceable real.

Proposition 4.6. *Every Δ^1_1-traceable real is Δ^1_1-dominated and Δ^1_1-semi-traceable.*

Proof. Obviously every Δ^1_1-traceable real is Δ^1_1-dominated.

Suppose we are given a Δ^1_1-traceable real x and $\Delta^1_1(x)$ function f. Let $g(n) = \langle f(2^n), f(2^n+2), \dots, f(2^{n+1}-1)\rangle$ for all $n \in \omega$. Then there is a Δ^1_1 trace T for g so that $|T_n| \leq n$ for all n.

Then for all $2^n + 1 \leq m \leq 2^{n+1}$, let $\hat{f}(m) =$ the $(m - 2^n)$-th entry of the tuple of the $(m - 2^n)$-th element of T_n if there exists such an m; otherwise, let $\hat{f}(m) = 1$. It is not difficult to see that for every n there is at least one $m \in [2^n, 2^{n+1})$ so that $f(m) = \hat{f}(m)$. □

From the proof above, one can see the following corollary.

Corollary 4.7. *A real x is Δ^1_1-traceable if and only if for every x-hyperarithmetic $\hat{f}$, there is a hyperarithmetic function f so that for every n, there is some $m \in [2^n, 2^{n+1})$ so that $f(m) = \hat{f}(m)$.*

The following proposition will be used in Theorem 4.13 to disprove the converse of Proposition 4.6.

Proposition 4.8. *For any real x, the following are equivalent.*

(1) *x is Δ^1_1-semi-traceable and Δ^1_1-dominated.*
(2) *For every function $g \leq_h x$, there exist an increasing Δ^1_1 function f and a Δ^1_1 function $F : \omega \to [\omega]^{<\omega}$ with $|F(n)| \leq n$ so that for every n, there exists some $m \in [f(n), f(n+1))$ with $g(m) \in F(m)$.*

Proof. (1) $\Longrightarrow$ (2): Immediate because $1 \leq n$.

(2) $\Longrightarrow$ (1). Suppose we are given a function $\hat{g} \leq_h x$. Without loss of generality, $\hat{g}$ is nondecreasing. Let f and F be the corresponding Δ^1_1 functions. Let $j(n) = \sum_{i \leq f(n+1)} \sum_{k \in F(i)} k$ and note that j is a Δ^1_1 function dominating $\hat{g}$.

To show that x is Δ^1_1-traceable, suppose we are given a function $\hat{g} \leq_h x$. Let $h(n) = \langle g(2^n + 1), g(2^n + 2), \ldots, g(2^{n+1} - 1)\rangle$. Then by assumption there are corresponding Δ^1_1 functions f_h and F_h. For every n and $m \in [2^n, 2^{n+1})$, let $g(m) =$ the $(m - 2^n)^{\text{th}}$ column of the $(m - 2^n)^{\text{th}}$ element in $F_h(n)$ if such an m exists; let $g(m) = 1$ otherwise. Then g is a Δ^1_1 function semi-tracing $\hat{g}$. □

To separate Δ^1_1-traceability from the conjunction of Δ^1_1-semi-traceability and Δ^1_1-dominability, we have to modify Sacks' perfect set forcing.

Definition 4.9. (1) *A Δ^1_1 perfect tree $T \subseteq 2^{<\omega}$ is* fat *at n if for every $\sigma \in T$ with $|\sigma| \in [2^n, 2^{n+1})$, we have $\sigma^\frown 0 \in T$ and $\sigma^\frown 1 \in T$. Then we also say that n is a* fat number *of T.*

(2) *A Δ^1_1 perfect tree $T \subseteq 2^{<\omega}$ is* clumpy *if there are infinitely many n so that T is fat at n.*
(3) *Let $\mathbb{F} = (\mathcal{F}, \subseteq)$ be a partial order of which the domain $\mathcal{F}$ is the collection of clumpy trees, ordered by inclusion.*

Let φ be a sentence of $\mathfrak{L}(\omega_1^{\text{CK}}, \dot{x})$. Then we can define the forcing relation, $T \Vdash \varphi$, as done by Sacks in Section 4, IV [16].

(1) φ is ranked and $\forall x \in T(\mathfrak{A}(\omega_1^{\text{CK}}, x) \models \varphi)$, then $T \Vdash \varphi$.
(2) If $\varphi(y)$ is unranked and $T \Vdash \varphi(\psi(n))$ for some $\psi(n)$ of rank at most α, then $T \Vdash \exists y^\alpha \varphi(y^\alpha)$.
(3) If $T \Vdash \exists y^\alpha \varphi(y^\alpha)$, then $T \Vdash \exists y \varphi(y)$.
(4) If $\varphi(n)$ is unranked and $T \Vdash \varphi(m)$ for some number m, then $T \Vdash \exists n \varphi(n)$.

(5) If φ and ψ are unranked, $T \Vdash \varphi$ and $T \Vdash \psi$, then $T \Vdash \varphi \wedge \psi$.
(6) If φ is unranked and $\forall P(P \subseteq T \implies P \nVdash \varphi)$, then $T \Vdash \neg\varphi$.

The following lemma can be deduced as done in [16].

Lemma 4.10. *The relation $T \Vdash \varphi$, restricted to Σ^1_1 formulas φ, is Π^1_1.*

Lemma 4.11. (1) *Let $\{\varphi_i\}_{i\in\omega}$ be a hyperarithmetic sequence of Σ^1_1 sentences. Suppose for every i and $Q \subseteq T$, there exists some $R \subseteq Q$ so that $R \Vdash \varphi_i$. Then there exists some $Q \subseteq T$ so that for every i, $Q \Vdash \varphi_i$.*
(2) $\forall\varphi\forall T\exists Q \subseteq T(Q \Vdash \varphi \vee Q \Vdash \neg\varphi)$.

Proof. Using the notation $P \upharpoonright n = \{\tau \in 2^{\leq n} \mid \tau \in P\}$, define $\mathcal{R}$ by

$$\mathcal{R}(R, i, \sigma, P) \Leftrightarrow (\sigma \in R,\ P \subseteq R,\ P \Vdash \varphi_i,\ P \upharpoonright |\sigma| = \{\tau \mid \tau \prec \sigma\},$$
$$\text{and } \log|\sigma| - 1 \text{ is the } i^{\text{th}} \text{ fat number of } R).$$

Note that $\mathcal{R}$ is a Π^1_1 relation. Then $\mathcal{R}$ can be uniformized by a partial Π^1_1 function $F : \mathcal{F} \times \omega \times 2^{<\omega} \to \mathcal{F}$. Using F, a hyperarithmetic family $\{P_\sigma \mid \sigma \in 2^{<\omega}\}$ can be defined by recursion on σ.

$P_\emptyset = T$.
If $\log|\sigma| - 1$ is not a fat number of P_σ, then $P_{\sigma^\frown 0}, P_{\sigma^\frown 1} = P_\sigma$.
Otherwise: If $\sigma \notin P_\sigma$, then $P_{\sigma^\frown 0} = P_{\sigma^\frown 1} = \emptyset$.
Otherwise: $P_{\sigma^\frown 0} \cap P_{\sigma^\frown 1} = \emptyset, P_{\sigma^\frown 0} \cup P_{\sigma^\frown 1} \subseteq P_\sigma$,
$P_{\sigma^\frown 0} \upharpoonright |\sigma|, P_{\sigma^\frown 1} \upharpoonright |\sigma| = \{\tau \mid \tau \prec \sigma\}$ and
$P_{\sigma^\frown 0}, P_{\sigma^\frown 1} \Vdash \wedge_{j \leq i}\varphi_j$ where
i is the number so that $\log|\sigma| - 1$ is the i-th fat number of T.

Let $Q = \bigcap_n \bigcup_{|\sigma|=n} P_\sigma$. Then $Q \in \mathcal{F}$. It is routine to check that for every i, $Q \Vdash \varphi_i$.

The proof of (2) is the same as the proof of Lemma 4.4 IV [16]. □

We say that a real x is generic if it is the union of roots of trees in a generic filter; equivalently, for each Σ^1_1 sentence φ, there is a condition T such that $x \in T$ and either $T \Vdash \varphi$ or $T \Vdash \neg\varphi$. One can check (Lemma 4.8, IV [16]) that for every Σ^1_1-sentence φ,

$$\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi \Leftrightarrow \exists P(x \in P \wedge P \Vdash \varphi).$$

Lemma 4.12. *If x is a generic real, then*
(1) *$\mathfrak{A}(\omega_1^{\mathrm{CK}}, x)$ satisfies Δ^1_1-comprehension. So $\omega_1^x = \omega_1^{\mathrm{CK}}$.*
(2) *x is Δ^1_1-dominated and Δ^1_1-semi-traceable.*
(3) *x is not Δ^1_1-traceable.*

Proof. (1). The proof of (1) is exactly same as the proof of Theorem 5.4 IV, [16].

(2). By Proposition 4.8, it suffices to show that for every function $g \leq_h x$, there are an increasing Δ^1_1 function f and a Δ^1_1 function $F : \omega \to \omega^{<\omega}$ with $|F(n)| \leq n$ so that for every n, there exists some $m \in [f(n), f(n+1))$ so that $g(m) \in F(m)$. Since $g \leq_h x$ and $\omega_1^x = \omega_1^{\mathrm{CK}}$, there is a ranked formula φ so that for every n, $g(n) = m$ if and only if $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi(n, m)$. So there is a condition $S \Vdash \forall n\exists! m\varphi(n, m)$. Fix a condition $T \subseteq S$. As in the proof of Lemma 4.11, we can build a hyperarithmetic sequence of conditions $\{P_\sigma\}_{\sigma\in 2^{<\omega}}$ so that

$$P_{\sigma^\frown i} \Vdash \varphi(|\sigma|, m_{\sigma^\frown i}) \text{ for } i \leq 1$$

if $\log|\sigma|-1$ is a fat number of P_σ and $\sigma \in P_\sigma$. Let Q be as defined in the proof of Lemma 4.11. Let f be the Δ^1_1 function such that $f(0)=0$, and $f(n+1)$ is the least number $k > f(n)$ so that m_σ is defined for some σ with $f(n) < |\sigma| < k$. Let $F(n) = \{0\} \cup \{m_\sigma \mid |\sigma| = n\}$, and note that F is a Δ^1_1 function. Then

$$Q \Vdash \forall n |F(n)| \leq n \wedge \forall n \exists m \in [f(n), f(n+1)) \exists i \in F(m)(\varphi(m,i)).$$

So

$$Q \Vdash \exists F \exists f(\forall n |F(n)| \leq n \wedge \forall n \exists m \in [f(n), f(n+1)) \exists i \in F(m)(\varphi(m,i))).$$

Since T is an arbitrary condition stronger than S, this means

$$S \Vdash \exists F \exists f(\forall n |F(n)| \leq n \wedge \forall n \exists m \in [f(n), f(n+1)) \exists i \in F(m)(\varphi(m,i))).$$

Since $x \in S$,

$$\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \exists F \exists f(\forall n |F(n)| \leq n \wedge \forall n \exists m \in [f(n), f(n+1)) \exists i \in F(m)(\varphi(m,i))).$$

So x is Δ^1_1-dominated and Δ^1_1-semi-traceable.

(3). Suppose $f : \omega \to \omega$ is a Δ^1_1 function so that for every n, there is a number $m \in [2^n, 2^{n+1})$ with $f(m) = x(m)$. Then there is a ranked formula φ so that $f(n) = m \Leftrightarrow \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi(n,m)$. Moreover, $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \forall n \exists m \in [2^n, 2^{n+1})(\varphi(m, x(m)))$. So there is a condition $T \Vdash \forall n \exists m \in [2^n, 2^{n+1})(\varphi(m, \dot{x}(m)))$ and $x \in T$. Let n be a number so that T is fat at n and $\sigma \in 2^{2^n-1}$ be a finite string in T. Let μ be a finite string so that $\mu(m) = 1 - f(m + 2^n - 1)$. Define $S = \{\sigma^\frown\mu^\frown\tau \mid \sigma^\frown\mu^\frown\tau \in T\} \subseteq T$. Then $S \Vdash \forall m \in [2^n, 2^{n+1})(\neg\varphi(m, x(m)))$. But S is stronger than T, a contradiction. By Corollary 4.7, x is not Δ^1_1-traceable. □

We may now separate Δ^1_1-traceability from the conjunction of Δ^1_1-semi-traceability and Δ^1_1-dominability.

Theorem 4.13. *There are $2^{\aleph_0}$ many Δ^1_1-dominated and Δ^1_1-semi-traceable reals which are not Δ^1_1-traceable.*

Proof. This is immediate from Lemma 4.12. Note that there are $2^{\aleph_0}$ many generic reals. □

5. Lowness for higher Kurtz randomness

Given a relativizable class of reals $\mathcal{C}$ (for instance, the class of random reals), we call a real x *low for $\mathcal{C}$* if $\mathcal{C} = \mathcal{C}^x$. We shall prove that lowness for Δ^1_1-randomness is different from lowness for Δ^1_1-Kurtz randomness. A real x is *low for Δ^1_1-Kurtz tests* if every $\Delta^1_1(x)$ open set with measure 1 has a Δ^1_1 open subset of measure 1. Clearly, lowness for Δ^1_1-Kurtz tests implies lowness for Δ^1_1-Kurtz randomness.

Theorem 5.1. *If x is Δ^1_1-dominated and Δ^1_1-semi-traceable, then x is low for Δ^1_1-Kurtz tests.*

Proof. Suppose x is Δ^1_1-dominated and Δ^1_1-semi-traceable and U is a $\Delta^1_1(x)$ open set with measure 1. Then there is a real $y \leq_h x$ so that U is $\Sigma^0_1(y)$. Hence for some Turing reduction Φ, if for all z we write U^z for the domain of Φ^z, then we have $U = U^y$.

Define a $\Delta^1_1(x)$ function $\hat{f}$ by: $\hat{f}(n)$ is the shortest string $\sigma \prec y$ so that $\mu(U^\sigma[\sigma]) > 1 - 2^{-n}$. By the assumptions of the Theorem, there are an increasing Δ^1_1 function g and a Δ^1_1 function f so that for every n, there is an $m \in [g(n), g(n+1))$ so that $f(m) = \hat{f}(m)$. Without loss of generality, we can assume that $\mu(U^{f(m)}[m]) > 1 - 2^{-m}$ for every m.

Define a Δ^1_1 open set V so that $\sigma \in V$ if and only if there exists some n so that $[\sigma] \subseteq \bigcap_{g(n)\leq m<g(n+1)} U^{f(m)}[m]$. By the property of f and g, $V \subseteq U^y = U$. But for every n,

$$\mu(\bigcap_{g(n)\leq m<g(n+1)} U^{f(m)}[m]) > 1 - \sum_{g(n)\leq m<g(n+1)} 2^{-m} \geq 1 - 2^{-g(n)+1}.$$

So

$$\mu(V) \geq \lim_n \mu\left(\bigcap_{g(n)\leq m<g(n+1)} U^{f(m)}[m]\right) = 1.$$

Hence x is low for Δ^1_1-Kurtz tests. □

Corollary 5.2. *Lowness for Δ^1_1-randomness differs from lowness for Δ^1_1-Kurtz randomness.*

Proof. By Theorem 4.13, there is a real x that is Δ^1_1-dominated and Δ^1_1-semi-traceable but not Δ^1_1-traceable. By Theorem 5.1, x is low for Δ^1_1-Kurtz randomness. Chong, Nies and Yu [1] proved that lowness for Δ^1_1-randomness is the same as Δ^1_1-traceability. Thus x is not low for Δ^1_1-randomness. □

Corollary 5.3. *There is a non-zero hyperdegree below $\mathcal{O}$ which is not a base for a cone of Δ^1_1-Kurtz randoms.*

Proof. Clearly there is a real $x <_h \mathcal{O}$ which is Δ^1_1-dominated and Δ^1_1-semi-traceable. Then the hyperdegree of x is not a base for a cone of Δ^1_1-Kurtz randoms. □

Actually the converse of Theorem 5.1 is also true.

Lemma 5.4. *If x is low for Δ^1_1-Kurtz randomness, then x is Δ^1_1-dominated.*

Proof. Firstly we show that if x is low for Δ^1_1-Kurtz tests, then x is Δ^1_1-dominated.

Suppose $f \leq_h x$ is an increasing function. Let $S_f = \{z \mid \forall n(z(f(n)) = 0)\}$. Obviously S_f is a $\Delta^1_1(x)$ closed null set. So there is a Δ^1_1 closed null set $[T] \supseteq S_f$ where $T \subseteq 2^{<\omega}$ is a Δ^1_1 tree. Define

$$g(n) = \min\{m \mid \frac{|\{\sigma \in 2^m \mid \sigma \in T\}|}{2^m} < 2^{-n}\} + 1.$$

Since $\mu([T]) = 0$, g is a well defined Δ^1_1 function. We claim that g dominates f.

For every n, $S_{f(n)} = \{\sigma \in 2^{f(n)} \mid \forall i \leq n(\sigma(f(i)) = 0)\}$ has cardinality $2^{f(n)-n}$. But if $g(n) \leq f(n)$, then since $S \subseteq [T]$, we have

$$|S_{f(n)}| \leq 2^{f(n)-g(n)} \cdot |\{\sigma \in 2^{g(n)} \mid \sigma \in T\}| < 2^{f(n)-g(n)} \cdot 2^{g(n)-n} = 2^{f(n)-n}.$$

This is a contradiction. So x is Δ^1_1-dominated.

Now suppose x is not Δ^1_1-dominated witnessed by some $f \leq_h x$. Then S_f is not contained in any Δ^1_1 closed null set. Actually, it is not difficult to see that for any σ with $[\sigma] \cap S_f \neq \emptyset$, $[\sigma] \cap S_f$ is not contained in any Δ^1_1 closed null set (otherwise, as proved above, one can show that f is dominated by some Δ^1_1 function). Then, by an induction, we can construct a Δ^1_1-Kurtz random real $z \in S_f$ as follows:

Fix an enumeration $P_0, P_1, \ldots$ of the Δ^1_1 closed null sets.

At stage $n+1$, we have constructed some $z \upharpoonright l_n$ so that $[z] \upharpoonright l_n \cap S_f \neq \emptyset$. Then there is a $\tau \succ z \upharpoonright l_n$ so that $[\tau] \cap S_f \neq \emptyset$ but $[\tau] \cap S_f \cap P_n = \emptyset$. Fix such a τ, let $l_{n+1} = |\tau|$ and $z \upharpoonright l_{n+1} = \tau$.

Then $z \in S_f$ is Δ^1_1-Kurtz random.

So x is not low for Δ^1_1-Kurtz randomness. □

Lemma 5.5. *If x is low for Δ^1_1-Kurtz randomness, then x is Δ^1_1-semi-traceable.*

Proof. The proof is analogous to that of the main result in [7].

Firstly we show that if x is low for Δ^1_1-Kurtz tests, then x is Δ^1_1-semi-traceable.

Suppose that x is low for Δ^1_1-Kurtz tests and $f \leq_h x$. Partition ω into finite intervals $D_{m,k}$ for $0 < k < m$ so that $|D_{m,k}| = 2^{m-k-1}$. Moreover, if $m < m'$, then $\max D_{m,k} < \min D_{m',k'}$ for any $k < m$ and $k' < m'$. Let $n_m = \max\{i \mid i \in D_{m,k} \wedge k < m\}$ for every $m \in \omega$. Note that $\{n_m\}_{m \in \omega}$ is a recursive increasing sequence.

For every function h, let

$$P^h = \{x \in 2^\omega \mid \forall m(x(h \upharpoonright n_m) = 0)\}$$

be a closed null set. Obviously P^f is a $\Delta^1_1(x)$ closed null set. Then there is a Δ^1_1 closed null set $Q \supseteq P^f$. We define a Δ^1_1 function g as follows.

For each $k \in \omega$, let d_k be the least number d so that

$$|\{\sigma \in 2^d \mid \exists x \in Q(x \succ \sigma)\}| \leq 2^{d-k-1}.$$

Note that $\{d_k\}_{k \in \omega}$ is a Δ^1_1 sequence. Define

$$Q_k = \{\sigma \mid \sigma \in 2^{d_k} \wedge \exists x \in Q(x \succ \sigma)\}.$$

Then $\{Q_k\}_{k \in \omega}$ is a Δ^1_1 sequence of clopen sets and $|Q_k| \leq 2^{d_k-k-1}$ for each $k < d_k$. Then Greenberg and Miller [7] constructed a finite tree $S \subseteq \omega^{<\omega}$ and a finite sequence $\{S_m\}_{k<m\leq l}$ for some l with the following properties:

(1) $[S] = \{h \in \omega^\omega \mid P^h \subseteq [Q_k]\}$;
(2) $S_m \subseteq S \cap \omega^{n_m}$;
(3) $|S_m| \leq 2^{m-k-1}$;
(4) every leaf of S extends some string in $\bigcup_{k<m\leq l} S_m$.

Moreover, both the finite tree S and sequence $\{S_m\}_{k<m\leq l}$ can be obtained uniformly from Q_k.

Now for each m with $k < m \leq l$ and $\sigma \in S_m$, we pick a distinct $i \in D_{m,k}$ and define $g(i) = \sigma(i)$. For the other undefined $i \in D_{m,k}$, let $g(i) = 0$.

So g is a well-defined Δ^1_1 function.

For each k, $P^f \subseteq Q \subseteq [Q_k]$. So $f \in [S]$. Hence there must be some $i > n_k$ so that $f(i) = g(i)$.

Thus x is Δ^1_1-semi-traceable.

Now suppose x is not Δ^1_1-semi-traceable as witnessed by $f \leq_h x$. Then P^f is not contained in any Δ^1_1 closed null set. It is shown in [7] that for any σ, assuming that $[\sigma] \cap P^f \neq \emptyset$, $[\sigma] \cap P^f$ is not contained in any Δ^1_1 closed null set. Then by an easy induction, one can construct a Δ^1_1-Kurtz random real in P^f.

So x is not low for Δ^1_1-Kurtz randomness. □

So we have the following theorem.

Theorem 5.6. *For any real $x \in 2^\omega$, the following are equivalent:*

(1) *x is low for Δ^1_1-Kurtz tests;*
(2) *x is low for Δ^1_1-Kurtz randomness;*
(3) *x is Δ^1_1-dominated and Δ^1_1-semi-traceable.*

It is unknown whether there exists a nonhyperarithmetic real which is low for Π^1_1-Kurtz randomness. However, we can prove the following containment.

Proposition 5.7. *If x is low for Π^1_1-Kurtz randomness, then x is low for Δ^1_1-Kurtz randomness.*

Proof. Assume that x is low for Π^1_1-Kurtz randomness, y is Δ^1_1-Kurtz random and there is a $\Delta^1_1(x)$ closed null set A with $y \in A$. By Theorem 2.7, the set

$$B = \bigcup \{C \mid C \text{ is a } \Delta^1_1 \text{ closed null set}\}$$

is a Π^1_1 null set. So $A - B$ is a $\Sigma^1_1(x)$set. Since y is Δ^1_1-Kurtz random, $y \notin B$. Hence $y \in A - B$ and so $A - B$ is a $\Sigma^1_1(x)$ nonempty set. Thus there must be some real $z \in A - B$ with $\omega^z_1 = \omega^x_1 = \omega^{\mathrm{CK}}_1$. Since $z \notin B$, z is Δ^1_1-Kurtz random. So by Proposition 3.3, z is Π^1_1-Kurtz random. This contradicts the fact that x is low for Π^1_1-Kurtz randomness. □

References

[1] Chi Tat Chong, André Nies, and Liang Yu. Higher randomness notions and their lowness properties. *Israel journal of mathematics*, 2008.

[2] Paul J. Cohen. *Set theory and the continuum hypothesis.* W. A. Benjamin, Inc., New York-Amsterdam, 1966.

[3] Rodney G. Downey, Evan J. Griffiths, and Stephanie Reid. On Kurtz randomness. *Theor. Comput. Sci.*, 321(2-3):249–270, 2004.

[4] S. Feferman. Some applications of the notions of forcing and generic sets. *Fund. Math.*, 56:325–345, 1964/1965.

[5] S. Feferman and C. Spector. Incompleteness along paths in progressions of theories. *J. Symbolic Logic*, 27:383–390, 1962.
[6] R. O. Gandy. Proof of Mostowski's conjecture. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 8:571–575, 1960.
[7] Noam Greenberg and Joseph S. Miller. Lowness for Kurtz randomness. *J. Symbolic Logic*, 74(2):665–678, 2009.
[8] G Hjorth and A Nies. Randomness in effective descriptive set theory. *J. London. Math. Soc.*, 75(2):495–508, 2007.
[9] Alexander S. Kechris. Measure and category in effective descriptive set theory. *Ann. Math. Logic*, 5:337–384, 1972/73.
[10] Wolfgang Kjos-Hanssen, Bjorn; Merkle and Frank Stephan. Kolmogorov complexity and the recursion theorem. *Symposium on Theoretical Aspects of Computer Science 2006, LNCS 3884, Springer*, 149–161, 2006.
[11] Per Martin-Löf. On the notion of randomness. In *Intuitionism and Proof Theory (Proc. Conf., Buffalo, N.Y., 1968)*, pages 73–78. North-Holland, Amsterdam, 1970.
[12] Yiannis N. Moschovakis. *Descriptive set theory*, volume 100 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1980.
[13] A. Nies. *Computability and Randomness*. Oxford University Press, 2009. 443+xvi pages.
[14] Gerald E. Sacks. Measure-theoretic uniformity in recursion theory and set theory. *Trans. Amer. Math. Soc.*, 142:381–420, 1969.
[15] Gerald E. Sacks. Countable admissible ordinals and hyperdegrees. *Advances in Math.*, 20(2):213–262, 1976.
[16] Gerald E. Sacks. *Higher recursion theory*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1990.
[17] C. Spector. Hyperarithmetical quantifiers. *Fund. Math.*, 48:313–320, 1959/1960.
[18] Hisao Tanaka. A basis result for Π^1_1-sets of postive measure. *Comment. Math. Univ. St. Paul.*, 16:115–127, 1967/1968.
[19] Sebastiaan A. Terwijn and Domenico Zambella. Computational randomness and lowness. *J. Symbolic Logic*, 66(3):1199–1205, 2001.

Department of Mathematics, University of Hawaii at Manoa, 2565 McCarthy Mall, Honolulu, HI 96822, USA
E-mail address: bjoern@math.hawaii.edu

Department of Computer Science, University of Auckland, Private Bag 92019, Auckland, New Zealand
E-mail address: andrenies@gmail.com

Departments of Mathematics and Computer Science, National University of Singapore, 2 Science Drive 2, Singapore 117543, Republic of Singapore.
E-mail address: fstephan@comp.nus.edu.sg

Institute of Mathematical Science, Nanjing University, Nanjing, JiangSu Province, 210093, P.R. of China
E-mail address: yuliang.nju@gmail.com