# A computational approach to the Borwein-Ditor Theorem

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**Abstract.** Borwein and Ditor (Canadian Math. Bulletin 21 (4), 497-498, 1978) proved the following. Let $\mathcal{A} \subset \mathbb{R}$ is a measurable set of positive measure and let $\langle r_m \rangle_{m \in \omega}$ be a null sequence of real numbers. For almost all $z \in \mathcal{A}$, there is $m$ such that $z + r_m \in \mathcal{A}$.
In this note we mainly consider the case that $\mathcal{A}$ is $\Pi^0_1$ and the null sequence $\langle r_m \rangle_{m \in \omega}$ is computable. We show that in this case every Oberwolfach random real $z \in \mathcal{A}$ satisfies the conclusion of the theorem. We extend the result to finitely many null sequences. The conclusion is now that for almost every $z \in \mathcal{A}$, the same $m$ works for all of them.
We indicate how this result could separate Oberwolfach randomness from density randomness.

## 1 Introduction

Our paper is based on the following result, which extends a previous weaker result by Kestelman [11].

**Theorem 1 (D. Borwein and S. Z. Ditor [4], Thm 1(i)).** *Suppose $\mathcal{A} \subset \mathbb{R}$ is a measurable set of positive measure and $\langle r_m \rangle_{m \in \omega}$ is a sequence of real numbers converging to 0. For almost all $z \in \mathcal{A}$, there is an $m$ such that $z + r_m \in \mathcal{A}$.*

Since one can consider the tails of a given null sequence of reals, for almost every $z \in \mathcal{A}$ there are in fact infinitely many $m$ such that $z + r_m \in \mathcal{A}$. (This is the form in which they actually stated the result.) We thank A. Ostaszewski for pointing out the Borwein-Ditor theorem to Nies during his visit at the London School of Ecomonics in June 2015. Ostaszewski's 2007 book with Bingham provides some background related to topology [3].

Nies' colloquium at LSE was about the study of effective versions of "almost everywhere" theorems via algorithmic randomness. The goal for that direction of study is to pin down the level of algorithmic randomness needed for a point $x$ so that the conclusion of a particular effective version of the theorem holds. For instance, Pathak et al. [15] study effective versions of the Lebesgue differentiation theorem, Brattka et al [5] look at the a.e. differentiability of nondecreasing functions, Galicki and Turetsky [10] study the a.e. differentiability of Lipschitz functions on $\mathbb{R}^n$, and Miyabe et al [13] consider the Lebesgue density theorem, recalled in Thm. 4 below.

Unless stated otherwise, we will consider effectively closed (i.e. $\Pi^0_1$) sets $\mathcal{A} \subseteq \mathbb{R}$. Without also imposing an effectiveness condition on the null sequences, the points $z$ for which the Borwein-Ditor property holds for all $\Pi^0_1$ sets are precisely the 1-generics. Recall that a real is 1-generic if it is not on the boundary of any $\Sigma^0_1$ set.

**Proposition 2** *$z \in \mathbb{R}$ is 1-generic $\iff$ for every $\Pi^0_1$ set $\mathcal{A} \subset \mathbb{R}$ containing $z$ and for every null sequence of real numbers $\langle r_m \rangle_{m\in\omega}$, $z \in \mathcal{A}+r_m$ for some $m$.*

*Proof.* ($\Rightarrow$) Suppose $z \notin \mathcal{A}+r_m$ for all $m$. Then $z$ belongs to the boundary of the complement of $\mathcal{A}$, $B = \mathbb{R} \setminus \mathcal{A}$, a $\Sigma^0_1$ class.
($\Leftarrow$) Suppose $z \in \mathbb{R}$ is not 1-generic. Then it belongs to the boundary of some $\Sigma^0_1$ set $\mathcal{B} \subset \mathbb{R}$. Let $\langle z_m \rangle_{m\in\omega}$ be a sequence of points in $\mathcal{B}$ converging to $z$. Define $r_m = z - z_m$ for all $m$. Then $\mathcal{A} = \mathbb{R} \setminus \mathcal{B}$ is a $\Pi^0_1$ class, $z \in \mathcal{A}$, $\langle r_m \rangle_{m\in\omega}$ is a sequence converging to 0, and $z \notin \mathcal{A} + r_m$ for all $m$.

## 2 Comparison of Lebesgue density and the Borwein-Ditor property

The definitions below follow [2]. Let $\lambda$ denote Lebesgue measure on $\mathbb{R}$.

**Definition 3.** *We define the lower Lebesgue density of a set $\mathcal{C} \subseteq \mathbb{R}$ at a point $z$ to be the quantity*

$$\underline{\varrho}(\mathcal{C}|z) := \liminf_{\gamma,\delta\to 0^+} \frac{\lambda([z-\gamma, z+\delta] \cap \mathcal{C})}{\gamma+\delta}.$$

Note that $0 \le \underline{\varrho}(\mathcal{C}|z) \le 1$.

**Theorem 4 (Lebesgue [12]).** *Let $\mathcal{C} \subseteq \mathbb{R}$ be a measurable set. Then $\underline{\varrho}(\mathcal{C}|z) = 1$ for almost every $z \in \mathcal{C}$.*

The Borwein-Ditor theorem is analogous to the Lebesgue density theorem. Both results say that for almost every point in a measurable class there are, in a specific sense, many arbitrarily close other points in the class.

An open set $\mathcal{C}$ clearly has lower Lebesgue density 1 at each of its members. Thus, the simplest non-trivial case is when $\mathcal{C}$ is closed. We say that a real $z \in [0,1]$ is a density-one point if $\underline{\varrho}(\mathcal{C}|z) = 1$ for every effectively closed class $\mathcal{C}$ containing $z$. Similar to the implication $(\Rightarrow)$ of Proposition 2, every 1-generic is a density-one point. So being a density-one point is by itself not a randomness notion, and neither is the Borwein-Ditor property for effectively closed sets. In both cases, to remedy this one has to add as an additional condition that the real is Martin-Löf random.

**Definition 5.** *Let $z \in \mathbb{R}$ be ML-random. We say that $z$ is* density random *if $z$ is also a density-one point.*

**Definition 6.** *Let $z$ be ML-random. We say that $z$ is* Borwein-Ditor (BD) random *if for each $\Pi^0_1$ set $\mathcal{A} \subseteq \mathbb{R}$ with $z \in \mathcal{A}$, and each computable null sequence of reals $\langle r_m \rangle_{m\in\omega}$, there is an $m$ such that $z + r_m \in \mathcal{A}$.*

Neither of the two randomness notions is equivalent to ML-randomness, because the least element of a non-empty effectively closed set of ML-randoms is neither density random nor BD-random. Density randomness is in fact known to be stronger than difference randomness (i.e. ML-randomness together with Turing incompleteness) by Bienvenu et al. [2] together with Day and Miller [6]. Much less is known at present about the placement of BD-randomness within the established notions.

## 3 Oberwolfach randomness implies BD randomness

Ah! the ancient pond
as a frog takes the plunge
sound of water

(Matsuo Basho)

To simplify notation, we identify the unit interval with Cantor space ${}^{\omega}2$ in what follows, ignoring dyadic rationals. For a string $\sigma$, as usual by $[\sigma]$ we denote the corresponding basic dyadic interval; for example [101] denotes the interval $[5/8, 3/4]$.

Bienvenu et al. [1] introduced Oberwolfach (OW) randomness, and also gave the following equivalent definition. A *left-c.e. bounded test* is a descending sequence $\langle \mathcal{V}_n \rangle$ of uniformly $\Sigma^0_1$ classes in Cantor space such that for some nondecreasing computable sequence of rationals $\langle \beta_s \rangle$ with $\beta = \sup_s \beta_s < \infty$, we have $\lambda(\mathcal{V}_n) \leq \beta - \beta_n$ for all $n$. $Z$ is *OW-random* iff $Z$ passes each such test in the sense that $Z \notin \bigcap_n \mathcal{V}_n$.

OW-randomness implies density randomness [1]. The converse implication remains unknown. The question is intriguing. All the equivalent characterizations of OW-randomness are within the, by now, almost classical framework of computability and randomness [14,7]. For instance, if $Z$ is ML-random, $Z$ is OW-random iff it does not compute every $K$-trivial set [1]. On the other hand, the seemingly very close notion of density randomness is defined analytically, and mainly has analytical characterizations such as via differentiability of interval-c.e. functions in [13, Thm. 4.2].

Using Theorem 1 it is easy to check that weak 2-randomness implies BD-randomness. We show that the much weaker notion of OW-randomness already implies BD-randomness.

While density and BD randomness are analogous, it seems unlikely that density implies BD. This provides evidence that OW-randomness is strictly stronger than density randomness.

**Theorem 7** *Let $Z$ be Oberwolfach random. Then $Z$ is BD random.*

*Proof.* Suppose we are given a $\Pi^0_1$ class $\mathcal{P} \subseteq {}^\omega 2$ with $Z \in \mathcal{P}$, and a computable null sequence of reals $\langle r_m \rangle_{m\in\omega}$. We may assume that $r_m \leq 2^{-m}$. Let $\langle \sigma_m \rangle_{m\in\omega}$ be a computable prefix-free sequence of strings such that $\mathcal{S} = {}^\omega 2 \setminus \mathcal{P} = [\{\sigma_m \colon m \in \mathbb{N}\}]^\prec$, and let $\mathcal{S}_m = [\sigma_0, \ldots, \sigma_{m-1}]^\prec$, the class of all bit sequences extending one of the $\sigma_i$. Let $q(m) = 1 + \max(m, \max_{i<m} |\sigma_m|)$. Now define a left-c.e. bounded test by

$$G_m = \bigcap_{i \leq q(m)} (\mathcal{S} + r_m) \setminus \mathcal{S}_m.$$

Clearly this is a descending sequence of uniformly $\Sigma^0_1$ sets. Let

$$\beta = \lambda\mathcal{S} \text{ and } \beta_m = \lambda\mathcal{S}_m - m2^{-m}$$

so that $\beta = \sup_m \beta_m$.

*Claim.* $\lambda G_m \leq \beta - \beta_m$.

We actually show this bound for $\mathcal{S} + r_{q(m)} \setminus \mathcal{S}_m$ instead of $G_m$. Since $A \setminus C \subseteq (A \setminus B) \cup (B \setminus C)$ for sets $A, B, C$, and by the translation invariance of $\lambda$,

$$\lambda(\mathcal{S} + r_{q(m)} \setminus \mathcal{S}_m) \leq \lambda(\mathcal{S} \setminus \mathcal{S}_m) + \lambda(\mathcal{S}_m + r_{q(m)} \setminus \mathcal{S}_m).$$

Recall that $r_k \leq 2^k$. Hence by definition of $q(m)$, for each $i < m$ we have

$$\lambda([\sigma_m] + r_{q(m)} \setminus [\sigma_m]) \leq 2^{-q(m)}.$$

Therefore $\lambda(\mathcal{S}_m + r_{q(m)} \setminus \mathcal{S}_m) \leq m2^{-q(m)} \leq m2^{-m}$ as required for the claim.

If $Z + r_n \notin \mathcal{P}$ for each $n$ then $Z \in \bigcap_m G_m$, so $Z$ is not OW-random.

We note that this proof works in much greater generality for an abelian group $(S, +)$ that is also a computable probability space $(S, \mu)$ with a translation invariant measure, such that $\lim_{r\to 0} \mu((A + r)\triangle A) = 0$ effectively for every basic open set $A$. For instance, the general theorem also applies to Cantor space with the usual ultrametric and the group structure of the 2-adic integers $(\mathbb{Z}_2, +)$.

Finally, similar to [4] we extend the foregoing theorem to the case of finitely many null sequences, and show that for an OW-random $Z$, one position works for all of them. This is in the spirit of multiple recurrence in ergodic theory, initiated by Furstenberg and others in the 1970s [9].

**Theorem 8** *Let $Z$ be Oberwolfach random. For each $\Pi^0_1$ class $\mathcal{P} \subseteq {}^\omega 2$ with $Z \in \mathcal{P}$, and $k$ many computable null sequences of reals $\langle r_{m,v}\rangle_{m\in\omega}$, $0 \leq v < k$, there is $m$ such that $Z + r_{m,v} \in \mathcal{P}$ for each $v < k$.*

*Proof.* We may assume that $r_{m,v} \leq 2^{-m}$ for each $v$. Let $\langle \sigma_m\rangle_{m\in\omega}$, $\mathcal{S}_m$, $q(m)$ and $\beta_m$ be defined as above. Let

$$G_m = \bigcap_{i\leq q(m)} \bigcup_{v<k} (\mathcal{S} + r_{i,v}) \setminus \mathcal{S}_m.$$

*Claim.* $\lambda G_m \leq k(\beta - \beta_m)$.

We actually show this bound for $\bigcup_{v<k}(\mathcal{S} + r_{q(m),v}) \setminus \mathcal{S}_m$ instead of $G_m$. By the translation invariance of $\lambda$,

$$\lambda \bigcup_{v<k}(\mathcal{S} + r_{q(m),v}) \setminus \mathcal{S}_m \leq k\lambda(\mathcal{S} \setminus \mathcal{S}_m) + \lambda(\bigcup_{v<k} \mathcal{S}_m + r_{q(m),v} \setminus \mathcal{S}_m).$$

By the definition of $q(m)$, we have

$$\lambda(\bigcup_{v<k} \mathcal{S}_m + r_{q(m),v} \setminus \mathcal{S}_m) \leq km2^{-q(m)} \leq km2^{-m},$$

which establishes the claim.

If for each $n$ there is $v < k$ such that $Z + r_{n,v} \notin \mathcal{P}$, then $Z \in \bigcap_m G_m$, so $Z$ is not OW-random.

## 4 Open questions

Due to the novelty of the concept of BD-randomness, a number of natural questions remain; they are not necessarily hard. Our first two questions have been tried by a number of researchers; the third has not been considered in any detail so far.

*Question 9.* Does density randomness imply BD randomness?

*Question 10.* Does BD-randomness imply difference randomness?

*Question 11.* Does lowness for BD-randomness coincide with lowness for ML-randomness?

Miyabe et al. [13, Thm. 2.6] show that lowness for density randomness coincides with lowness for ML-randomness. The containment "$\subseteq$" is immediate from the result of Downey et al. [8] that $\text{Low}(\mathsf{W2R}, \mathsf{MLR}) = \text{Low}(\mathsf{MLR})$ (where $\mathsf{W2R}$ is the class of weakly 2-randoms). This proof works the same for BD-randomness. Thus, low for BD-random implies low for ML-random. However, the proof of the converse containment in the case of density randomness cannot be adapted in any obvious way, because we now have to consider a null sequence computable in the oracle.

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