LOW UPPER BOUNDS IN THE TURING DEGREES REVISITED

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Abstract. We give an alternative proof of a result of Kučera and Slaman [KS09] on low bounds of ideals in the Δ^0_2 Turing degrees. This is a characterization of the ideals in the Δ^0_2 degrees which have a low upper bound. It follows that there is a low upper bound for the ideal of the K-trivial degrees. Our proof is direct, in the sense that it does not use universal classes of PA degrees.

1. Introduction

When one studies ideals in the Turing degrees, an interesting question concerns their upper bounds, that is, the degrees that are above all members of the ideal. In particular, given an ideal which is definable in a certain way, what can we say about the complexity of its upper bounds?

For ideals in the computably enumerable degrees, this was the theme in [BN09]. It was shown that every proper[1] Σ^0_3 ideal in the c.e. Turing degrees has a low$_2$ upper bound. Moreover, every Σ^0_4 proper ideal in the c.e. Turing degrees has an incomplete upper bound.

The motivation for most of these results and questions came from the study of a particular ideal, the K-trivial degrees. Recall that a set A is K-trivial if its initial segments have prefix free complexity as low as those of a computable set. Formally, $K(A \upharpoonright n) \leq K(0^n) + c$ for all $n \in \mathbb{N}$ and some constant c, where K denotes the prefix free complexity.[2] In [Nie05] Nies showed that this notion is closed under Turing equivalence and in fact, the c.e. K-trivial degrees form a Σ^0_3 ideal in the c.e. Turing degrees. In [Nie06] Nies proved that there is no low c.e. upper bound for this ideal. This was the motivation behind Question 4.3 in [MN06], which asked whether there is a merely Δ^0_2 low upper bound for the ideal of c.e. K-trivial degrees. By a result in [Nie05], any such upper bound would also be an upper bound for the ideal of all K-trivial degrees.

This question was answered by Kučera and Slaman in [KS09], where they characterized the ideals in the Turing degrees which have a low upper bound. In Section 2 we give a simpler and more direct proof of their result. It is hoped that our argument and presentation will make this result and the methods behind its proof more accessible.

2. Low upper bounds for ideals in the Δ^0_2 degrees

In [MN06] it was asked whether some low set is Turing above all the K-trivial sets. Kučera and Slaman gave conditions for an ideal which suffice to be contained in a principal ideal given by a low degree. These conditions hold for the ideal induced by the K-trivial sets. In this way they answered the question in [MN06] in the affirmative.

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[1] A proper ideal in the c.e. Turing degrees is one that forms a proper subset of the c.e. degrees.

[2] For background on prefix free complexity and algorithmic randomness see [Nie09].

Let ψ be a partial function. We say that a function f dominates ψ if there is $n_0 \in \mathbb{N}$ such that $\psi(n) \downarrow \Rightarrow f(n) > \psi(n)$ for all $n > n_0$.

To understand the conditions of Kučera and Slaman, let us first consider a principal ideal given by a low degree $\mathbf{b}$. Let $B \in \mathbf{b}$. Then $\emptyset'$ can decide whether a computation with oracle B halts. Formally, $\{\langle x, e\rangle \mid \Phi_e^B(x) \downarrow\} \leq_T \emptyset'$.[3] Therefore (a) the principal ideal given by $\mathbf{b}$ is generated by a $\emptyset'$-computable sequence. Further, (b) $\emptyset'$ can compute a total function which dominates all partial computable functions relative to B, namely $f(x) = \max\{\Phi_e^B(x) \mid \Phi_e^B(x) \downarrow \wedge e \leq x\}$.

The following theorem shows the sufficiency of these conditions.

Theorem 2.1 (Kučera and Slaman [KS09]). *Suppose the ideal $\mathbb{I}$ in the Δ^0_2 Turing degrees is generated by a uniformly $\emptyset'$-computable sequence of sets, and some function $f \leq_T \emptyset'$ dominates all partial computable functions relative to any degree in $\mathbb{I}$. Then some low degree $\mathbf{b}$ is above all the degrees in $\mathbb{I}$.*

In the following we give a proof of Theorem 2.1. Our proof follows the same basic idea as in [KS09] but uses more direct coding, avoiding the use of universal Π^0_1 classes (i.e. classes whose paths code complete extensions of Peano arithmetic). We will construct a low set B whose degree $\mathbf{b}$ is an upper bound for the ideal $\mathbb{I}$. Let (A_e) be a uniformly $\emptyset'$-computable sequence of sets which generates $\mathbb{I}$.

We build B by an effective forcing argument using Π^0_1 classes. The construction is relative to $\emptyset'$. For each $e \in \mathbb{N}$ there is a stage where we decide whether $e \in B'$. If we fix an effective list of all Π^0_1 classes, this is equivalent to deciding whether B is in the eth Π^0_1 class. This ensures that B is low.

We will also have a simple coding to ensure that $A_e \leq_T B$ for all $e \in \mathbb{N}$. Namely, we use a pairing function $\langle ., . \rangle$ and ensure that for each $e \in \mathbb{N}$ there exists some $i \in \mathbb{N}$ such that $x \in A_e$ iff $\langle e, i, x\rangle \in B$, for all $x \in \mathbb{N}$. The lowness requirements impose various Π^0_1 conditions on B which could destroy the coding.

If we had lowness indices for all A_e in a uniform way, then we could decide B' stage-by-stage without destroying the coding of A_e into B. Roughly speaking, instead of merely asking whether we can restrict B by an additional Π^0_1 condition, we would ask whether we can do this while committing to coding A_e into B.

Since we do not have a uniform sequence of the lowness indices for all A_e, we will have to use f instead. Given that the coding will be very simple (a many-one reduction) the event that some Π^0_1 condition is incompatible with an existing coding procedure is Σ^0_1 relative to the sets A_e that are involved in the existing coding. The function f will be used to obtain bounds on searches aimed at discovering the incompatibility of the existing coding with new Π^0_1 conditions. The domination property of f will allow us to argue that for every given set A_e, the search by f will be almost always long enough so that the right decision is made. When a wrong decision is made, this mistake will be found at a later stage, because incompatibility is Σ^0_1 relative to the construction. Upon realizing the mistake, the associated coding will have to be abandoned and a new coding for the same set will start.

To sum up, the decisions we make about whether $e \in B'$ are irreversible but the coding procedures can sustain a finite injury.

2.1. **Coding.** For each $e \in \mathbb{N}$ we wish to have infinitely many sets of codes for the coding of A_e into B. This is because upon a wrong judgement in the construction, we may commit to a Π^0_1 condition which does not allow to decide the membership of the current family of codes in B. In that case we need to start using the next family of codes. We will ensure

[3] Here and in the following we let $\{\Phi_e\}_{e \in \mathbb{N}}$ be an effective list of all Turing functionals.

that the reserved families of codes for future use are always usable under the current Π^0_1 conditions.

Let $\langle \cdot, \cdot \rangle$ be the usual pairing function. The ith family of codes for A_e is

$$M_e(i) = \{\langle e, i, j\rangle \mid j \in \mathbb{N}\}$$

where $\langle e, i, j\rangle := \langle\langle e, i\rangle, j\rangle$. Also let

$$N_e(i) = \bigcup_{k>i} M_e(k).$$

This is the reservoir of future families of codes with respect to $M_e(i)$.

The coding with respect to $M_e(i)$ is straightforward: for all $j \in \mathbb{N}$ we wish to ensure that $j \in A_e$ iff $\langle e, i, j\rangle \in B$.

2.2. **Lowness.** Let P_e be the Π^0_1 class of reals X such that $\Phi^X_e(e) \uparrow$. Hence $e \in B'$ is equivalent to $B \notin P_e$. We often view Π^0_1 classes as computable trees without explicitly indicating it.

To indicate our decision about whether $B \in P_t$, we have a parameter q_t. If we decide $B \in P_t$ we set $q_t = t$. Otherwise we restrict B into a clopen subset of $2^\omega - P_t$ and let $q_t \neq t$ be an index of the clopen set as a Π^0_1 class. Then the Π^0_1 condition on B that has been built up in stages prior to stage s is $Q_s := \cap_{i<k} P_{q_i}$, where k is the least such that $q_k \uparrow$ (and the empty intersection is 2^ω). Since the sequence (q_i) completely determines the jump of B, there is no need to build B explicitly. There will be a unique B which satisfies the forcing conditions, and since the construction is computable in $\emptyset'$, B' is also computable in $\emptyset'$.

For each A_e we have a parameter $p_e[s]$ which indicates the coding of A_e that is active at stage s. In other words, at stage s the set A_e is coded through $M_e(p_e[s])$. For simplicity we let $M_e[s] := M_e(p_e[s])$. The reserved coding apparatus (for future use) is the set of positions $N_e := N_e(p_e)$. A *configuration* is a finite or infinite binary sequence. A real X *agrees* with a configuration g on a set (of positions) $Y = \{y_0 < y_1 < \dots\}$ from position n if $X(y_i) = g(i)$ for all $i < |g|$ such that $g(i) \geq n$.

In order to keep the reserved coding positions usable, we will make sure that at each stage s the current condition Q_s is compatible with *any* possible configuration on $N_e[s]$ *from a certain position* ℓ_s *on.* In other words, for any configuration there exists a path through Q_s which agrees with the configuration on N_e from position ℓ_s on. By the compactness of the Cantor space, we will only need to deal with finite configurations. This will help in keeping the construction computable in $\emptyset'$. The threshold ℓ_s will be constantly updated in a monotone way, thus ensuring that at any stage we can start a new coding from *some* position on.

An additional parameter r_s will be defined at stage s of the construction. This provides an argument for f which induces a suitably large bound for the searches for possible incompatibilities between a Π^0_1 condition and the current coding.

The parameters of the construction are displayed in Table 1.

2.3. **Coding one A_e.** In order to give the idea behind the full construction, we show how to code a single set A_e of the ideal into a low set B that we construct. This atomic version of the construction yields the following.

Proposition 2.2 (Kučera and Slaman [KS09]). *Let $\mathbb{I}$ be an ideal satisfying the hypothesis of Theorem 2.1 and let (A_e) be a uniformly $\emptyset'$-computable sequence generating it. Given $e \in \mathbb{N}$ we can uniformly construct a low set B and its lowness index such that $A_e \leq_T B$.*

TABLE 1. Parameters in the construction

A_e	eth set of the given ideal
B	Constructed set
P_e	eth Π^0_1 class
q_e	Indicates if $B \in P_e$
$M_e(i)$	ith family of codes for A_e
p_e	Indicates which family of codes is currently used for A_e
N_e	Families of codes for A_e that may be used in the future
ℓ_s	Position where coding using N_e can start at stage s
f	Given dominating function
r_s	Argument on which f gives a suitably large search bound
Q_s	Π^0_1 class in the forcing argument for the construction of B

To show Proposition 2.2 we construct the sequence (q_i) determining the jump of B, as discussed above.

Atomic construction: At stage $s+1$ consider the least i such that $q_i \uparrow$. Let $r_{s+1} := \langle \ell_s, m, p_e[s] \rangle$ where $m > s$ is an index for $Q_s \cap P_i$. See if

(2.1) there is a finite configuration σ such that all binary sequences which agree with A_e on $M_e[s] \upharpoonright f(r_s)$ and with σ on $N_e[s]$ from ℓ_s are terminal in Q_s.

In that case redefine p_e to be a *large* value (i.e. larger than any parameter up to the current stage of the construction), say that A_e was injured and end this stage. Otherwise, see if there is a finite configuration σ such that all paths which agree with A_e on $M_e[s] \upharpoonright f(r_s)$ and with σ on $N_e[s]$ from ℓ_s are terminal in $Q_s \cap P_i$.

(i) If not, let $q_i = i$, $\ell_{s+1} = \ell_s$ and go to the next stage.
(ii) If yes, let ℓ_{s+1} be the $|\sigma|$th element of $N_e[s]$. Also, let q_i be an index $> i$ of the clopen set of strings τ of length ℓ_{s+1} which agree with σ on $N_e[s]$ from ℓ_s and agree with A_e on $M_e[s] \upharpoonright f(r_s)$.

Atomic verification: First, we show by induction that at all stages s of the construction the following holds:

(2.2) for every configuration, there exists a path through Q_s which agrees with the configuration on $N_e[s]$ from ℓ_s.

For $s = 0$ it is trivial. Assume that the claim holds at stage s. If no new definition of some q_i is made at stage $s+1$, it clearly holds at this stage. If some q_i was defined through (i), by the criterion behind this clause the claim holds at stage $s+1$. If a definition of q_i is made through (ii), the claim holds since (2.1) was answered in the negative. This concludes the induction step and the proof of the claim.

Second, we show that the coding parameter p_e reaches a limit. Consider the following partial computable functional Θ. Given $\ell, j \in \mathbb{N}$ and an index i of a Π^0_1 class, $\Theta^{A_e}(\langle \ell, i, j \rangle)$ equals the least number $k > \ell$ such that for some configuration all paths of length k which agree with A_e on $M_e(j)$ and with the configuration on $N_e(j)$ from ℓ are terminal in P_i. By the properties of $f(r_s)$, there is a stage s_0 such that for all $s > s_0$ the number $f(r_s)$ is larger than $\Theta^{A_e}(r_s)$ when the latter is defined. It suffices to show that after s_0 the coding parameter p_e can change value at most once.

Suppose that $p_e[s_1+1] \neq p_e[s_1]$ for some $s_1 > s_0$. Since (2.2) holds for $s = s_1$, for every configuration there is a path in Q_{s_1} which agrees with A_e on $M_e[s_1+1]$ and with the configuration on $N_e[s_1+1]$ from ℓ_{s_1+1}. This holds because $p_e[s_1+1]$ will get a *large* value and $\langle e, p_e, 0\rangle$ will be larger than the threshold ℓ_{s_1}. By induction we show that this holds for the remaining stages. Suppose that it holds at $s \geq s_1$. If some q_i is defined according to (i) at stage $s+1$, let m be the index of $Q_s \cap P_i$. If the claim did not hold at $s+1$, $\Theta^{A_e}(\langle \ell_s, m, p_e\rangle)\downarrow$ and since $f(r_s)$ is larger than this value we would proceed through (ii), a contradiction. If the construction proceeds through (ii), we only restrict Q_s to the paths of length ℓ_{s+1} that agree with A_e on $M_e[s]$ and with a (finite) configuration on $N_e[s]$ after ℓ_s. Since the claim holds at s, it will also hold after this restriction. This finishes the induction and shows that for $s > s_1$ the class Q_s contains paths of any configuration on $N_e[s]$ from ℓ_s which agree with A_e on $M_e[s]$. In particular, the construction will never change p_e after stage s_1.

Third, notice that since $p_e[s]$ reaches a limit p_e, the parameter q_i will be defined for all $i \in \mathbb{N}$. Now it is clear that there is a unique path $B \in \cap_s Q_s$ whose jump is determined by (q_i). Since the construction is computable in $\emptyset'$, it uniformly provides the reduction $B' \leq_T \emptyset'$. Moreover, since B agrees with A_e on $M_e(p_e)$, we have '$n \in A_e$ iff $\langle e, p_e, n\rangle \in B$' for all $n \in \mathbb{N}$. Therefore, $A_e \leq_m B$.

2.4. **Coding all** A_e**,** $e \in \mathbb{N}$**.** Let $N_e[s]$ be defined as in the previous section. If $p_e[s]\uparrow$ let $N_e[s] := \cup_{i>x} M_e(i)$ where x is the latest value of p_e. Also, let $N[s] := \cup_e N_e[s]$ be the set of all reserved positions at stage s. The parameter ℓ_s of the previous section will be used here in the same way.

Construction: At stage $s+1$, let t be the largest number such that $p_t\downarrow$. Also, let i be the least such that $q_i\uparrow$ and $m > s$ be an index of $Q_s \cap P_i$. Define $r_s = \langle \ell_s, m, \bar{p}, N[s]\rangle$, where $\bar{p} = \langle p_0, \dots, p_t\rangle[s]$. (By $N[s]$ we actually mean an index of this recursive set, i.e. an index of an increasing function enumerating its elements.) For each $e \leq t$ let $v_e > i$ be an index of the Π^0_1 class Q_s restricted to the paths which agree with A_j on $M_j[s] \restriction f(r_s)$ for all $j \leq e$. Let k be the largest number $\leq s$ such that the following holds

(2.3) for all configurations σ there is a path in P_{v_k} which agrees with σ on $N[s]$ from ℓ_s.

If $k < t$ let $p_j\uparrow$ and say that A_j is injured for all $j > k$, and end this stage. Otherwise consider the least $n \leq k$ such that the following holds:

(2.4) there is a configuration σ such that all paths which agree with σ on $N[s]$ from ℓ_s are terminal in $P_{v_n} \cap P_i$.

(i) If there is no such n, let $q_i = i$ and $\ell_{s+1} = \ell_s$.
(ii) Otherwise let $q_i > i$ be an index of the Π^0_1 class consisting of the paths of P_{v_n} which agree with σ on $N[s]$ from ℓ_s. Also let ℓ_{s+1} be the $|\sigma|$th element of $N[s]$.

Define p_{t+1} to be a *large* (i.e. larger than any parameter of the construction at the current stage) number and go to the next stage.

Verification. First, we show by induction that at all stages s of the construction the following holds:

(2.5) for every configuration, there exists a path through Q_s which agrees with the configuration on $N[s]$ from ℓ_s.

For $s = 0$ it is trivial. Assume that the claim holds at stage s. If no new definition of any q_i is made at stage $s+1$, it clearly holds at this stage. If some q_i was defined through (i), by the criterion behind this clause, for every configuration there exists a path in $P_{v_k} \cap P_i$

which agrees with it on $N[s]$ from ℓ_s. If a definition of q_i is made through (ii) at stage $s+1$, we restrict Q_{s+1} to the paths of P_{v_k} which agree with a certain configuration σ on $N[s]$ from ℓ_s. By the property (2.3) of P_{v_k} (and the choice of k), the class Q_{s+1} will have paths which agree with any given configuration from the $|\sigma|$th member of $N[s]$. This concludes the induction step and the proof of the claim.

Second, we show that for each $e \in \mathbb{N}$ there is a stage s_* after which the coding parameters p_n, $n < e$ remain constantly defined and the following holds for all $s > s_*$:

(2.6) for all configurations σ there exists a path through Q_s which agrees with A_n on $M_n(p_n)$ for $n < e$ and with σ on $N[s]$ from ℓ_s.

Inductively suppose that p_n, $n < e$ remain constantly defined after stage s_* and (2.6) holds at all $s > s_*$. Consider the following partial computable functional Θ_e. Given $\bar{j} = \langle j_0, \dots, j_e\rangle$, a computable set N_*, a threshold $\ell \in \mathbb{N}$ and an index i of a Π^0_1 class, $\Theta_e^{\oplus_{n\leq e} A_n}(\langle \ell, i, \bar{j}, N_*\rangle)$ equals the least number $x > \ell$ such that for some configuration all paths of length x which agree with A_n on $M_n(j_n)$ for $n \leq e$ and with the configuration on N_* from ℓ are terminal in P_i. We note that the computable set is given in the form of an index of a computable function that enumerates it monotonically. Moreover, if during the search, the computation finds that N_* is not disjoint from $M_n(j_n)$ for $n \leq e$, then the functional does not converge. Notice that by definition, $r_s > s$ for all $s \in \mathbb{N}$. By the properties of f, there is a stage $s_0 > s_*$ such that for all $s > s_0$ the number $f(r_s)$ is larger than $\Theta_e^{\oplus_{n\leq e} A_n}(r_s)$, when this is defined. For the induction step, it suffices to show that after s_0 the coding parameter p_e can be redefined at most once.

Suppose that p_e becomes undefined at some stage $s_1 > s_0$. According to the construction, at stage $s_1 + 1$ it will receive a large value. Since (2.5) holds for $s = s_1$, for every configuration there is a path in Q_{s_1} which agrees with A_e on $M_e[s_1 + 1]$ and with the configuration on $N[s_1 + 1]$ from ℓ_{s_1}. By induction we show that for all $s \geq s_1$ and every configuration there exists a path through Q_s which agrees with A_e on $M_e[s_1 + 1]$ and with the configuration on $N[s]$ from ℓ_s.

Suppose that it holds at $s \geq s_1$. If some q_i is defined according to (i) at stage $s + 1$, let m be the index of $Q_s \cap P_i$. If the claim did not hold at $s + 1$, $\Theta_e^{\oplus_{n\leq e} A_n}(\langle \ell_s, m, \bar{p}, N[s]\rangle)\downarrow$ for $\bar{p} = \langle p_0, \dots, p_e\rangle[s]$. Since $f(r_s)$ is larger than this value we would proceed through (ii), a contradiction. If the construction proceeds through (ii), then for some $t \geq e$ and a (finite) configuration we restrict Q_s to the paths that agree with A_n on $M_n[s]$ for $n \leq t$ and with the configuration on $N[s]$ after ℓ_s. Since the claim holds at s, it will also hold after this restriction. This finishes the induction and shows that for $s > s_1$ the class Q_s contains paths of any configuration on N_e from ℓ_s which agree with A_e. In particular, the construction will never change p_e after stage s_1.

Third, notice that since $p_e[s]$ reaches a limit p_e, the parameters q_i will be defined for all $i \in \mathbb{N}$. Now it is clear that there is a unique path $B \in \cap_s Q_s$ whose jump is determined by (q_i). Since the construction is computable in $\emptyset'$, it uniformly provides the reduction $B' \leq_T \emptyset'$. Moreover, since for all e, B agrees with A_e on $M_e(p_e)$ we have $n \in A_e$ iff $\langle e, p_e, n\rangle \in B$ for all $e, n \in \mathbb{N}$. Therefore, $A_e \leq_m B$ for all $e \in \mathbb{N}$.

References

[BN09] G. Barmpalias and André Nies. Upper bounds on ideals in the computably enumerable Turing degrees. Submitted, 2009.

[KS09] Antonín Kučera and Theodore Slaman. Lower upper bounds of ideals. *J. Symbolic Logic*, 74(2):517–534, 2009.

[MN06] J. Miller and A. Nies. Randomness and computability: open questions. *Bull. Symbolic Logic*, 12(3):390–410, 2006.

[Nie05] André Nies. Lowness properties and randomness. *Adv. Math.*, 197(1):274–305, 2005.

[Nie06] André Nies. Reals which compute little. In *Logic Colloquium '02*, volume 27 of *Lect. Notes Log.*, pages 261–275. Assoc. Symbol. Logic, La Jolla, CA, 2006.

[Nie09] André Nies. *Computability and Randomness*. Oxford University Press, 444 pp., 2009.

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