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UNIVERSALITY FOR LEFT-COMPUTABLY ENUMERABLE METRIC SPACES

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ABSTRACT. There exists a universal object in the class of left-computably enumerable (left-c.e.) metric spaces with diameter bounded by a constant under effective isometric embeddings. There is no such universal object in the class of all left-c.e. metric spaces.

What is an appropriate effective version of the notion of a Polish metric space? We view Polish metric spaces as triples $\mathcal{M} = (M, d, \langle p_k \rangle_{k\in\mathbb{N}})$ where d is the distance function, and $\langle p_k \rangle_{k\in\mathbb{N}}$ is a designated dense sequence. We call the points p_i the special points. We say that $\mathcal{M}$ is *computable* if $d(p_i, p_k)$ is a computable real uniformly in i, k. There has been recent work on computable metric spaces, for instance by Nies and Melnikov [2, 3].

Natural generalisations of the notion of a computable metric space are the following.

Definition 1. We say that a Polish metric space $\mathcal{M} = (M, d, \langle p_k \rangle_{k\in\mathbb{N}})$ is a left-c.e. [right-c.e.] metric space if $d(p_i, p_k)$ is a left-c.e. real [right-c.e. real] uniformly in numbers $i, k \in \mathbb{N}$. We write $d_s(p, q)$ for the distance of special points p, q at stage s.

For left-c.e. metric spaces, intuitively speaking, the distance between points can increase over time. For right-c.e. metric spaces, the distance decreases.

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In this paper we study the left-c.e. case. We were motivated by a seemingly unrelated area: of research on Π^0_1 equivalence structures with domain the set $\mathbb{N}$ of natural numbers in Ianovski et al. [1]. An equivalence structure can be seen as a pseudometric space, where the distance of two points is 0 if they are equivalent, and 1 otherwise. Thus the notion of a left-c.e. metric space generalizes Π^0_1 equivalence structures. Similarly, right-c.e. metric spaces generalize Σ^0_1 equivalence structures.

It turns out that the left-c.e. case yields the more interesting computability theoretic properties. There exists a universal object in the class of left-c.e. metric spaces with diameter bounded by a left-c.e. constant under effective embeddings. In contrast, there is no such universal object in the class of all left-c.e. metric spaces. For right-c.e. metric spaces, it is easily shown that a universal object exists in both the bounded and the unbounded case.

Example 1. Let β be a left-c.e. real. Then $[0, \beta]$ is a left-c.e. metric space; so is the circle of radius β with the Euclidean metric inherited from $\mathbb{R}^2$. In fact $[0, \beta]$ has a computable presentation even if β is noncomputable. (Note that this presentation is not effectively compact.) In contrast, the circle of radius β does not have a computable presentation at all if β is noncomputable. To see this, given n, find points $q_0, \ldots, q_{n-1}$ among the p_i such that $\forall i, k \, |d(q_i, q_{i+1}) - d(q_k, q_{k+1})| \leq 2^{-n}$ (where addition is taken mod n). This means the points q_i are the corners of an n-gon up to small error. Hence $\sum_{i<n} d(q_i, q_{i+1})$ converges to $2\pi\beta$ effectively. So $2\pi\beta$ is computable. (In more detail, each side of the n-gon has length within 2^{-n} of $2\beta \cos(\pi/n)$, so the error for the n-th approximation is at most $n2^{-n} + 2\beta|\pi - n\cos(\pi/n)|$.)

Let E be a Π^0_1 equivalence relation on $\mathbb{N}$. Define $d(x, y) = 1 - 1_E(\langle x, y\rangle)$, namely 0 if Exy and 1 otherwise. Clearly this metric makes $\mathbb{N}$ a left-c.e. metric space (with $p_k = k$). This space has a computable presentation, though not uniformly. Note that the Π^0_1 equivalence relations can be uniformly identified with left-c.e. metric spaces where the set of possible distances is $\{0, 1\}$.

Definition 2. A *Cauchy name* in a metric space with distinguished dense sequence $\mathcal{M} = (M, d, \langle p_k\rangle_{k\in\mathbb{N}})$ is a function g on $\mathbb{N}$ such that $d(p_{g(i)}, p_{g(k)}) \leq 2^{-i}$ for each $i \leq k$.

The Cauchy names in a left-c.e. metric space form a Π^0_1 subclass of Baire space. This together with the examples seems to suggest it is more natural to look at left-c.e. spaces, rather than at right-c.e. ones.

Definition 3. An isometric embedding g from a left-c.e. [right-c.e.] metric spaces $\mathcal{M} = (M, d, \langle p_k \rangle_{k\in\mathbb{N}})$ to another $\mathcal{N} = (N, d, \langle q_k \rangle_{k\in\mathbb{N}})$ is called *computable* if uniformly in k one can compute a Cauchy name for $g(p_k)$.

Theorem 2. *Let $\gamma > 0$ be a left-c.e. real. Within the class of left-c.e. metric spaces of diameter at most γ, there is a left-c.e. metric space $\mathcal{U}$ which is universal with respect to computable isometric embeddings.*

Proof. For the duration of the proof, a *left-c.e. pre-metric function* will be a symmetric function $h\colon \mathbb{N}\times\mathbb{N} \to [0,\gamma]$ such that $h(v, v) = 0$ and $h(v, w)$ is a left-c.e. real, uniformly in v, w. Such a function is presented by its c.e. undergraph $G = \{\langle v, w, q\rangle \colon q \in \mathbb{Q}_0^+ \wedge v \le w \wedge q \le h(v, w)\}$. Let $h_t(x, y) = \max\{q\colon \langle v, w, q\rangle \in G_t\}$ so that $h(v, w) = \sup_t h_t(v, w)$.

Lemma 1. *Given a pre-metric function h, one can effectively determine a c.e. set W that is an initial segment of $\mathbb{N}$, and a pre-metric function g such that g satisfies the triangle inequality on W; if h satisfies the triangle inequality then $W = \mathbb{N}$ and $g = h$.*

Proof of lemma. Define a partial computable sequence of stages by $t_{-1} = -1$, $t_0 = 0$ and

$$t_{i+1} \simeq \mu t > t_i \quad \forall v, w \le t_i\big[h_t(v, w) - \widetilde{h}_t(v, w) \le 2^{-i}\big] \wedge \tag{1}$$

$$\forall v, w \le t_{i-1}[\widetilde{h}_t(v, w) \ge \widetilde{h}_{t_i}(v, w)], \tag{2}$$

where

$$\widetilde{h}_t(v, w) = \inf\{\sum_{r<t_i} h_t(q_r, q_{r+1})\colon\ q_0, \dots, q_{t_i} \le t_i \wedge q_0 = v \wedge q_{t_i} = w\}.$$

Note that $\widetilde{h}_t(v, w) \le h_t(v, w)$, and $\widetilde{h}_t$ satisfies the triangle inequality on $[0, t_i]$. Informally, the sequence of stages can only be continued at a stage $t > t_i$ if the "improved version" $\widetilde{h}_t$ that satisfies the triangle inequality is close enough to h_t; furthermore, for values $v, w \le t_{i-1}$ its value at t must be at least the one at the last relevant stage t_i.

If we define t_{i+1} we also put $(t_{i-1}, t_i]$ into W. Finally, for $v, w \in \mathbb{N}$, if $v \in W \wedge w \in W$ we let $g(v, w) = \sup\{\widetilde{h}_{t_{i+1}}(v, w)\colon\ v, w \le t_i\}$, and otherwise $g(v, w) = 0$.

Each $\widetilde{h}_{t_i+1}$ satisfies the triangle inequality on $[0, t_i]$, so from the monotonicity in t_i it is clear that the pre-metric function g satisfies the triangle inequality on W.

Claim 3. *Suppose that h satisfies the triangle inequality. Then there are infinitely many stages t_i, so $W = \mathbb{N}$ and $g = h$.*

We proceed by induction on i. Suppose that t_i is defined. Assume that t_{i+1} remains undefined. Given $\epsilon > 0$ let $s > t_i$ be a stage such that $\forall x, y \le t_i\, [h(x,y) - h_s(x,y) \le \epsilon/t_i$. Then for each $v, w \le t_i$ we have $h(v,w) \le \widetilde{h}_s(v,w) + \epsilon$. Thus $\lim_{s>t_i} \widetilde{h}_s(v,w) = h(v,w)$. This shows that both (1) and (2) are satisfied for sufficiently large s. Hence t_{i+1} will be defined, contradiction. This concludes the inductive step and the proof of the lemma. □

Clearly one can effectively list all the left-c.e. pre-metric functions as $h_0, h_1, \ldots$ Let $\langle W_n, g_n \rangle_{n\in\mathbb{N}}$ be the pairs of a c.e. initial segment of $\mathbb{N}$ and a pre-metric function and given by the Lemma. Note that (W_n, g_n) presents a metric space for each n. The required universal left-c.e. metric space $\mathcal{U}$ is an effective disjoint union of these spaces, where the distance between points in different components is γ. More formally, let f be a 1-1 computable function with range $\bigcup_n \{n\} \times W_n$. Let $p_k = k$ be the special points of $\mathcal{U}$, and define

$d_{\mathcal{U}}(i,k) = \gamma$ if $f(i)_0 \ne f(k)_0$; otherwise, let
$d_{\mathcal{U}}(i,k) = g_p(f(i)_1, f(k)_1)$, where $p = f(i)_0 = f(k)_0$.

Given a left-c.e. metric space $\mathcal{M} = (M, d, \langle q_r \rangle_{r\in\mathbb{N}})$, there is n such that $h_n(r,s) = d(q_r, q_s)$. This h_n satisfies the triangle inequality. So $q_r \to f^{-1}(\langle n, r \rangle)$ is a computable isometric embedding $\mathcal{M} \to \mathcal{U}$ as required. Note that the range of the embedding only contains special points of $\mathcal{U}$, which can be thought of as Cauchy names that are constant. □

From the proof of the foregoing theorem for $\gamma = 1$, it is clear that $\{\langle i,k \rangle \colon d_{\mathcal{U}}(i,k) = 0\}$ is a universal Π^0_1 equivalence relation under computable reducibility in the sense of [1]. In this way we have re-obtained their existence result for such an object.

The bound γ on the diameter in Theorem 2 is necessary. If we discard that bound, there is no universal object in the class of all left-c.e. metric spaces with respect to computable isometric embeddings by the following fact.

Proposition 1. *Let $\mathcal{M} = (M, d, \langle p_k \rangle)$ be a left-c.e. metric space. There exists a left-c.e. ultrametric space $\mathcal{A}$ such that $\mathcal{A}$ can not be isometrically embedded into $\mathcal{M}$.*

Proof. We make $\mathcal{A}$ a discrete metric space with domain consisting solely of special points $q_0, q_1, \ldots$ We use the pair q_{2n}, q_{2n+1} to destroy the n-th potential computable embedding into $\mathcal{M}$. Such an embedding f would be given by computably associating to q_k a Cauchy name for $f(q_k)$. In particular, the map sending q_k to the first component of this Cauchy name is computable. Note that this component is a special point p_r of $\mathcal{M}$ which has distance at most 1 from $f(q_k)$.

Let $\langle \phi_n \rangle_{n \in \mathbb{N}}$ be an effective listing of partial computable functions with domain an initial segment of $\mathbb{N}$. By the discussion above, it suffices to meet the requirements

$$R_n : \phi_n \, \text{total} \Rightarrow d_{\mathcal{A}}(q_{2n}, q_{2n+1}) > d(\phi_n(q_{2n}), \phi_n(q_{2n+1})) + 2.$$

Construction. At stage s, for $n = 0, \ldots, s$, if $\phi_n(q_{2n}), \phi_n(q_{2n+1})$ are defined, let $v_{n,s} = d_s(\phi_n(q_{2n}), \phi_n(q_{2n+1})) + 3$; otherwise $v_{n,s} = 0$. Set $d_{\mathcal{A},s}(q_{2n}, q_{2n+1}) = v_{n,s}$ towards satisfying R_n.

For each $n \neq k$, and $a = 2n, 2n+1$, $b = 2k, 2k+1$, set

$$d_{\mathcal{A},s}(q_a, q_b) = \max(v_{n,s}, v_{k,s}).$$

It is clear that the distances in $\mathcal{A}$ remain finite. Also the ultrametric inequality holds because trivially for reals u, v, w we have

$$\max(u, w) \leq \max(\max(u, v), \max(v, w)).$$

We apply this to distances of the form $d_{\mathcal{A}}(q_{2n}, q_{2n+1})$ to get the ultrametric inequality between points from three different pairs. Finally, each requirement R_n is met. □

The structure of our universal objects appears to be quite arbitrary. We ask whether an extra condition can be satisfied that would make them unique under computable isomorphism. This condition is effective homogeneity in the sense of Fraïssé theory.

Question 4. *Can a universal Π^0_1 equivalence relation, or left-c.e. metric space, be effectively homogeneous?*

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