THE JOURNAL OF SYMBOLIC LOGIC
Volume 76, Number 2, June 2011

BOREL STRUCTURES AND BOREL THEORIES

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Abstract. We show that there is a complete, consistent Borel theory which has no "Borel model" in the following strong sense: There is no structure satisfying the theory for which the elements of the structure are equivalence classes under some Borel equivalence relation and the interpretations of the relations and function symbols are uniformly Borel.

We also investigate Borel isomorphisms between Borel structures.

§1. Introduction.

The completeness theorem states that each consistent first-order theory T has a model $\mathscr{M}$ no larger than the size of the language of T. If this language is countable then $\mathscr{M}$ can be defined from T. If the language is in fact effectively given, then the elementary diagram of $\mathscr{M}$ is computable relative to T. On the other hand, if the language is uncountable, the proof of the completeness theorem relies on the axiom of choice. So in general we cannot expect to bound the complexity of $\mathscr{M}$ in terms of T.

We investigate theories and structures the size of the continuum. Here a natural way to impose an effectivity condition is to use a first-order language that can be seen as a standard Borel space, and to require that the theory is Borel. A Borel structure is one where the elements of the structure are equivalence classes under some Borel equivalence relation E, and the interpretations of the relations and function symbols are uniformly Borel. For example, the field of reals with constants naming each element, and a unary function symbol naming each continuous function, is a Borel structure. A further example of a Borel structure is the Boolean algebra $\mathscr{P}(\omega)$ modulo the ideal of finite sets. In this case it is unknown whether there is an injective Borel representation, namely one where E is equality.

Our main result is that an effective version of the completeness theorem fails at the Borel level: we build a complete consistent Borel theory without a Borel model.

Received June 17, 2009.

Hjorth was supported by an Australian professorial fellowship through the ARC. Nies was partially supported by the Marsden Fund of New Zealand, grants 03-UOA-130 and 08-UOA-187.

Shortly before the publication of this paper, the news of Greg's tragic death reached me. I was shocked, and couldn't believe it in the beginning. In fact I had looked forward to visiting Greg in Melbourne a few weeks later.

Most of this work was done when we met in Wellington, end of 2008. It is my second paper with Greg, the first being on a quite different subject, randomness and effective descriptive set theory. Each time, I more than enjoyed meeting him, and marvelled at his insights. I will miss Greg as a wonderful colleague, and as a friend.

0022-4812/11/7602-0009/$2.60

We also begin to investigate Borel isomorphism between Borel structures. For instance, there are continuum many Borel presentations of $(\mathbb{R}, +)$ that are not Borel isomorphic. We leave to future investigations a closer look at the "Borel dimension", the number of Borel representations that are not Borel equivalent.

For more background see the survey paper [8]. We begin with some basic definitions and facts.

Definition 1.1. A set A equipped with a σ-algebra $\mathscr{B}$ is said to be a *standard Borel space* if there is Polish topology on A for which $\mathscr{B}$ is the resulting class of Borel sets.

Theorem 1.2 (Kuratowski, see [5]). *Let X be a standard Borel space and let $B \subset X$ be a Borel set. Then B equipped with the canonical σ-algebra*

$$\{A \subset B \colon A \text{ is Borel in } X\}$$

is a standard Borel space.

For us, *Borel set* will always mean a Borel subset of some Polish or standard Borel space equipped with the above σ-algebra of Borel subsets. The next few observations are completely routine. Details can be found in [5].

Lemma 1.3. *Every countable set S equipped with the collection of* all *subsets, $\mathscr{P}(S)$, is a standard Borel space.*

Consequently, we will always think of a countable set equipped with the collection of all its subsets as a standard Borel space.

Lemma 1.4. *A finite product of standard Borel spaces is standard Borel.*

Here we equip the finite product of standard Borel spaces, $\prod_{i\leq N} X_i$, with the σ-algebra generated by all cylinder sets

$$\prod_{i\leq N} A_i,$$

where each A_i is a Borel subset of X_i.

Lemma 1.5. *The countable disjoint union of standard Borel spaces is again a standard Borel space.*

Here we equip the countable disjoint union of standard Borel spaces, $\dot{\bigcup}_{i\in\mathbb{N}} X_i$ with the σ-algebra consisting of all sets of the form $\dot{\bigcup}_{i\in\mathbb{N}} A_i$, where each A_i is a Borel subset of X_i.

Putting the above lemmas together, if B is a standard Borel space and S is a countable set, then we first obtain that $B \dot{\cup} S$ is a standard Borel space in the canonical Borel structure indicated above, then that each

$$(B \dot{\cup} S)^N = \prod_{i\leq N} B \dot{\cup} S$$

is standard Borel, and finally that

$$(B \dot{\cup} S)^{<\infty} = \dot{\bigcup_{N\in\mathbb{N}}} (B \dot{\cup} S)^N$$

is a standard Borel space.

DEFINITION 1.6. A *Borel signature* is a Borel set $\mathscr{L}$ such that the sets

$$\{R \in \mathscr{L} : R \text{ is a relation symbol of arity } n\}$$

and

$$\{f \in \mathscr{L} : f \text{ is a function symbol of arity } n\}$$

are all Borel.

Using Polish notation one can naturally identify formulas in the resulting first-order language with finite strings in

$$\mathscr{L} \cup \{\neg, \vee, \wedge, \forall, \exists, v_0, v_1, \dots\},$$

where $v_0, v_1, \dots$ are our variable symbols. In other words, the collection of well formed first-order formulas, $\mathscr{L}_{\omega,\omega}$, is a subset of

$$(\mathscr{L} \cup \{\neg, \vee, \wedge, \forall, \exists, v_0, v_1, \dots\})^{<\omega}.$$

It is then easily verified that $\mathscr{L}_{\omega,\omega}$ is a Borel subset of $(\mathscr{L} \cup \{\neg, \vee, \wedge, \forall, \exists, v_0, v_1, \dots\})^{<\omega}$.

DEFINITION 1.7. Let $\mathscr{L}$ be a Borel signature. Then a *Borel first-order theory in* $\mathscr{L}$ is a Borel subset T of $\mathscr{L}_{\omega,\omega}$, where we equip

$$\mathscr{L}_{\omega,\omega} \subset (\mathscr{L} \cup \{\neg, \vee, \wedge, \forall, \exists, v_0, v_1, \dots\})^{<\omega}$$

with the σ-algebra of Borel subsets in its canonical standard Borel structure.

DEFINITION 1.8. We say that an equivalence relation H on a standard Borel space X is *Borel* if it is Borel as a subset of $X \times X$, in the canonical Borel structure on the product space $X \times X$. For $x \in X$ we use $[x]_H$ to denote the equivalence class of x.

We proceed to the main definition of this paper.

DEFINITION 1.9. Let $\mathscr{L}$ be a Borel signature. Let $\mathscr{M}$ be an $\mathscr{L}$ structure. We say that $\mathscr{M}$ is a *Borel* $\mathscr{L}$ *presentation* if there is a standard Borel space X and a Borel equivalence relation $H \subset X \times X$ such that

$$\mathscr{M} = X/H = \{[x]_H : x \in X\},$$

and

$$\{(a_0, \dots, a_{n-1}, R) \in X^n \times \mathscr{L} : \\ R \text{ is an } n\text{-ary relation symbol of } \mathscr{L} \ \ \& \ \ \mathscr{M} \models R([a_0]_H, \dots, [a_{n-1}]_H)\}$$

is Borel as a subset of $X^n \times \mathscr{L}$; further,

$$\{(a_0, \dots, a_{n-1}, b, f) \in X^{n+1} \times \mathscr{L} : \\ f \text{ is an } n\text{-ary function symbol of } \mathscr{L} \ \ \& \ \ \mathscr{M} \models f([a_0]_H, \dots, [a_{n-1}]_H) = [b]_H\}$$

is Borel as a subset $X^{n+1} \times \mathscr{L}$. We say that a structure $\mathscr{N}$ is *Borel* if there is a Borel presentation $\mathscr{M}$ which is isomorphic to $\mathscr{N}$.

We will usually denote presentations as $(X, H; \dots)$ where $\mathscr{M} = X/H$ and the $(\dots)$ refers to the interpretations of the various non-logical symbols of $\mathscr{L}$.

In previous studies such as [9, 4], the language was assumed to be countable. If the language is uncountable, the present definition is more restrictive than merely requiring that each individual relation or function be Borel: the relations and functions need to be "uniformly Borel". This uniformity will be important below.

In many cases our Borel presentations arise as structures on actual standard Borel spaces X, rather than as quotient object of the form X/H. In this case we say that the model is an *injective Borel presentation*. These can also be thought of as Borel models in the above sense by taking H to be the identity relation.

In [4] it is shown that some Borel structure in a finite signature fails to have an injective presentation. The real point of the non-completeness theorem is not just that there is a complete, consistent Borel theory with no Borel injectively presented model, but the theory in question has no Borel model even in our more generous sense.

To illustrate the main definition 1.9 we give some examples of Borel structures. Clearly, each countable structure in a countable language is Borel.

(1) The fields $(\mathbb{R}, +, \times)$ and $(\mathbb{C}, +, \times)$ are Borel structures. So are these fields in the extended language with names for all elements and for all continuous functions from the field to itself. (In the second example, to satisfy the uniformity condition of the main definition, we use that the continuous functions form a standard Borel space $\mathscr{C}$, and that the evaluation map $K \times \mathscr{C} \to K$ is continous, where K is either $\mathbb{R}$ or $\mathbb{C}$)
(2) All Büchi automatic structures (see [4]) are Borel structures.
(3) Let $=^*$ denote the equivalence relation of eventual agreement on infinite binary sequences. By (2), the Boolean algebras $\mathscr{B} = (\mathscr{P}(\omega), \subset)$ and $\mathscr{B}/{=^*}$ are Borel structures. As mentioned already, it is unknown whether $\mathscr{B}/{=^*}$ has an injective presentation. However, the example in [4] of a Büchi automatic structure without an injective Borel presentation is closely related to $\mathscr{B}/{=^*}$.
(4) Second order arithmetic, namely, the structure $(\omega, \mathscr{P}(\omega), 0, 1, +, \times, \in)$, is Borel.

In fact, most structures one finds in books related to analysis such as [6] are Borel. To obtain a structure of size the continuum that is not Borel, one can use tools from mathematical logic. Let κ be the size of the continuum.

Proposition 1.10. [4] *The linear order $(\kappa, \leq)$ is not Borel.*

Proof. Assume for a contradiction that $(\kappa, \leq)$ is Borel. Then so is $(\omega_1, \leq)$, so let $\mathscr{B}$ be a Borel presentation of $(\omega_1, \leq)$. Then the class of linear orderings of $\mathbb{N}$ which embed in $\mathscr{B}$ is $\utilde{\Sigma}^1_1$. This contradicts the boundedness theorem for WF [5, Thm 31.2] which implies that for every $\utilde{\Sigma}^1_1$ set of well-orderings, some ordinal $\gamma < \omega_1$ bounds the order type of each member. ⊣

A stronger result of Harrington and Shelah [2] states that every Borel presentable preorder can be mapped in an order preserving way into 2^α, the functions $\alpha \to 2$ with the lexicographical order, for some countable ordinal α. Note that 2^α is separable. Hence such a preorder has no subset of order type ω_1.

The structure $\mathscr{B}/{=^*}$ in (3) above is an ω_1-saturated Borel model for the theory of dense Boolean algebras of size the continuum. (This determines the Boolean algebra $\mathscr{B}/{=^*}$ up to isomorphism if the continuum hypothesis holds.) An example to the contrary is the theory of dense linear order without end points. Any of its ω_1-saturated models of size the continuum has an ω_1-chain, and is therefore not Borel by [2].

There are a few basic facts about Borel structures to which we will make repeated appeal. For instance, recalling that a set is $\underset{\sim}{\Sigma}^1_1$ if it is the Borel image of a Borel set, one has that a set is Borel if and only if both it and its complement are $\underset{\sim}{\Sigma}^1_1$. Further, a function between two standard Borel spaces is Borel measurable if and only if its graph is Borel as a subset of the product space. See [5].

The following well-known theorem from descriptive set theory will be essential. For a proof see Example 1.6 in [3].

THEOREM 1.11. *There is no Borel function* $F : \mathscr{P}(\omega) \to \mathscr{P}(\omega)$ *such that*

$$X =^* Y \Leftrightarrow F(X) = F(Y)$$

for each $X, Y \subseteq \omega$.

§2. Borel isomorphism.

DEFINITION 2.1. Two Borel presentations $(X, H; \dots), (Y, L; \dots)$ are said to be *Borel isomorphic* if there is an isomorphism $\Phi\colon X/H \to Y/L$ such that the preimage on $X \times Y$

$$\widehat{\Phi} = \{\langle x, y\rangle : \Phi([x]_H) = [y]_L\}$$

is Borel.

Borel isomorphism is easily verified to be an equivalence relation on Borel presentations; for transitivity, one uses the Lusin separation theorem to show that the composition of to isomorphisms with Borel preimage also has a Borel preimage.

A Borel structure $\mathscr{M}$ is *Borel categorical* if any two Borel presentations of it are Borel isomorphic. More generally, one can define the *Borel dimension* of a Borel structure $\mathscr{M}$ to be the number of equivalence classes modulo Borel isomorphism on the set of Borel presentations of $\mathscr{M}$. This is analogous to the notion of computable dimension in the area of recursive model theory. It was suggested by Bakhadyr Khoussainov.

Note that $\mathscr{M}$ is Borel categorical iff it has Borel dimension 1. Presently we only know examples of Borel dimensions 1 or $2^{\aleph_0}$.

Examples of Borel categorical structures are:

(1) The linearly ordered set $(\mathbb{R}, \leq)$.
(2) The Boolean algebra $(\mathscr{P}(\mathbb{N}), \subset)$.
(3) The field $(\mathbb{R}, +, \times)$.

For (1), given $(X, H, \leq_X)$, $(Y, L, \leq_Y)$ two Borel presentations which yield linear orders isomorphic to $(\mathbb{R}, \leq)$, we choose countable dense subsets $P \subset X$, $Q \subset Y$ such that P contains at most one element of each H equivalence class, and similarly for Q and L. Both subsets induce models of dense linear orderings without endpoints. Hence we can find some order isomorphism $\theta\colon P \to Q$. This isomorphism induces an isomorphism $\Phi\colon X/H \to Y/L$. Then $\widehat{\Phi} = \bigcap_{q \in P} R_q$, where R_q is the Borel relation on $X \times Y$ defined by

$$xR_q y \text{ iff } [q \leq_X x \leftrightarrow \theta(q) \leq_Y y].$$

Hence $\widehat{\Phi}$ is Borel.

(2) is similar; see [4, Lemma 4.1].

For (3), we first show that the ordering relation is Borel for each Borel presentation (X, H, f, g) of $(\mathbb{R}, +, \times)$. For, $x \geq y$ iff $\exists z\,[f(g(z,z)), y)Hx]$, namely, $x/H - y/H$ is a square. Further, $x < y$ if $\neg xHy \;\;\&\;\; \exists z\,[f(g(z,z)), x)Hy]$, namely, $y/H - x/H$ is a positive square. Since both the ordering relation and its complement are $\underset{\sim}{\Sigma}^1_1$, this shows that the ordering relation is Borel by the Lusin separation theorem.

Secondly, we build an isomorphism of two Borel presentations of $(\mathbb{R}, +, \times)$ by the same argument as in (1), but taking as countable dense subsets sets representing the rationals. Then the induced order isomorphism with Borel preimage is in fact an isomorphism of the presentations of the field.

We show that the structure of reals under addition has the maximal Borel dimension $2^{\aleph_0}$.

Theorem 2.2. *There are continuum many injective, Borel presentations of* $(\mathbb{R}, +)$ *that are not Borel isomorphic.*

Proof. We exploit that any isomorphism between the group structure of two Polish groups that is Borel must be a homeomorphism (see for instance Section 1.2 of [1]).

For each $p > 1$ recall the Banach space

$$\ell^p = \{\vec{x} \in \mathbb{R}^{\mathbb{N}} \colon \sum_n |x_n|^p < \infty\},$$

where the norm is $|\vec{x}|_p = (\sum_n |x_n|^p)^{1/p}$. Let G_p be (canonical injective Borel presentation of) the abelian group underlying ℓ_p. Clearly as abstract groups these are all isomorphic, being vector spaces of dimension $2^{\aleph_0}$ over $\mathbb{Q}$. It suffices to show that G_p is not Borel isomorphic to G_q for $p \neq q$. Otherwise let $\phi \colon G_p \cong G_q$ be a Borel isomorphism of Polish groups. By the above remark, ϕ will be continuous, and then linear. But for $1 < p < q$ there is no continuous linear bijection between ℓ^p and ℓ^q. See [6, top of p. 54]. ⊣

§3. A Borel theory without a Borel completion.

In the following two sections we study the question when Borel theories have Borel models the size of the continuum. The case of countable languages is special due to the following theorem of Harvey Friedman, 1979, published in [9]. The idea is to use indiscernibles to obtain a large model. See [8] for a recent proof.

Theorem 3.1. *Let T be a consistent first-order theory in a countable language. Then T has an injective Borel model of size the continuum. The model can be chosen so that its elementary diagram is Borel.*

Theorem 3.1 cannot be extended to Borel theories in a first-oder language the size of the continuum. If we allow the theory to be incomplete, it is easy to find a counterexample.

Theorem 3.2. *There exists a consistent Borel theory which has no Borel model and no Borel completion.*

Proof. Let $\mathscr{U}$ be a free ultrafilter on $\mathbb{N}$. We will consider a Borel subset of the atomic diagram of the structure $(P(\mathbb{N}), \mathscr{U})$, such that any model of it codes a free ultrafilter on $\mathbb{N}$. This contradicts the easy fact that there are no free Borel ultrafilters on $\mathbb{N}$; see for instance [5, Exercise 8.50].

The signature of our theory contains a unary predicate U and a constant symbol c_A for each $A \subseteq \mathbb{N}$. It can be turned into a Borel signature in the sense of Definition 1.6 in a canonical way. The theory consist of the following sentences:

- $c_A \neq c_B$, for every $A \neq B \subseteq \mathbb{N}$;
- $U(c_{\mathbb{N}})$;
- $U(c_A) \to U(c_B)$, for every pair of sets such that $A \subset B \subset \mathbb{N}$;
- $U(c_A) \leftrightarrow \neg U(c_{\mathbb{N}\setminus A})$, for every $A \subset \mathbb{N}$;
- $U(c_A)\ \&\ U(c_B) \to U(c_{A\cap B})$, for every $A, B \subset \mathbb{N}$;
- $\neg U(c_A)$, for each finite set $A \subset \mathbb{N}$.

Clearly, this theory is Borel (even Π^0_1). The theory is consistent as it has the model $(P(\mathbb{N}), \mathscr{U})$ extended by constants naming each subset of $\mathbb{N}$. However, it does not have any Borel presentable model $\mathscr{X}$. Otherwise, $\{A \subset \mathbb{N} \colon c_A^{\mathscr{X}} \in U^{\mathscr{X}}\}$ is a Borel free ultrafilter on $\mathbb{N}$.

Likewise, if T is a completion of our theory which is Borel, then

$$\{A \subset \mathbb{N} \colon T \models c_A \in U\}$$

is a Borel free ultrafilter. ⊣

Note that the argument depends heavily on the manner in which the signature is made into a standard Borel spaces: we used that the map $A \to c_A$ is Borel.

We mention another example of a theorem that is proved using the axiom of choice and has no version for Borel objects: each partial order can be extended to a linear order on the same domain. The counterexample for Borel objects was pointed out by Antonio Montalbán.

PROPOSITION 3.3. *There is a Borel relation $R \subseteq X \times X$, where $X = \{0,1\}^\omega$, such that R is a partial order without a Borel linear extension.*

PROOF. Let P be any Borel preorder with an ω_1-chain, such as $\mathscr{P}(\mathbb{N})$ with almost inclusion $\subset^*$. We equip each equivalence class for this preorder with the lexicographical order $\leq_{lex}$. Thus we define

$$Rxy \Leftrightarrow (Pxy\ \&\ \neg Pyx) \vee (Pxy\ \&\ Pyx\ \&\ x \leq_{lex} y),$$

where $x, y \subseteq \mathbb{N}$. Then R is a Borel partial order that has no Borel linear extension by [2], as remarked after Prop. 1.10. ⊣

§4. A complete Borel theory without a Borel model.

Our main result is an "anti-completeness theorem" at the Borel level:

THEOREM 4.1. *There is a complete, consistent Borel theory with no Borel model.*

Part of the difficulty in proving this theorem stems from the fact that we also have to rule out non-injective Borel presentations. Note that the language of the theory will necessarily be uncountable by Theorem 3.1.

The proof works in four steps. Firstly, we construct a theory $\mathbb{S}$ with a Borel axiom system. Secondly, we show that all its countable omega models have the same theory. Omega models are certain models of the $\mathbb{S}$ restricted to an appropriate countable sub-signature. Thirdly, we let the desired theory $\mathbb{S}^*$ be the theory that coincides with the theory of these countable omega models for each relevant sub-signature. We show that it is a complete Borel theory. Fourthly, we observe that $\mathbb{S}^*$ has no Borel model: such a model would contain a function that contradicts Theorem 1.11.

In the following we identify 2^ω with the collection of all subsets of ω.

4.1. The Borel axiom system $\mathbb{S}$. We first define a Borel axiom system $\mathbb{S}$ with four sorts denoted $\mathscr{C}, \mathscr{F}, \mathscr{A}, \mathscr{T}$.

Formally speaking, the sorts are unary predicates, and we require that their disjoint union is the whole domain of a model. We will generally write

$$x \in \mathscr{C},$$

and such like, rather than the perhaps more formally correct $\mathscr{C}(x)$. We will be thinking of the elements in $\mathscr{F}$ as providing functions from $\mathscr{C}$ to elements of $\mathscr{T}$. Formally speaking one would represent this in first-order logic by introducing a new relation symbol $G(\cdot, \cdot, \cdot)$ and write

$$G(f, x, b)$$

when $f \in \mathscr{F}$, $x \in \mathscr{C}$, $b \in \mathscr{T}$ and we have in mind that the function indicated by $\mathscr{F}$ will map x to b. Moreover, in this formal presentation one would also prescribe that elements of $\mathscr{F}$ are encoding functions by adding the axioms

$$\forall f \in \mathscr{F}\, \forall x \in \mathscr{C}\, \exists! b \in \mathscr{T}\; G(f, x, b)$$

and

$$\forall f, x, b\, (G(f, x, b) \Rightarrow (f \in \mathscr{F} \ \& \ x \in \mathscr{C} \ \& \ b \in \mathscr{T})).$$

We will refrain from formally discussing a predicate G along these lines, and instead use the more intuitive and natural expression

$$f(x) = b,$$

but the reader should understand that in the background there is an intended formalized version involving a predicate such as G.

As well as the predicates corresponding to $\mathscr{C}, \mathscr{F}, \mathscr{A}, \mathscr{T}$ and some hidden predicate G used to unravel the way in which the elements of $\mathscr{F}$ are thought of as functions, we have in the signature binary relation symbols Q and S, a ternary relation symbol R, a function symbol L, and unary predicates B and T, as well as an armada of constant symbols indexed by natural numbers and elements of 2^ω and an array of unary predicates indexed by elements of 2^ω: e_n for $n \in \omega$, c_x for $x \subset \omega$, and unary predicates U_x for $x \subset \omega$. This can be naturally viewed as a Borel signature $\mathscr{L}$ in the sense of Definition 1.6 – for instance if one lets $\mathscr{L}$ be the disjoint copy of $\{0, 1\} \times 2^\omega$, ω, and the discrete Polish space

$$\{p_{\mathscr{C}}, p_{\mathscr{F}}, p_{\mathscr{A}}, p_{\mathscr{T}}, p_Q, p_R, p_S, p_L, p_B, p_T\},$$

then we could identify each c_x with the corresponding $(0, x) \in \{0\} \times 2^\omega$, each U_x with the corresponding $(1, x) \in \{1\} \times 2^\omega$, each e_n with the corresponding $n \in \omega$, and each of the language symbols in

$$\{\mathscr{C}, \mathscr{F}, \mathscr{A}, \mathscr{T}, Q, R, S, L, B, T\}$$

with their obvious counterparts in $\{p_{\mathscr{C}}, p_{\mathscr{F}}, p_{\mathscr{A}}, p_{\mathscr{T}}, p_Q, p_R, p_S, p_L, p_B, p_T\}$.

We now consider the sorts and other symbols in more detail. Two tables at the end of this subsection summarize them.

(1): $\mathscr{C}$. This includes interpretations of the constant symbols $\{c_x : x \in 2^\omega\}$. There is a relation E defined on $\mathscr{C}$. If $x =^* y$ the system $\mathbb{S}$ contains the axiom

$$c_x E c_y.$$

If $x \neq^* y$, the system $\mathbb{S}$ contains the axiom

$$\neg c_x E c_y.$$

Further, $\mathbb{S}$ expresses that E is an equivalence relation on $\mathscr{C}$ with all equivalence classes infinite.

Aside. The purpose of this in the later construction will be to provide a Borel copy of 2^ω with the equivalence relation $=^*$.

(2): $\mathscr{F}$. Every element of $\mathscr{F}$ is a function from $\mathscr{C}$ onto B (where B is discussed in the section on $\mathscr{T}$ below). We require that $\mathscr{F}$ is non-empty and

$$\forall f \in \mathscr{F}\, \forall x, y \in \mathscr{C}\, [xEy \Leftrightarrow f(x) = f(y)].$$

We require that identity in $\mathscr{F}$ is determined by its value as a function:

$$\forall f, g \in \mathscr{F}\, (f \neq g \Rightarrow \exists d (f(d) \neq g(d))).$$

We also require that the functions in $\mathscr{F}$ be closed under finite perturbation: For each $n \in \omega$ and each permutation $\alpha\colon n \to n$ we will have the axiom

$$\forall f \in \mathscr{F}\, \forall d_0, d_1, \dots d_{n-1} \in \mathscr{C}\, [\bigwedge_{i \neq j} \neg E d_i d_j \Rightarrow$$
$$\exists g \in \mathscr{F}\, [(\bigwedge_i f(d_i) = g(d_{\alpha(i)})) \wedge \forall d \in \mathscr{C}\, [\bigwedge_i \neg E d d_i \Rightarrow f(d) = g(d)]].$$

Aside. We will also later engage in a trick which will ensure that in any Borel model of our theory, any two elements of $\mathscr{F}$ can *only* disagree on finitely many elements – this will be important in terms of locating a *complete* Borel theory and appears as part of 4.6 and its application in defining $\mathbb{T}^*$ via the lemma 4.9.

The following notion will be important for the rest of the proof.

Definition 4.2. Let $\mathscr{L}'$ be a sub-signature of $\mathscr{L}$ containing the unary predicate $\mathscr{A}$ and the constant symbols e_n $(n \in \omega)$. We say that a model $\mathscr{M}$ for $\mathscr{L}'$ is an *omega model* if $\mathscr{A}^{\mathscr{M}} = \{e_n^{\mathscr{M}} : n \in \omega\}$.

We will be able to ensure that all Borel models of the theory are omega models. With this completed, we ultimately obtain that any interpretation of $\mathscr{F}$ in a Borel model will consist of a single equivalence class, under the equivalence relation of agreeing off of a finite set, of bijections from $\mathscr{C}/E$ to B.

To ensure that all Borel models of the theory are omega models, we need a further collection of axioms.

(3): $\mathscr{A}$. This has constant symbols $(e_n)_{n\in\omega}$ and unary predicates $(U_x)_{x\in 2^\omega}$ defined over elements of $\mathscr{A}$. We will have the further sentences in our axiom system:

(a) Whenever $x, y, z \in 2^\omega$ with $z = x \cap y$,
then $\forall e \in \mathscr{A}\, ((U_x(e) \wedge U_y(e)) \Leftrightarrow U_z(e))$.
(b) Whenever $z = \omega \setminus x$ we have $\forall e \in \mathscr{A}\, (U_x(e) \Leftrightarrow \neg U_z(e))$.
(c) Whenever $z \subset y$ we have $\forall e \in \mathscr{A}\, (U_z(e) \Rightarrow U_y(e))$.
(d) If $n \notin x$ then $\neg U_x(e_n)$.

(e) If $n \in x$ then $U_x(e_n)$.
(f) For each n, $\forall e \in \mathscr{A}\ (e \in U_{\{n\}} \to e = e_n)$.

Aside. We can now explain how we want to achieve that all Borel models of $\mathbb{S}^*$ are omega models. Intuitively, a given $x \in 2^\omega$ is represented by the corresponding U_x, with the e_n's representing the natural numbers. In advance we cannot rule out non-standard interpretations of our theory, where $\mathscr{A}$ has some elements other than $\{e_n : n \in \omega\}$, but at (a)-(c) we have demanded that *even if* $\mathscr{A} \setminus \{e_n : n \in \omega\} \neq \emptyset$, then we still obtain the U_x's behaving in the same boolean arrangement as the original x's from which they were derived. This in turn gets used in Lemma 4.4, since we can ensure that any non-standard element of $\mathscr{A}$ would generate a non-principal ultrafilter on ω. Since there are no Borel non-principal ultrafilters, we can guarantee that $\mathscr{A}$ will only have $\{e_n : n \in \omega\}$ as its elements. This critical step, with reference to the remarks at the end of (2) above, will enable us to correctly interpret the notion of finite difference inside any Borel models.

(4): $\mathscr{T}$. This in turn consists of two types, T and B. There is a function L defined on T, taking values in $\mathscr{A}$. We also have a relation Q defined between elements of B and elements of T. There is a binary relation S on T.

The intuition is as follows.

- We think of T as elements of a complete binary tree.
- S provides the successor relation.
- L assigns levels to the elements of T.
- B is a collection of infinite branches of the tree.
- Q tells us which of the branches are above which of the nodes.

Thus we have the following axioms.

$\exists! t \in T(L(t) = e_0)$.
$\forall t \in T\ \exists t_0, t_1\ (t_0 S t \wedge t_1 S t \wedge t_0 \neq t_1 \wedge \forall t'\ (t' S t \Rightarrow (t' = t_0 \vee t' = t_1)))$.
$\forall t, t' \in T\ ((t' S t \wedge L(t) = e_n) \Rightarrow L(t') = e_{n+1})$.
$\forall b \in B\ \forall e \in \mathscr{A}\ \exists! t \in T\ (L(t) = e \wedge Q(b, t))$.
$\forall t \in T\ \forall b \in B\ (Q(b, t) \Rightarrow \exists! t'(t' S t \wedge Q(b, t)))$.
$\forall t \in T\ \exists b \in B\ (Q(b, t))$.
$\forall b, b' \in B\ (b \neq b' \Rightarrow \exists t \in T\ (Q(b, t) \wedge \neg Q(b', t')))$.

Aside. Thus an interpretation of our theory will in particular give a function from $\mathscr{C}/E$ to elements of B which will be branches through T. We want to use the existence of a Borel model to then obtain a contradiction with 1.11. However, the simple existence of a function in the Borel model from $\mathscr{C}$ to B with the property that

$$x =^* y \Leftrightarrow f(c_x) = f(c_y)$$

is not yet sufficient, since the model might not be what we have called injective and the elements of B might be equivalence classes rather than actual points in a Polish space.

However, since we have a trick to ensure that all the Borel models will be omega-models, we can guarantee that in any Borel model, T will be a countable tree of height ω. Thus the elements of B can be identified with branches through a tree and hence with actual *points*, not equivalence classes, in a suitably chosen Polish space.

(5): We will have one last list of axioms for our theory which is designed to make the functions in $\mathscr{F}$ behave in a highly homogeneous manner. For this we introduce one further ternary relation R which should be thought of as measuring the disagreement between elements of $\mathscr{F}$. R will only hold when the first two coordinates are in $\mathscr{F}$ and the last in $\mathscr{A}$.

$\forall f, g \in \mathscr{F}\ \exists! e \in \mathscr{A}\ (R(f, g, e))$.

For each $n \in \omega$ we further introduce the sentence which says that, for all $f, g \in \mathscr{F}$, if $R(f, g, e_n)$ then there are exactly n equivalence classes of E on which f and g disagree.

Aside. This is related to our desire to make $\mathscr{F}$ induce a single equivalence class of bijections $\mathscr{C}/E \to B$ under the equivalence relation of agreement off a finite set. Since we have a trick to ensure that all the Borel models will be ω-models, for each $f, g \in \mathscr{F}$ there must exist an actual e_n with

$$R(f, g, e_n).$$

Hence f and g disagree on only n many elements.

The reader might however, through all this, be puzzled by the effort taken to engineer that in the Borel models of our theory the functions in $\mathscr{F}$ are to be so constrained to represent a single equivalence class under almost everywhere agreement. This in turn relates to our desire to make the theory Borel and complete. In some sense, the countable ω models of sufficiently rich fragments of our theory restricted to a countable language will all be highly homogeneous as shown in Lemma 4.6 . This in turn will allow a calculation at Lemma 4.9 showing that $\mathbb{S}^*$ is Borel.

The symbols with their intended use are summarized in the following tables.

Sort	Intended meaning	constants or unary predicates
$\mathscr{C}$	$\mathscr{C}/E$ "contains" $2^\omega/=^*$	c_x $(x \in 2^\omega)$
$\mathscr{A}$	set of natural numbers	e_n $(n \in \omega)$, U_x $(x \in 2^\omega)$
$\mathscr{T} = T \cup B$	T is the tree $2^{<\omega}$	
	B is the set of paths on T	
$\mathscr{F}$	functions $\mathscr{C} \to B$ inducing	
	a bijection $\mathscr{C}/E \to B$	

Relation symbol	Field	Intended meaning
E	$\mathscr{C} \times \mathscr{C}$	equivalence relation on $\mathscr{C}$
S	$T \times T$	successor relation on the tree T
G	$\mathscr{F} \times \mathscr{C} \times B$	function application
L	$T \times \mathscr{A}$	level of a node
Q	$B \times T$	branch contains node
R	$\mathscr{F} \times \mathscr{F} \times \mathscr{A}$	measures disagreement of two functions

Definition 4.3. Let $\mathbb{S}$ be the above Borel axiom system in the language with signature $\mathscr{L}$.

4.2. Some properties of the axiom system $\mathbb{S}$.

LEMMA 4.4. *Each Borel model $\mathscr{M}$ of $\mathbb{S}$ is an ω-model in the sense of Definition* 4.2.

PROOF. Otherwise choose $e \in (\mathscr{A})^{\mathscr{M}}$ which does not equal $(e_n)^{\mathscr{M}}$ for any $n \in \omega$. We obtain a Borel free ultrafilter $\mathscr{U}$ on 2^ω by letting

$$x \in \mathscr{U} \Leftrightarrow \mathscr{M} \models U_x(e),$$

which is impossible as already mentioned in the proof of Theorem 3.2. ⊣

In the following let T be a complete binary tree (that is to say, T is isomorphic to $2^{<\omega}$ under the prefix relation). Let $[T]$ denote the branches through T. We say that $S \subset [T]$ is dense if $\forall t \in T\, \exists b \in S\, [t \preceq b]$.

LEMMA 4.5 (Malitz [7]). *Suppose the countable sets $D, \widehat{D} \subset [T]$ are dense. Then there is an automorphism π of T with $\{\pi(b) : b \in D\} = \widehat{D}$.*

Malitz's result is used to prove the following key lemma.

LEMMA 4.6. *Let $\mathscr{L}_0$ be a finite sub-signature of $\mathscr{L}$ including $\mathscr{C}, \mathscr{F}, \mathscr{A}, \mathscr{T}, E, R, L, S, Q$. Let $\mathscr{L}_1 = \mathscr{L}_0 \cup \{e_n : n \in \omega\}$. Let $\mathscr{M}_0, \mathscr{M}_1$ be countable models of $\mathbb{S}|_{\mathscr{L}_1}$ which are ω-models in the sense of Definition* 4.2. *Then $\mathscr{M}_0 \cong \mathscr{M}_1$.*

PROOF. Throughout we will use that $\mathscr{M}_0$ and $\mathscr{M}_1$ satisfy the relevant axioms in the language of $\mathscr{L}_1$, and in particular all the axioms in (4). Let $[x_0]_{=^*}, [x_1]_{=^*}, \dots, [x_M]_{=^*}$ enumerate the equivalence classes of $\{x : c_x \in \mathscr{L}_0\}$. Choose $f \in \mathscr{F}^{\mathscr{M}_0}$. By the axioms in (4), choose $N \in \omega$ sufficiently large so that for all $i \neq j \leq M$ there exists $t_0 \neq t_1 \in T^{\mathscr{M}_0}$ with

$$\mathscr{M}_0 \models L(t_0) = e_N, \mathscr{M}_0 \models L(t_1) = e_N,$$
$$\mathscr{M}_0 \models Q(f(c_{x_i}), t_0), \mathscr{M}_0 \models Q(f(c_{x_j}), t_1).$$

Let $T_N^{\mathscr{M}_0}, T_N^{\mathscr{M}_1}$ respectively be the sets

$$\{t \in T^{\mathscr{M}_0} : \exists n \leq N \mathscr{M}_0 \models L(t) = e_n\},$$
$$\{t \in T^{\mathscr{M}_1} : \exists n \leq N \mathscr{M}_1 \models L(t) = e_n\}.$$

Let

$$\sigma : T_N^{\mathscr{M}_0} \cong T_N^{\mathscr{M}_1}$$

preserve the successor relation. We choose $b_0, \dots, b_M \in T^{\mathscr{M}_1}$ that will the isomorphic images of the $f(c_{x_i})$ under a suitable isomorphism of trees. This isomorphism will extend σ. So we choose the b_i in such a way that, for all t with

$$\mathscr{M}_0 \models L(t) = e_N,$$

and all $i \leq M$, we have

$$\mathscr{M}_0 \models Q(f(c_{x_i}), t) \text{ iff } \mathscr{M}_1 \models Q(b_i, \sigma(t)).$$

Now let $(t_i)_{i \in \omega}$ enumerate the distinct nodes in $T^{\mathscr{M}_0}$ such that

(a) t_i is not on a branch associated to any of the c_{x_j}'s (i.e., $\forall j \leq M\, \mathscr{M}_0 \models \neg Q(f(c_{x_j}), t_i)$).

(b) t_i is an immediate successor of a node on some branch associated to c_{x_j} (i.e., $\exists t, j\, \mathscr{M}_0 \models t_i S t, Q^{\mathscr{M}_0}(f(c_{x_j}), t)$).

Let $(\widehat{t}_i)_{i \in \omega}$ be defined similarly in $T^{\mathscr{M}_1}$ but for the branches $b_0, \dots, b_M$:

(a) $\forall j \leq N\, \mathscr{M}_1 \models \neg Q(b_j, \widehat{t_i})$.
(b) $\exists t, j \mathscr{M}_1 \models \widehat{t_i} S t, \mathscr{M}_1 \models Q^{\mathscr{M}_0}(b_j, t)$.

We can do this so that, for each n, i,

$$(\mathscr{M}_0 \models L(t_i) = e_n) \Rightarrow (\mathscr{M}_1 \models L(\widehat{t_i}, e_n)),$$

and if there exists t with

$$\mathscr{M}_0 \models Q(f(c_{x_j}), t) \wedge t_i S t$$

then there exists $\widehat{t}$ with

$$\mathscr{M}_1 \models Q(b_j, \widehat{t}) \wedge \widehat{t_i} S \widehat{t}.$$

At each i, let T^i be the set of t in $T^{\mathscr{M}_0}$ which have t_i as an ancestor. With the structure endowed by $\mathscr{M}_0$ this becomes a perfect binary tree – think of this intuitively as the tree of points whose last contact with one of the branches associated to one of the x_j's is equal to t_i. Similarly we let $\widehat{T}^i$ be the set of $\widehat{t} \in T^{\mathscr{M}_1}$ such that $\widehat{t_i}$ is an ancestor of $\widehat{t}$. Let

$$D_i = \{b : \mathscr{M}_0 \models Q(b, t_i)\},$$
$$\widehat{D}_i = \{b : \mathscr{M}_1 \models Q(b, \widehat{t_i})\}.$$

Since $\mathscr{M}_0$ and $\mathscr{M}_1$ are countable, the sets D_i and $\widehat{D}_i$ are countable. Further, by the sixth axiom in (4), D_i is dense in T_i and $\widehat{D}_i$ is dense in $\widehat{T}_i$. We can then apply Malitz's lemma to find

$$\pi : T^i \cong \widehat{T}^i$$

with induced

$$\pi_i[D_i] = \widehat{D}_i.$$

Let

$$\pi : \mathscr{T}^{\mathscr{M}_0} \cong \mathscr{T}^{\mathscr{M}_1}$$

be the result of patching these together and assigning

$$\pi(f(c_{x_j})) = b_j.$$

Fix $\widehat{f} \in \mathscr{F}^{\mathscr{M}_1}$ with $\widehat{f}(c_{x_j}) = b_j$ all $j \leq M$. (The axioms listed under (2) above make this possible.)

We now extend π to become an isomorphism of structures. For $c \in \mathscr{C}^{\mathscr{M}_0}$ we let

$$\pi(c) = (c_x)^{\mathscr{M}_1}$$

if $c = (c_x)^{\mathscr{M}_0}$ for some x, and otherwise simply choose $\pi(c)$ so that

$$\widehat{f}(\pi(c)) = \pi(f(c));$$

this is possible since π at the level of $B^{\mathscr{M}_0} \to B^{\mathscr{M}_1}$ is one to one and onto, and the axioms at (1) give $\widehat{f}$ as a bijection between $\mathscr{C}/E$ and B. Since each E class is infinite, we can do this so that π provides a bijection

$$([c]_E)^{\mathscr{M}_0} \cong ([\pi(c)]_E)^{\mathscr{M}_1}.$$

The fact that $\mathscr{M}_0, \mathscr{M}_1$ are ω-models in terms of $\mathscr{A}$, along with axioms at (2) and (5), ensure that $\mathscr{F}^{\mathscr{M}_0}$ and $\mathscr{F}^{\mathscr{M}_1}$ is given exactly by all the bijections

$$(\mathscr{C}/E)^{\mathscr{M}_0} \to B^{\mathscr{M}_0}$$

and

$$(\mathscr{C}/E)^{\mathscr{M}_1} \to B^{\mathscr{M}_1}$$

which agree with the functions induced by f and $\widehat{f}$, respectively, on all but finitely many values. From this it is easy to extend π to an isomorphism between $\mathscr{F}^{\mathscr{M}_0}$ and $\mathscr{F}^{\mathscr{M}_1}$: given $g \in \mathscr{F}^{\mathscr{M}_0}$, by the axioms in (5) there is $e \in \mathscr{A}^{\mathscr{M}_0}$ such that $R^{\mathscr{M}_0}(f, g, e)$. Since $\mathscr{M}_0$ is an omega-model, $e = e_n$ for some $n \in \omega$. Thus by (5) f, g disagree on exactly n equivalence classes of $E^{\mathscr{M}_0}$. Let these be the equivalence classes of $d_1, \dots, d_n \in \mathscr{C}^{\mathscr{M}_0}$. Since f, g induce bijections $(\mathscr{C}/E)^{\mathscr{M}_0} \to B^{\mathscr{M}_0}$, there is a permutation α of n such that $f(d_i) = g(d_{\alpha(i)})$ for each $i \in n$. Now let $d_i' = \pi(d_i)$. By the axioms in (2), there is $\widehat{g} \in \mathscr{F}^{\mathscr{M}_1}$ such that $\widehat{g}(d'_{\alpha(i)}) = \widehat{f}(d_i')$ for each $i \in n$, and $\widehat{g}(d') = \widehat{f}(d')$ if $\neg E^{\mathscr{M}_1} d' d_i'$ for each i. The extension of π maps g to $\widehat{g}$. Clearly this extension of π preserves function application.

It only remains to extend π to $\mathscr{A}^{\mathscr{M}_0}$ – but here we simply send

$$(e_n)^{\mathscr{M}_0} \mapsto (e_n)^{\mathscr{M}_1},$$

and for the predicate symbols of the form U_x in the common language, our axiomatization at (3) ensures we have at each x, n

$$\mathscr{M}_0 \models U_x(e_n) \Leftrightarrow \mathscr{M}_1 \models U_x(e_n). \qquad \dashv$$

4.3. The theory $\mathbb{T}$.

Notation 4.7. Let $A_{\mathbb{S}}$ be the set of $\mathscr{M}$ as described in Lemma 4.6: that is to say, $\mathscr{M}$ is a countable ω-model of $\mathbb{S}|_{\mathscr{L}_1}$ for some finite subset $\mathscr{L}_0$ of $\mathscr{L}$ including $\mathscr{C}, \mathscr{F}, \mathscr{A}, \mathscr{T}, E, R, L, S, Q$, and for $\mathscr{L}_1 = \mathscr{L}_0 \cup \{e_n : n \in \omega\}$. Note that $A_{\mathbb{S}}$ can be seen as a Borel set in a standard Borel space.

Definition 4.8. Let $\mathbb{T}$ be the set of $\phi \in \mathscr{L}$ such that there exists $\mathscr{M} \in A_{\mathbb{S}}$ with

$$\mathscr{M} \models \phi.$$

Lemma 4.9. $\mathbb{T}$ *is Borel.*

Proof. In the light of Lemma 4.6 we have $\phi \in \mathbb{T}$ if and only if

$$\exists \mathscr{M} \in A_{\mathbb{S}} [\mathscr{M} \models \phi],$$

if and only if

$$\forall \mathscr{M} \in A_{\mathbb{S}} [\mathscr{L}(\phi) \subset \mathscr{L}(\mathscr{M}) \Rightarrow \mathscr{M} \models \phi].$$

Thus $\mathbb{T}$ is Borel as both it and its complement are $\underset{\sim}{\Sigma}^1_1$. $\dashv$

Lemma 4.10. $\mathbb{T}$ *is complete.*

Proof. This is immediate by the structure of the definition of $\mathbb{T}$. $\dashv$

Lemma 4.11. $\mathbb{T}$ *is consistent.*

Proof. From 4.6 and the definition of $\mathbb{T}$, we have that for any finite $\mathscr{L}_0 \subset \mathscr{L}(\mathbb{T})$ there is a countable model of $\mathbb{T}|_{\mathscr{L}_0}$. $\dashv$

Theorem 4.12. $\mathbb{T}$ *has no Borel model.*

PROOF. First we show that there is no injective Borel model $\mathscr{M}$ of $\mathbb{T}$. Assume otherwise. By Lemma 4.4 and $\mathbb{T} \supset \mathbb{S}$,

$$(\mathscr{A})^{\mathscr{M}} = \{(e_n)^{\mathscr{M}} : n \in \omega\}.$$

Let $(t_n)_{n\in\omega}$ enumerate the elements of $(T)^{\mathscr{M}}$. For each $b \in B^{\mathscr{M}}$, we let

$$\rho(b) = \{n : \mathscr{M} \models Q(b, t_n)\}.$$

Fix $f \in (\mathscr{F})^{\mathscr{M}}$. Define $\theta : 2^\omega \to 2^\omega$ by

$$\theta(x) = \rho((f(c_x))^{\mathscr{M}}).$$

Then for all $x, y \in 2^\omega$,

$$x =^* y \Leftrightarrow \mathscr{M} \models c_x E c_y,$$

by the axioms at (1),

$$\Leftrightarrow \mathscr{M} \models f(c_x) = f(c_y),$$

by the axioms at (2),

$$\Leftrightarrow \rho((f(c_x))^{\mathscr{M}}) = \rho((f(c_y))^{\mathscr{M}}),$$

by the axioms at (4) describing B as a collection of branches through T. Thus we obtain a Borel function θ with

$$x =^* y \Leftrightarrow \theta(x) = \theta(y),$$

contradicting Theorem 1.11.

Now consider the general case where we allow models that are Borel presentations with a nontrivial equivalence relation H (i.e., equality in $\mathscr{M}$ does not actually correspond to true = in the outside world). There still is no Borel model of $\mathbb{T}$. The only adjustment to the above argument is to let $(t_n)_{n\in\omega}$ be a complete sequence of *representatives* with respect to H for elements of $T^{\mathscr{M}}$. ⊣

Acknowledgements. We thank Bakhadyr Khoussainov for his suggestion to include Borel dimension, and Antonio Montalbán for helpful discussions. We also thank the referee for suggestions on how to improve the paper.

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