UNIVERSAL RECURSIVELY ENUMERABLE SETS OF STRINGS

CRISTIAN S. CALUDE, ANDRÉ NIES,
LUDWIG STAIGER AND FRANK STEPHAN

Abstract. The main topic of the present work are universal machines for plain and prefix-free description complexity and their domains. It is characterised when an r.e. set W is the domain of a universal plain machine in terms of the description complexity of the spectrum function s_W mapping each non-negative integer n to the number of all strings of length n in W; furthermore, a characterisation of the same style is given for supersets of domains of universal plain machines. Similarly the prefix-free sets which are domains or supersets of domains of universal prefix-free machines are characterised. Furthermore, it is shown that the halting probability Ω_V of an r.e. prefix-free set V containing the domain of a universal prefix-free machine is Martin-Löf random, while V may not be the domain of any universal prefix-free machine itself. Based on these investigations, the question whether every domain of a universal plain machine is the superset of the domain of some universal prefix-free machine is discussed. A negative answer to this question had been presented at CiE 2010 by Mikhail Andreev, Ilya Razenshteyn and Alexander Shen, while this paper was under review.

A preliminary version of this paper was presented at DLT 2008 and has appeared in M. Ito, M. Toyama (eds.), *Developments in Language Theory*, Lectures Notes in Computer Science 5257, Springer-Verlag, pages 170–182, Heidelberg, 2008.

Cristian S. Calude is supported in part by the Hood Foundation. Address: Department of Computer Science, The University of Auckland, Private Bag 92019, Auckland, New Zealand, `cristian@cs.auckland.ac.nz`.

André Nies is partially supported by the Marsden Fund of New Zealand, grant no. 03-UOA-130. Address: Department of Computer Science, The University of Auckland, Private Bag 92019, Auckland, New Zealand, `andrenies@gmail.com`.

Ludwig Staiger is an external researcher of The University of Auckland. Address: Martin-Luther-Universität Halle-Wittenberg, Institut für Informatik, 06099 Halle, Germany, `staiger@informatik.uni-halle.de`.

Frank Stephan is partially supported by the National University of Singapore grants R146-000-114-112, R252-000-308-112 and R252-000-420-112; he is also an external researcher of The University of Auckland. Address: Department of Mathematics and Department of Computer Science, National University of Singapore, Singapore 119076, Republic of Singapore, `fstephan@comp.nus.edu.sg`.

1. Introduction

Universal machines U play a central role in algorithmic information theory. A universal plain machine is used to define the plain complexity C. For a string x one lets $C(x)$ be the length of a shortest string p such that $U(p) = x$. One defines $H(x)$ in a similar way when U is a universal prefix-free machine. For details see [2, 14].

Which r.e. sets can be the domains of a universal plain [prefix-free] machine? The main results of the paper give combinatorial characterisations. They are based on the number of strings of each length in the set. Further, the r.e. [prefix-free] supersets of such domains are characterised.

The motivation comes in part from the facts proven in [5] and [17, Exercise 2.2.12] : a prefix-free r.e. superset of the domain of a universal prefix-free machine is a prefix code coding all positive integers in an optimal way (up to a fixed constant). Such a code is Turing complete, has maximal density, but is not maximal.

In spite of obvious differences, there is an interesting similarity between the (supersets of) domains of plain and universal prefix-free machines. The present paper explores these facts combining recursion theoretic arguments with (combinatorial) algorithmic information theory. This is necessary because recursion theory alone does not yield a sufficiently fine distinction between recursively enumerable prefix codes, as, for example, the prefix-free set $V = \{0^n 1 : n \in A\}$ has the same complexity as the subset $A \subseteq \mathbb{N}$ and all these prefix codes are indistinguishable by their entropy. A special role will be played by the spectrum function s_W mapping a non-negative integer n to the number of all strings of length n in the set W. The results proven in this paper deal with the following topics.

(a) Combinatorial characterisations of domains and supersets of domains of plain and universal prefix-free machines based on the spectrum function are given. These investigations led to one major question in this paper: is every domain of a universal plain machine the superset of the domain of some universal prefix-free machine?

(b) The halting probability Ω_M of a prefix-free machine M whose domain contains the domain of a universal prefix-free machine is Martin-Löf random. However, $\mathrm{dom}(M)$ itself may fail to be the domain of any universal prefix-free machine.

The paper is organised as follows. In the next section we will present the notation and background. In Section 3 we discuss the case of universal plain machines. In Section 4 we study universal prefix-free machine, while in Section 5 some relations between (supersets of) domains of universal plain and prefix-free machines are investigated. The last section is devoted to conclusions and further studies.

2. Background and notation

Let X^* be the set of all strings over $X = \{0,1\}$: $X^* = \{\lambda, 0, 1, 00, 01, 10, 11, 000, \ldots\}$. A subset $W \subseteq X^*$ is prefix-free if there are no non-empty strings p, q such that $p, pq \in W$. The ordering $\leq_{\text{qlex}}$ is called the quasi-lexicographical, length-lexicographical or military ordering of X^*: $\lambda <_{\text{qlex}} 0 <_{\text{qlex}} 1 <_{\text{qlex}} 00 <_{\text{qlex}} 01 <_{\text{qlex}} 10 <_{\text{qlex}} 11 <_{\text{qlex}} 000 <_{\text{qlex}} 001 \ldots$ and so on. Furthermore, the sets of non-negative integers $\mathbb{N}$ and strings X^* are identified by letting $n \in \mathbb{N}$ represent the unique string x with $\#\{y \in X^* : y <_{qlex} x\} = n$. This is particularly useful in order to extend concepts like complexity to natural numbers without defining these concepts twice.

The function $a, b \mapsto \langle a, b \rangle$ is Cantor's pairing function of a and b: $\langle a, b \rangle = (a+b)(a+b+1)/2 + b$.

A machine M is a partial recursive function from X^* to X^*. We use machine and function synonymously. The description complexity $C_M(x)$ based on M is $C_M(x) = \inf\{|p| : M(p) = x\}$. The machine U is called *universal* if for every machine M there is a constant c with $\forall x\, [C_U(x) \leq C_M(x) + c]$.

A prefix-free machine M is a partial recursive function mapping X^* to X^* such that its domain $\text{dom}(M) \subseteq X^*$ is prefix-free. Analogously, a prefix-free machine U is referred to as universal if for every prefix-free machine M there is a constant c with $\forall x\, [C_U(x) \leq C_M(x) + c]$.

If U is prefix-free and universal, we write $H_U(x)$ for $\inf\{|p| : U(p) = x\}$. Further unexplained notation can be found in the books of Odifreddi [18], Calude [2] and Li and Vitányi [14].

A basic result of algorithmic information theory says that such universal machines exist [2, 14]. Here are some examples for prefix-free machines. Given a uniformly r.e. listing $M_0, M_1, M_2, \ldots$ of all the prefix-free machines, let $U_{ad}(1^n 0x) = M_n(x)$ for all n and $x \in \text{dom}(M_n)$; then U_{ad} is a universal machine. This is the standard example and machines of this type are called "universal by adjunction". Furthermore, from a given universal machine U one can build a machine U_{ev} such that the domain of U_{ev} only contains strings of even length: the idea is to define that $U_{ev}(x0) = U(x)$ for all x is in the domain of U with odd length; $U_{ev}(x) = U(x)$ for all x in the domain of U with even length; $U_{ev}(x)$ is undefined for all other x. This machine U_{ev} is not universal by adjunction. Figueira, Stephan and Wu [10] constructed a universal prefix-free machine U such that for each x and each length $n \geq H_U(x)$ there is exactly one string p of length n with $U(p) = x$.

In the sequel, we assume the underlying machine be fixed to some default and the complexities C (plain) and H (prefix-free) are written without any subscript [9].

A real $r = 0.r_1r_2 \dots r_n \dots$ ($r_i \in \{0, 1\}$) is Martin-Löf random if there is a constant c such that for all $n \geq 1$, $H(r_1r_2 \dots r_n) \geq n - c$. For a prefix-free set $V \subseteq X^*$, let $\Omega_V = \sum_{p \in V} 2^{-|p|}$. Every real Ω_V is left-r.e. that is, the limit of an increasing computable sequence of rationals. Chaitin [6] proved that if U is the domain of a universal prefix-free machine, then the left-r.e. real Ω_U is Martin-Löf random. Combining the results of Calude, Hertling, Khoussainov and Wang [3] and Kučera and Slaman [15] shows that the converse is also true: every left-r.e. Martin-Löf random real is the halting probability of some universal prefix-free machine.

A left-r.e. real number r is Solovay reducible to a left-r.e. real number $\tilde{r}$ if there is a computable approximation $a_0, a_1, a_2, \dots$ of r from below, a computable approximation $b_0, b_1, b_2, \dots$ of $\tilde{r}$ from below and a positive real constant $c > 0$ such that $(r - a_s) < c(\tilde{r} - b_s)$ for all s. A sufficient criterion is also that the above approximations and constant $c > 0$ exists and satisfy that $a_{s+1} - a_s < c(b_{s+1} - b_s)$ for all s. Furthermore, a set A is Solovay reducible to B if $\sum_{n \in A} 2^{-n}$ is Solovay reducible to $\sum_{n \in B} 2^{-n}$ as real numbers.

The spectrum function of a set $W \subseteq X^*$ is the function $s_W : X^* \to \mathbb{N}$ defined as $s_W(n) = \#(W \cap X^n)$ and $s_W(n, m) = \sum_{i=n}^{n+m} s_W(i)$.[1] Furthermore, for a machine M, $s_M(n)$ is just $s_{\mathrm{dom}(M)}(n)$.

The following facts are folklore. Let κ be one of the descriptive complexities C or H, let U be the corresponding plain or prefix-free universal machine and let $\sigma_{\mathrm{qlex}} : X^* \to X^*$ be the computable function such that $\sigma_{\mathrm{qlex}}(x)$ is the successor of x in the length-lexicographical order of X^*.

Fact 1.

(1) **Functions preserve complexity:** *Let φ be a partial recursive function from X^* to X^*. Then there is a constant c_φ depending only on φ such that $\kappa(\varphi(w)) \leq \kappa(w) + c_\varphi$ for all $w \in \mathrm{dom}(\varphi)$. In the case of H this holds also for functions in several arguments, for example $H(x + y) \leq H(x) + H(y) + c$ for some constant c.*

(2) **Continuity:** *There is a constant c_{qlex} such that for all $x \in X^*$ it holds $|\kappa(x) - \kappa(\sigma_{\mathrm{qlex}}(x))| \leq c_{\mathrm{qlex}}$.*

[1] The spectrum function s_W is also known as cardinality profile and the function $n \mapsto s_W(0, n)$ is also known as census function.

This implies $\min\{\kappa(w)-\kappa(v) : \kappa(w) > \kappa(v) \wedge w >_{\mathrm{qlex}} v\} \leq c_{\mathrm{qlex}}$.

(3) **Spectrum Function:** *If* $W \subseteq X^*$ *is r.e. then the spectrum function* s_W *is recursively approximable from below.*

(4) **Mapping Sets:** *If* $W, W' \subseteq X^*$ *are r.e. and if there is a* $c \in \mathbb{N}$ *such that* $\forall n(s_W(n,c) \leq s_{W'}(n,c))$ *then there is a partial-recursive one-to-one function* $\varphi : W \to W'$ *such that* $W = \mathrm{dom}(\varphi)$ *and* $||\varphi(x)| - |x|| \leq c$ *for all* $x \in W$.

(5) **Kraft-Chaitin:** *Let* $f : \mathbb{N} \to \mathbb{N}$ *be a recursive function such that* $\sum_{i=0}^{\infty} 2^{-f(n)} \leq 1$. *Then there is a partial-recursive one-to-one function* $g : \mathbb{N} \to X^*$ *such that* $|g(n)| = f(n)$ *and* $g(\mathbb{N}) \subseteq X^*$ *is prefix-free.*

(6) **Kraft-Chaitin (second variant):** *Let* $f : \mathbb{N} \to \mathbb{N}$ *be a function recursively approximable from below such that* $\sum_{i=0}^{\infty} f(n) \cdot 2^{-n} \leq 1$. *Then there is a partial-recursive one-to-one function* $g : \mathbb{N} \to X^*$ *such that* $\#\{g(i) : |g(i)| = n\} = f(n)$ *and* $g(\mathbb{N}) \subseteq X^*$ *is prefix-free.*

Note that, due to recoding by Fact 1.4, an r.e. set W is the domain of a plain universal machine iff $s_W(n) = s_U(n)$ for some plain universal machine U and all n; similarly, a prefix-free r.e. set W is the domain of a plain prefix-free machine iff $s_W(n) = s_U(n)$ for some prefix-free universal machine U and all n. This first observation, as pointed out by an anonymous referee, motivates further research about the connections between the domains of universal machines and the spectrum function s_W.

3. Universal plain r.e. sets

In this section the domains of universal plain machines and their supersets are characterised in terms of the spectrum function.

Theorem 2. *An r.e. set* W *is the superset of the domain of a plain universal machine* $\Leftrightarrow$ *there is a constant* c *such that* $s_W(n, c') \geq 2^n$ *for all* n *and* $c' \geq c$.

Proof. $(\Rightarrow)$: There is a constant c such that every string of length $n+1$ has a plain description complexity of at most $n+c$. At least half of these strings do not have plain description complexity below n. Thus it follows that for at least half of the 2^{n+1} strings x of length $n+1$ there is a $p \in W$ with $n \leq |p| \leq n+c$ and $U(p) = x$. Thus $s_W(n,c) \geq 2^n$.

$(\Leftarrow)$: For every n which is a multiple of $c+1$ and uniformly recursively in n, one can construct a one-one function from $A_n = X^n \cup X^{n+1} \cup \ldots \cup X^{n+c}$ into W such that every $p \in A_n$ is mapped into $W \cap A_{n+c+1}$; these functions just enumerate the first 2^{n+c+1} elements of $W \cap A_{n+c+1}$ and then map the elements of A_n in a one-one manner into the enumerated elements. This function has a partial recursive and one-one inverse f whose domain is a subset of W and whose range is the full set X^*; note that $|f(p)| \geq |p| - 2c - 2$ for all p where $f(p)$ is defined.

If U is a universal plain machine, then the mapping $p \mapsto U(f(p))$ is also a universal plain machine and its domain is a subset of W; this completes the proof. □

Theorem 3. *An r.e. set $W \subseteq X^*$ is the domain of a universal plain machine $\Leftrightarrow$ there is a constant c such that $C(s_W(n,c)) \geq n$ for each n.*

Proof. $(\Rightarrow)$: Let $W = \mathrm{dom}(U)$ for some universal plain machine U. One defines a three-place partial recursive function $\varphi : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to X^*$ with inputs m, n and c as follows. The function simulates U until U has halted on a set V of m strings q with $n \leq |q| \leq n + c$ and it then outputs the length-lexicographic first $q' \in R$ for which $U(q')$ is length-lexicographically maximal: $U(q') \geq_{\mathrm{qlex}} U(q)$ for all $q \in V$.

This function terminates whenever $m \leq s_U(n,c)$. If $m = s_U(n,c)$ let $p_{n,c}$ denote its output. In this case, in view of Theorem 2, $2^{n+c+1} > s_U(n,c) \geq 2^n$. So the value of n can be obtained from $s_U(n,c)$ and a constant $c_0 \leq c+1$. Thus, in this case, $C(p_{n,c}) \leq C(s_U(n,c)) + 4\log c + c'$ for some constant c'.

Next it is shown that $C(U(p_{n,c})) = |p_{n,c}|$ for $c \geq c_{\mathrm{qlex}}$. Assume $C(U(p_{n,c})) < |p_{n,c}|$, that is, there is a p such that $|p| < |p_{n,c}|$ and $U(p) = U(p_{n,c})$. Then, by the definition of φ, $p \notin V$ and hence $|p| < n$. Now following Fact 1 (2), there is a $q \in \mathrm{dom}(U)$ such that $|p| < |q| \leq |p| + c_{\mathrm{qlex}}$, $C(U(q)) = |q|$ and $U(p_{n,c}) = U(p) <_{\mathrm{qlex}} U(q)$ whence $|q| < n + c$. Repeating this argument gives that there is a $q_0 \in \mathrm{dom}(U)$ such that $U(p_{n,c}) <_{\mathrm{qlex}} U(q_0)$, $C(U(p_{n,c})) < C(U(q_0))$ and $n \leq |q_0| < n + c$ which contradicts the choice of $p_{n,c}$.

Note that, in the same way, Fact 1 (2) implies $|p_{n,c}| \leq n + c - c_{\mathrm{qlex}}$. Finally, it follows from Fact 1 (1) that the inequality $C(p_{n,c}) \geq C(U(p_{n,c})) - c_U \geq |p_{n,c}| - c_U$ holds. Putting the inequalities together, one obtains $C(s_U(n,c)) \geq n + c - c_U - c_{\mathrm{qlex}} - 4\log c - c'$ which satisfies $C(s_U(n,c)) \geq n$ for a sufficiently large constant c.

$(\Leftarrow)$: Let the condition be satisfied. Now a plain machine M is built as follows. Let b be a coding constant for M given by the recursion theorem. Let $p_0, p_1, p_2, \ldots$ be a recursive one-one enumeration of the domain of a universal plain machine U. Define a computable sequence $t_0, t_1, t_2, \ldots \in \mathbb{N}$ in stages.

At stage s let $m = |p_s|+b+1$. Let W_t be the subset of W enumerated at stage t and

$$M(p_s) = s_{W_{t_s}}(m, c).$$

Then $C(s_{W_{t_s}}(m, c)) < m$, so there must be a $t_{s+1} > t_s$ such that

$$s_{W_{t_{s+1}}}(m, c) > s_{W_{t_s}}(m, c).$$

Now a machine M is defined on the domain of W. Let $M(q) = U(p_s)$ for all $q \in W_{t_{s+1}} \setminus W_{t_s}$.

Indeed, $U(p_s)$ has an M-description which is at most $b+c$ bits longer than p_s, hence M is a universal plain machine with domain W. □

Recall that a string w is compressible (with respect to C) iff $C(w) < |w|$. A consequence of Theorem 3 is that the compressible strings form the domain of a universal plain machine. This is interesting because shortest descriptions cannot be compressed by more than a constant.

Corollary 4. *Let $W = \{p \in X^* : C(p) < |p|\}$. Then there is a universal plain machine with domain W.*

Proof. Let C_s be an approximation of the complexity C from above and let U be the underlying universal plain machine. Now define a partial-recursive function $\varphi : X^* \to X^*$ as follows: If the input has the form $0^i1^j0p,\ j \geq 1$, then do

(1) Let $n = |p| + i + 1$.
(2) Determine $m = U(p)$.
(3) If m is found, search for the first stage s such that there are at least m strings in the set $\{q : n \leq |q| \leq n + 2j \wedge C_s(q) < |q|\}$.
(4) If m, s are found, let $\varphi(0^i1^j0p) = r$ be the lexicographic first string of length $n + 2j$ with $C_s(r) \geq |r|$.

Let φ be undefined on all other inputs.

Note that $\varphi(0^i1^j0p)$ is defined iff the second and third step of this algorithm terminate. Then Fact 1 (1) yields a constant $c_\varphi \geq 1$ such that

$$\forall i, j > 0\ \left[C(\varphi(0^i1^j0p)) < i + j + |p| + c_\varphi\right].$$

Let $c = 2 \cdot c_\varphi$ and assume by way of contradiction that there is a number n with $C(s_W(n, c)) < n$. Then there would be a p with $|p| < n$ and $U(p) = s_W(n, c)$. Let $i = n - |p| - 1$ and let $j = c_\varphi$. By construction, $\varphi(0^i1^j0p)$ is a string of length $n + c$ not in W and

$$C(\varphi(0^i1^j0p)) \leq i + j + |p| + c_\varphi = n + c - 1 < n + c.$$

These two facts together contradict the definitions of c, c_φ and W. Hence W is the domain of a universal machine by Theorem 3. □

One can also show that one can remove the incompressible strings from a given domain of a universal machine.

Theorem 5. *Assume that U is a given universal machine. Then there is a further universal machine N such that* $\mathrm{dom}(N) = \{p \in \mathrm{dom}(U) : C(p) < |p|\}$.

Proof. First define a machine M such that $M(p) = y$ whenever $U(p) \in y10^*$ and $M(p) = \lambda$ whenever $U(p) \in 0^*$. Now there is a partial-recursive function f such that $f(p,c)$ is the first q found with $|p| + c \leq |q| \leq |p| + 2c$ and $M(q) = M(p)$. As there is a constant bounding the differences of the Kolmogorov complexities of y and $y1$ as well as the differences of the Kolmogorov complexities of $y10^k$ and $y10^{k+1}$ for all y and k, it follows that for all sufficiently large c and all $p \in \mathrm{dom}(U)$, $f(p,c)$ is defined. Now $C(f(p,c)) \leq |p| + |c|/2 + d$ for a constant d and all p, c where $f(p,c)$ is defined; hence one can choose c such that $f(p,c)$ is defined and $C(f(p,c)) < |p| + c$ for all $p \in \mathrm{dom}(U)$. So there is for every $p \in \mathrm{dom}(U)$ an index $f(p,c)$ which is at most $2c$ bits longer than p and which satisfies $M(f(p,c)) = U(p)$. It follows that the machine N with $\mathrm{dom}(N) = \{q \in \mathrm{dom}(U) : C(q) < |q|\}$ and $N(q) = M(q)$ on this domain is a universal machine as well. □

4. Universal r.e. prefix codes

Recall that a universal prefix-free machine U is a prefix-free machine such that for every further prefix-free machine M there is a constant c such that for every $p \in \mathrm{dom}(M)$ there is a $q \in \mathrm{dom}(U)$ with $U(q) = M(p)$ and $|q| \leq |p| + c$. Following [5], an r.e. prefix-free set $W \subseteq X^*$ containing the domain of a universal prefix-free machine is referred to as a *universal r.e. prefix code.*

It has been shown in [5] that though universal r.e. prefix codes $W \subseteq X^*$ are not maximal prefix-free sets they satisfy the same density condition as the whole set X^* namely

$$\lim_{n\to\infty} \frac{\log s_W(0,n)}{n} = 1\,.$$

However, this density condition does not specify universal r.e. prefix codes among r.e. prefix-free sets: a simple recursive prefix-free set $L \subseteq X^*$ satisfying the same condition was obtained in [5]. The next theorem gives a necessary and sufficient condition on the spectrum function which specifies the universal r.e. prefix codes among r.e. prefix-free sets.

Theorem 6. *Let W be an r.e. prefix-free set. Then W is an universal r.e. prefix code $\Leftrightarrow$ there exist constants $c, d \in \mathbb{N}$ such that*

$$\forall n \left[2^{n-H(n)-d} \leq s_W(n,c) \leq 2^{n-H(n)+d}\right].$$

Proof. $(\Rightarrow)$: Let W be a universal r.e. prefix code. It is well-known that for each r.e. set $W \subseteq X^*$ satisfying $\sum_{w \in W} 2^{-|w|} < \infty$ and for every $c \in \mathbb{N}$ there is a constant $d_c \in \mathbb{N}$ such that

$$\forall n \left[s_W(n,c) \leq 2^{n-H(n)+d_c}\right].$$

This shows the upper bound.

The lower bound follows by an argument from Section 9 of [11] (see also Section 4 of the first edition of [14]) where it is shown that for any domain of a universal prefix-free machine U there are constants $c, d \in \mathbb{N}$ such that $2^{n-H(n)-d} \leq \left|\{y : n-c \leq H_U(y) \leq n+c\}\right|$.

$(\Leftarrow)$: Conversely, let $W \subseteq X^*$ be an r.e. prefix-free set satisfying $2^{n-H(n)-d} \leq s_W(n,c)$ for some constants $c, d \in \mathbb{N}$ and let U be a universal prefix-free machine. As shown in the first direction, the constant c and some further constant d_c satisfy that $s_U(n,c) \leq 2^{n-H(n)+d_c}$. Now consider the universal machine U_0 which is defined via $U_0(0^{d'+d_c+1} \cdot p) = U(p)$ and undefined elsewhere. Then $s_{U_0}(n+d', c) \leq 2^{n-H(n)-d_c}$.

On the other hand $s_W(n+d', c) \geq 2^{n+d'-H(n+d')-d}$. Using the fact that $H(n+d') \leq H(n) + 2\log d' + c'$ for some constant $c' \in \mathbb{N}$, it follows that for sufficiently large d' the inequality

$$\begin{aligned} n + d' - H(n+d') - d &\geq n + d' - H(n) - 2\log d' - c' - d \\ &\geq n - H(n) - d_c \end{aligned}$$

holds. Thus $s_W(n+d', c) \geq s_{U_0}(n+d', c)$ and $s_{U_0}(n) = 0$ for $n \leq d'$.

According to Fact 1 (4) there is a one-one partial recursive function $\varphi : \mathrm{dom}(U_0) \to W$ such that $\left||\varphi(x)| - |x|\right| \leq c+1$. Consequently, $U_0 \circ \varphi^{-1}$ is a universal machine having domain $\mathrm{dom}(U_0 \circ \varphi^{-1}) \subseteq W$. □

Corollary 7. *For every prefix-free r.e. set W and every constant c there is a constant d such that*

$$\forall n \left[H(\langle n, s_W(n,c)\rangle) \leq n + d\right].$$

Proof. Let W and c be given. There is a program p_n for n of length $H(n)$. Furthermore, there is by Theorem 6 a constant e such that $\forall n \left[s_W(n,c) \leq 2^{n-H(n)+e}\right]$ and $s_W(n,c)$ can be written down in a string σ_n of $n + e - |p|$ binary bits. Hence there is a prefix-free machine M with $M(p_n\sigma_n) = \langle n, s_W(n,c)\rangle$; M first finds n from a suitable prefix p_n and then takes the $n + e - |p_n|$ binary bits following p_n to produce

$s_W(n,c)$. Translating V into the underlying universal machine can be done by replacing the constant e by a new constant d. $\square$

The following sharper lower bound on the possible spectrum function $s_W(n,c)$ can be obtained in the case that W is the domain of a universal prefix-free machine.

Theorem 8. *The following conditions are equivalent for a prefix-free r.e. set W:*

(a) *W is the domain of a universal prefix-free machine;*
(b) *There is a constant c such that $H(\langle n, s_W(n,c)\rangle) \geq n$ for each n;*
(c) *There is a constant c such that $H(s_W(n,c)) \geq n$ for each n.*

Proof. This is shown by adapting the proof of the corresponding implications in Theorem 3 to a universal prefix-free machine U and prefix-free complexity H.

(a) $\Rightarrow$ (b): This follows the corresponding direction of Theorem 3. The only change to the direction ($\Rightarrow$) in the proof of Theorem 3 is that one replaces C by H; also, replace the equation defining M by $M(p_s) = \langle m, s_{W_{t_s}}(m,c)\rangle$. Now define the machine V as before. Note that the domain of V is prefix-free by hypothesis on W.

(b) $\Rightarrow$ (a): The proof is similar to the one of Theorem 3, direction ($\Leftarrow$). The main change is that one cannot determine the value of n from $s_U(n,c)$. Instead, this value comes from the pair $\langle n, s_U(n,c)\rangle$ and one uses this pair in place of $s_U(n,c)$ throughout the proof. One obtains that $H(p_{n,c}) \leq H(\langle n, s_U(n,c)\rangle) + 2\log c + c'$. Once taken into account this difference and using H in place of C, the proof runs exactly as the one for Theorem 3.

(a) and (b) $\Rightarrow$ (c): Assume that (a) and (b) hold; that U be the prefix-free universal machine with domain W and that c be the constant as in (b). Now, one can define a partial-recursive function ψ with prefix-free domain such that

$$\psi(0^{n-|p|}10^a1 \cdot p) = \langle n+2a, s_W(n+2a,c)\rangle$$

whenever $U(p) = s_W(n, 2a+c)$ and $|p| \leq n$. This is possible as n is $|p|$ plus the number of zeroes at the beginning of $0^{n-|p|}10^a1 \cdot p$, a is the number of zeroes between the first and second 1 and $s_W(n+2a,c)$ can be obtained by computing $U(p) = s_W(n, 2a+c)$ and then enumerating the $s_W(n, 2a+c)$ strings of W of length n to $n+2a+c$ and counting how many of them have length between $n+2a$ and $n+2a+c$. Now there is a constant b such that $H(\langle n+2a, s_W(n+2a,c)\rangle) \leq n+a+b$ where b is independent of a. Letting $a = b+1$, it follows that $n+a+b < n+2a$ and

$$H(s_W(n, 2a+c)) < n \Rightarrow H(\langle n+2a, s_W(n+2a, c)\rangle) < n+2a.$$

As $H(\langle n+2a, s_W(n+2a,c)\rangle) < n+2a$ does not hold for any n, $H(s_W(n, 2a+c)) \geq n$ for all n. So (c) is satisfied with the constant $2a+c$.

(c) $\Rightarrow$ (b): This follows directly from the definition. □

Theorem 9. *There exist a prefix-free r.e. set W and a universal prefix-free machine U such that* $\mathrm{dom}(U) \subset W$ *and W is not the domain of a universal prefix-free machine.*

Proof. Now it is shown that there is a prefix-free r.e. set $W \subseteq X^*$ which satisfies Theorem 6 but not Theorem 8. For this, one starts from a universal prefix-free machine U' such that $\sum_{n\in\mathbb{N}} s_{U'}(n) \cdot 2^{-n} < 1/2$ and $2^{n-H(n)-d} \leq s_{U'}(n,c) \leq 2^{n-H(n)+d}$ for suitable constants c, d.

Define $s(n) = 0$ if $s_{U'}(n) = 0$ and $s(n) = 2^{\lceil \log s_{u'}(n) \rceil}$ otherwise. Then $\sum_{n\in\mathbb{N}} s(n) \cdot 2^{-n} < 1$ and s is a function recursively approximable from below. According to Fact 1 (6) there is a prefix-free r.e. set W with $s_W = s$.

Since $s_{U'}(n) \leq s_W(n) \leq 2 \cdot s_{U'}(n)$, one has $2^{n-H(n)-d-1} \leq s_W(n,c) \leq 2^{n-H(n)+d+1}$ and W satisfies Theorem 6.

On the other hand $H(s_W(n)) = O(\log n)$, thus W cannot satisfy Theorem 8. □

Although the complexity of a universal r.e. prefix code might not be large up to a given length n, the next result shows that the number

$$\Omega_W = \sum_{x\in W} 2^{-|x|}$$

is Martin-Löf random, a property shared with the domains of prefix-free universal machines. Note that there is no contradiction as for every left-r.e. real number $\rho > 0$ one can find a recursive prefix-free set W such that $\Omega_W = \rho$, see [3] and [19].

Theorem 10. *Suppose U is a universal prefix-free machine with domain contained in a prefix-free r.e. set W. Then Ω_W is Martin-Löf random.*

Proof. The basic idea of the proof is to show that Ω_U is Solovay reducible to Ω_W. This is done by approximating the halting probability of U such that $\Omega_{U,0} = 0$ and for every u one can compute a natural number k_u with $\Omega_{U,u+1} - \Omega_{U,u} = 2^{-k_u}$. Next one constructs a sequence

$t_0, t_1, \ldots$ of integers such that there is a rational constant $\delta > 0$ with the property:

$$\forall u\, [\delta \cdot 2^{-k_u} \leq \Omega_{W,t_{u+1}} - \Omega_{W,t_u}]\,.$$

This property is a reformulation of the fact that there is a Solovay-reduction from Ω_U to Ω_W. If Ω_U is Solovay-reducible to a left-r.e. set the latter is Martin-Löf random [21], so the theorem follows.

The constant δ and the sequence $t_0, t_1, t_2, \ldots$ will come out of the following inductive construction. Using the recursion theorem together with the Kraft-Chaitin Theorem, one can obtain a constant c and an r.e. prefix-free set V such that for every $x \in V$ there is a $p \in \mathrm{dom}(U)$ with $U(p) = x \wedge |p| \leq |x| + c$. In more detail, given c, one constructs V. This yields the Kraft-Chaitin set $\{\langle |x|, x\rangle \colon x \in V\}$ and hence a corresponding prefix-free machine M, for which one can effectively obtain a coding constant $\widetilde{c}$ with respect to U. By the recursion theorem one can suppose that $\widetilde{c} = c$.

Given c, one defines V in stages:

(1) An invariant of the construction is $\Omega_{V,u} = \Omega_{U,u}$ for all u.

(2) The initialisation is $t_0 = 0$ and $V_0 = \emptyset$ which is consistent with the given invariant.

(3) At stage u, assume that t_u and V_u are defined. Let k_u be the unique integer with

$$2^{-k_u} = \Omega_{U,u+1} - \Omega_{U,u}\,.$$

Find a natural number m_u which is so large that $2|W_{t_u}| < 2^{m_u}$. By the Kraft-Chaitin Theorem one can select 2^{m_u} strings of length $k_u + m_u$ which are not yet in V_u and put them as new elements into V_{u+1}. This adds 2^{-k_u} to Ω_V giving

$$\Omega_{V,u+1} = \Omega_{V,u} + 2^{m_u} \cdot 2^{-k_u - m_u} = \Omega_{U,u} + 2^{-k_u} = \Omega_{U,u+1}\,.$$

Furthermore, one can select t_{u+1} to be the first stage beyond t_u where for every string $x \in V_{u+1}$ there is an $y \in \mathrm{dom}(U_{t_{u+1}}) \cap W_{t_{u+1}}$ such that $|y| \leq |x| + c$ and $U(y) = x$; as at least half of these strings y had not been in W_{t_u} it follows that

$$\Omega_{W,t_{u+1}} - \Omega_{W,t_u} \geq 2^{-k_u - c - 1}\,.$$

(4) The last equation in (3) permits to choose $\delta = 2^{-c-1}$.

Hence Ω_U is Solovay reducible to Ω_W. □

Remark 11. The anonymous referees of this paper suggested an alternative proof for the previous result using semimeasures. Here a *discrete semimeasure* is a function μ is a mapping from natural numbers to non-negative real numbers such that $\sum_n \mu(n) \leq 1$.

A semimeasure μ is referred to as *recursively approximable from below* provided the set $\{(n, m, k) : n, m, k \in \mathbb{N} \wedge m/k < \mu(n)\}$ is recursively enumerable.

A discrete semimeasure μ is called *universal* iff for every further discrete semimeasure ν recursively approximable from below there is a multiplicative constant c with $\nu(n) \leq c\mu(n)$ for all n. The anonymous referees provided the following result to the authors:

> ($*$) If μ is a universal discrete semimeasure then $\mu(\mathbb{N})$ is Martin-Löf random.

To see ($*$), let U be a universal prefix-free machine and let $f : \mathbb{N} \to dom(U)$ be a recursive one-one enumeration of $dom(U)$. Then $\nu(n) = 2^{-|f(n)|}$ is a (computable) discrete semimeasure and $\Omega_U = \sum_n \nu(n)$ is a Martin-Löf random real.

Since μ is universal there is a $c \in \mathbb{N}$ such that $\nu(n) \leq c \cdot \mu(n)$ for all n. Then, since $\nu : n \to \mathbb{Q}$ is a recursive function, the real

$$\beta = \sum_{n \in \mathbb{N}} (c \cdot \mu(n) - \nu(n))$$

is a real recursively approximable from below. Then $c \cdot \sum_n \mu(n) = \Omega_U + \beta$ as the sum of two reals recursively approximable from below where one of them is Martin-Löf random is also Martin-Löf random [2, Corollary 7.55]. Hence $\sum_n \mu(n)$ is also Martin-Löf random. This completes the proof of ($*$).

Using ($*$), one can then obtain Theorem 10 directly from Theorem 6 as follows: Let c be so large that $s_W(n, c) \geq 2^{n-H(n)}$ for all n. Then define the semimeasure

$$\begin{aligned} \mu(n) &= s_W((c+1)n) \cdot 2^{-(c+1)n} + s_W((c+1)n+1) \cdot 2^{-(c+1)n-1} + \\ & \quad \ldots + s_W((c+1)n+c) \cdot 2^{-(c+1)n-c}. \end{aligned}$$

Now $\mu(n) \geq 2^{-H((c+1)n)-c}$ and $2^{-H((c+1)n)-c} \geq 2^{-H(n)-d}$ for some constant d. Hence it follows that μ is a universal semimeasure. As $\Omega_W = \sum_n 2^{-n} \cdot s_W(n) = \sum_n \mu(n)$, the number Ω_W is Martin-Löf random.

If W is a universal r.e. prefix code, then one can use the constants c, d from Theorem 6 to compute relative to W for every n the value $H(n)$ up to an additive constant error. It follows that one can find for every number n a number m with $H(m) > n$: one just takes that m below 4^n for which $m - \log(s_W(m, c))$ is maximal and the choice is right in all but finitely many places. Using Merkle's result on complex sets [12] or Arslanov's completeness criterion for weak truth-table reducibility in combination with the fact that W has r.e. dnr Turing degree [18], one obtains that W is wtt-complete. For this, recall that a set A is wtt-complete iff it is r.e. and the halting problem $\mathbb{K}$ is wtt-reducible to A, that is, there is a recursive function f and a Turing machine M which computes $\mathbb{K}$ relative to A such that $\mathbb{K}(n)$ is computed without making any query to A above $f(n)$.

Corollary 12. *If W is a universal r.e. prefix code then W is weak truth-table complete, that is, $\mathbb{K} \leq_{wtt} W$.*

Remark 13. Corollary 4 does not carry over to prefix-free machines as the set of compressible strings is not prefix-free and cannot be the domain of a universal prefix-free machine. But Theorem 5 transfers as follows: If U is a universal prefix-free machine then there is a further universal prefix-free machine M such that $\mathrm{dom}(M) = \{x \in \mathrm{dom}(U) : H(x) < |x|\}$. Note that for every r.e. prefix-free set there is a constant d such that $H(x) \leq |x| + d$ for all its members; this fact is crucial for transferring the proof of Theorem 5 to the prefix-free case.

5. Supersets of domains of universal plain versus prefix-free machines

The domain W of a universal plain machine cannot be prefix-free because its density is too high. This section addresses the question whether such a domain always *contains* the domain of a universal prefix free machine. While we left the question open, we collect some interesting facts surrounding it. The first theorem gives some minimum requirement on the function s_W.

Theorem 14. *Suppose the r.e. set W contains the domain of a universal prefix-free machine U. Then either there is a constant c such that $s_W(n, c) \geq 2^n$ for all n or the Turing degree of s_W is that of the halting problem.*

Proof. Assume by way of contradiction that $W <_T \mathbb{K}$ and that there is a W-recursive function f such that $s_W(f(c), 2c) < 2^{f(c)}$ for all c and let f_t be a recursive approximation of f such that the convergence module of the approximation is also W-recursive, that is, the mapping $n \mapsto \min\{t : \forall t' \geq t\,[f_{t'}(n) = f(n)]\}$ is W-recursive. This is possible as W is an r.e. set. Furthermore, let $a_0, a_1, a_2, \ldots$ be a recursive one-to-one enumeration of $\mathbb{K}$. Now one defines a prefix-free machine M such that $M(0^{a_t}1\sigma) = \sigma$ for all $\sigma \in \{0,1\}^{f_t(a_t)+2}$. There is a constant d such that for every t and every $\sigma \in \{0,1\}^{f_t(a_t)}$ there is $p \in \mathrm{dom}(U)$ with $U(p) = \sigma \wedge |p| \leq a_t + |\sigma| + d$.

As W has r.e. Turing degree and $W <_T \mathbb{K}$, the convergence module of f cannot dominate that of $\mathbb{K}$; hence there are infinitely many t such that $f_t(a_t) = f(a_t)$. Now choose t such that $a_t > d$ and $f_t(a_t) = f(a_t)$. It follows that $H_U(\sigma) \leq f(a_t) + 2a_t$ for $2^{f(a_t)+2}$ strings; at least $2^{f(a_t)+1}$ of these strings must be the image of a $p \in \mathrm{dom}(U)$ with $f(a_t) \leq |p| \leq f(a_t) + 2a_t$ in contradiction to the choice of f. Hence it is not possible that both assumptions on W are true, so either there is a constant c such that $s_W(n, c) \geq 2^n$ for all n or $W \equiv_T \mathbb{K}$. □

Note that for each constant $c > 0$ the set $\{0^c p : |p| \text{ is a multiple of } c\}$ is a superset of the domain of some universal prefix-free machine; hence the "either-condition" Theorem 14 cannot be dropped. The next result shows that the "or-condition" is not sufficient to guarantee that some subset is the domain of a universal prefix-free machine.

Theorem 15. *Let W be an r.e. set such that for every c there is an n with $s_W(n, c) < 2^n$. Then there is an r.e. set W' with $s_W = s_{W'}$ such that W' does not contain the domain of any universal prefix-free machine.*

Proof. The central idea is to construct by induction relative to the halting problem a sequence $p_0, p_1, p_2, \ldots$ of strings such that each p_{e+1} extends p_e and $p_e \in W_e$ whenever this can be satisfied without violating the extension-condition. Furthermore, the set W' is constructed such that for each length n one enumerates $s_W(n)$ many strings of length n into W' and chooses each string $w \in X^n$ such that w is different from the strings previously enumerated into W' and one satisfies that w extends the approximations $p_{0,n}, p_{1,n}, \ldots, p_{e,n}$ of $p_0, p_1, \ldots, p_e$ for the largest possible e which can be selected.

For any fixed e it holds for almost all n that $p_{e,n} = p_e$ and that $s_W(n) \leq 2^{n-|p_e|}$ implies that all members of $W' \cap X^n$ extend p_e. By assumption there is for each constant $c > |p_e|$ a sufficiently large n such that $s_W(n, 4c) < 2^n$ and all members of W' of length $n + c, n + c + 1, \ldots, n + 4c$ extend p_e. Assume now that W_e is the domain of a universal machine. Then, for one of these constants c the corresponding n has in addition the property that there is a member of W_e of length between $n + c$ and $n + 2c$. If this member of W_e is not in W' then W_e is not a subset of W'. If this member of W_e is in W' then it is an extension of p_e and by the way p_e is chosen it follows that also $p_e \in W_e$, a contradiction with the assumption that W_e is prefix-free. Hence none of the W_e is a subset of W' and the domain of a universal prefix-free machine. □

The previous result is contrasted by the following example.

Example 16. *Assume that W is an r.e. set (not necessarily prefix-free) such that there is a real constant $c > 0$ with $s_W(n) \cdot 2^{-n} > c$ for all n*

and assume that f is a recursive function with $\sum_n 2^{-n} f(n) < c$. Then there is a prefix-free recursive subset $W' \subseteq W$ with $s_{W'}(n) = f(n)$ for all n.

The set W' can be constructed by simply picking, for $n = 0, 1, 2, 3, \ldots$, exactly $f(n)$ strings of length n out of W which do not extend previously selected shorter strings.

The main question is to find conditions on s_W which guarantee that W has a subset which is the domain of a universal prefix-free machine. In the light of Theorem 15 a necessary condition is that $\exists c \, \forall n \, [s_W(n, c) \geq 2^n]$. Is this condition also sufficient? By Theorem 3 this condition characterises the supersets of plain universal machines; hence in the conference version of this paper [4], the question was stated as follows:

Question 17. *Does the domain of every universal plain machine contain the domain of a universal prefix-free machine?*

While this paper was under review and revision, Andreev, Razenshteyn and Shen [1] solved the question to the negative by constructing a recursive set containing the third of all strings of each length which is not the superset of any domain of a prefix-free universal machine.

6. Discussion

One major topic of the paper were characterisations of (supersets of) domains of universal plain and prefix-free machines expressed in terms of the spectrum function s_V. Although the results were stated in form of $s_V(n, c)$, they can also stated as follows using $s_V(0, n)$ in place of $s_V(n, c)$:

(1) W is the superset of the domain of a universal plain machine iff there exists a natural number c such that $\forall n \, [s_W(0, n) \geq 2^{n-c}]$;
(2) W is the domain of a universal plain machine iff there exists a natural number c such that $\forall n \, [C(s_W(0, n)) \geq n - c]$;
(3) W is the superset of the domain of a universal prefix-free machine iff there exists a natural number c such that $\forall n \, [s_W(0, n) \geq 2^{n-H(n)-c}]$;
(4) W is the domain of a universal prefix-free machine iff there exists a natural number c such that $\forall n \, [H(\langle s_W(0, n), n) \geq n - c]$.

Furthermore, the halting probability Ω_M of a prefix-free machine M containing the domain of a universal prefix-free machine is Martin-Löf random, but $\text{dom}(M)$ may not be the domain of any universal prefix-free machine. Various relations between (supersets of) domains of universal plain and prefix-free machines have been investigated.

The analogy between the cases of plain and prefix-free description complexity is not perfect. Pursuing this analogy one might look at the property that every r.e. prefix-free superset of the domain of a universal prefix-free machine is also the subset of such a domain. Therefore one can ask which r.e. sets are the subset of the domain of a first universal machine and the superset of the domain of a second universal machine. The answer is that these are all r.e. sets V where there is a constant c such that

$$\forall n \; \left[2^n \leq s_V(n,c) \leq 2^{n+c} - 2^n\right]$$

and therefore this class is not really interesting. One might question whether the set is "isomorphic" to the prefix-free r.e. superset of the domain of a universal prefix-free machine. Although a good characterisation for the domains of universal machines had been found, the adequate question for the supersets was not found.

There are various definitions of universality and in this paper we considered the definition according to which U is universal if the description complexity based on U cannot be improved by more than a constant. A prominent alternative notion says that U is *universal by adjunction* if for every further machine M there is a finite string q such that $U(qp) = M(p)$ for all $p \in \mathrm{dom}(M)$. Universality by adjunction is quite restrictive and using the spectrum function s_V one cannot characterise when a prefix-free set V is the domain of a machine which is universal by adjunction; however, this is done for normal universal machines in Theorem 8. Nevertheless, due to the more restrictive nature, prefix-free machines U which are universal by adjunction have the property

$$\exists c \, \forall n \, [H(s_U(n)) \geq n - H(n) - c].$$

In other words, these machines are difficult on every level. This is not true for normal universal machines and one can use this method to obtain a machine which is universal but not universal by adjunction: the desired machine U is obtained from a given universal machine M such that for all $p \in \mathrm{dom}(M)$, $U(p0) = U(p1) = M(p)$ if $|p|$ is odd and $U(p) = M(p)$ if $|p|$ is even; it is easy to see that U inherits prefix-freeness and universality from M. Fact 5 in [5] provides more information about this topic.

As the topic of the paper are mostly supersets of domains of universal machines, it is natural to ask what can be said about the r.e. subsets of such domains. Indeed, these subsets are easy to characterise, where in the following two strings are comparable if one of them extends the other.

Proposition 18. *A prefix-free r.e. set $V \subseteq X^*$ is a subset of the domain of a universal prefix-free machine iff there is a string p such that no q comparable to p is in V; an r.e. set $V \subseteq X^*$ is the subset of the domain of a universal plain machine iff there is a constant c such that $s_{X^* \setminus V}(n, c) \geq 2^n$ for all n.*

Note that a subset of the domain of a prefix-free machine is also the subset of the domain of a universal plain machine, but not vice versa. Indeed, every prefix-free subset of X^* is a subset of the domain of a universal plain machine.

Acknowledgement. The authors would like to thank Alexander Shen and Wang Wei for discussions on the topic of this paper. Furthermore, special thanks go to the anonymous referees who provided numerous comments and suggestions.

References

[1] Mikhail Andreev, Ilya Razenshteyn and Alexander Shen: Not every domain of a plain decompressor contains the domain of a prefix-free one. F. Ferreira, H. Guerra, E. Majordomo, J. Rasga (eds). *Programs, Proofs, Processes, 6th Conference on Computability in Europe, CiE 2010, Abstract and Handbook Booklet*, Ponta Delgada, Azores, Portugal, University of Azores, pp. 31–37, 2010. Technical Report CoRR abs/1001.4462, 2010.

[2] Cristian S. Calude. *Information and Randomness: An Algorithmic Perspective*, Second Edition, Revised and Extended, Springer-Verlag, Berlin, 2002.

[3] Cristian S. Calude, Peter H. Hertling, Bakhadyr Khoussainov and Yongge Wang. Recursively enumerable reals and Chaitin Ω numbers, *Theoretical Computer Science*, 255:125–149, 2001.

[4] Cristian S. Calude, André Nies, Ludwig Staiger and Frank Stephan. Universal recursively enumerable sets of strings. *Developments in Language Theory*, Twelfth International Conference, DLT 2008, Kyoto, Japan. Proceedings. Springer LNCS 5257:170–182, 2008. Conference version of this paper.

[5] Cristian S. Calude and Ludwig Staiger. On universal computably enumerable prefix codes. *Mathematical Structures in Computer Science*, 19:45–57, 2009.

[6] Gregory J. Chaitin. A theory of program size formally identical to information theory. *Journal of the Association for Computing Machinery*, 22:329–340, 1975.

[7] Gregory J. Chaitin. Information-theoretic characterizations of recursive infinite strings. *Theoretical Computer Science*, 2:45–48, 1976.

[8] Gregory J. Chaitin. Algorithmic information theory, *IBM Journal of Research and Development*, 21:350–359+496, 1977.

[9] Rod Downey, Denis Hirschfeldt and Geoff LaForte. Randomness and reducibility. *Journal of Computer and System Sciences*, 68:96–114, 2004.

[10] Santiago Figueira, Frank Stephan and Guohua Wu. Randomness and universal machines. *Journal of Complexity*, 22:738–751, 2006.

[11] Peter Gács *Lecture notes on descriptional complexity and randomness.* Boston University, 1993.

[12] Bjørn Kjos-Hanssen, Wolfgang Merkle and Frank Stephan. Kolmogorov complexity and the recursion theorem. STACS 2006: *Twenty-Third Annual Symposium on Theoretical Aspects of Computer Science*, Marseille, France, February 23-25, 2006. Proceedings. Springer LNCS, 3884:149–161, 2006.

[13] Andrei N. Kolmogorov. Three approaches to the definition of the concept "quantity of information". *Problemy Peredachi Informacii*, 1:3-11, 1965.

[14] Ming Li and Paul Vitányi. *An Introduction to Kolmogorov Complexity and Its Applications*. Second Edition, Springer, 1997.

[15] Antonín Kučera and Theodore Slaman. Randomness and recursive enumerability. *SIAM Journal on Computing*, 31:199–211, 2001.

[16] Per Martin-Löf. The definition of random sequences. *Information and Control*, 9:602-619, 1966.

[17] André Nies. *Computability and Randomness*. Oxford University Press, 2009.

[18] Piergiorgio Odifreddi. *Classical Recursion Theory*. North-Holland, Amsterdam, 1989.

[19] Claus Peter Schnorr. *Zufälligkeit und Wahrscheinlichkeit.* Lecture Notes in Mathematics 218, Springer Verlag, Berlin 1971.

[20] Claus Peter Schnorr. Process complexity and effective random tests. *Journal of Computer and System Sciences*, 7:376–388, 1973.

[21] Robert Solovay. *Draft of paper on Chaitin's work.* Unpublished notes, 215 pages, 1975.