HIGHNESS PROPERTIES CLOSE TO PA-COMPLETENESS

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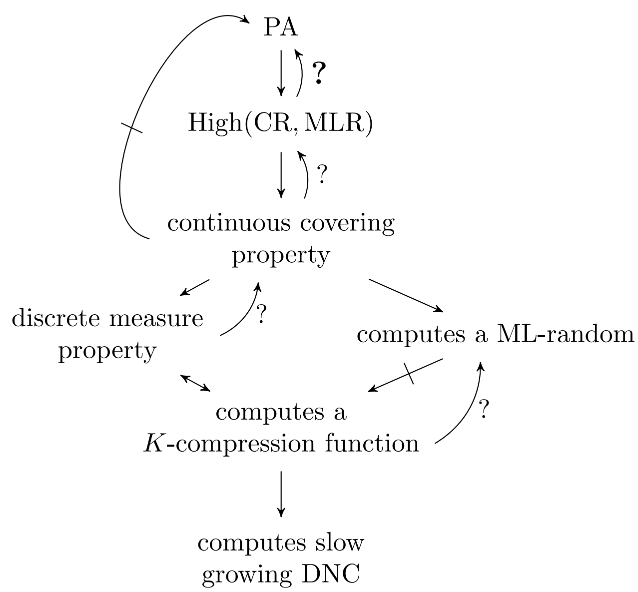
Abstract. We identify properties of oracles bringing them close to having PA degree. These properties are based on relativizing concepts from algorithmic information theory, computable analysis, and algorithmic randomness.

1. Introduction

Recall Medvedev reducibility $\leqslant_M$, Muchnik reducibility $\leqslant_w$.

For $f, g \in \omega^\omega$ we write $f \leqslant g$ to mean that f is majorized by g. In this case, we say that f is g-bounded. Let $\mathrm{id}^\omega = \{f \in \omega^\omega : (\forall n)\ f(n) \leqslant n\}$, in other words, the *identity bounded* functions.



Theme: approximating computably enumerable objects computably from an oracle. Which objects: things that come out of randomness.

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2. Some new Muchnik complete computably bounded Π^0_1 classes

Question: can we show Medvedev incompleteness? I.e. is the non-uniformity in the proofs necessary?

2.1. C-compression functions.
A *C-compression function* is an injective function $F\colon 2^{<\omega} \to 2^{<\omega}$ such that $|F(\sigma)| \leqslant C(\sigma)$ for all σ.

Lemma 2.1. *Every C-compression function has PA degree.*

Proof. Let $F\colon 2^{<\omega} \to 2^{<\omega}$ be a C-compression function. Let $\delta \in (0,1]$. Say that $\sigma \in 2^{<\omega}$ is *δ-heavy* (for F) if $|\{\tau \in 2^{|\sigma|}\colon F(\langle\sigma,\tau\rangle) \leqslant |\sigma|\}| \geqslant \lfloor \delta 2^{|\sigma|} \rfloor$. Note that at most not many strings are delta heavy ☐

2.2. Martingale domination.
Code martingales by Path on Π^0_1 class on 2^ω.

Proposition 2.2. *There is an atomless c.e. martingale M such that every martingale dominating M has PA degree.*

Proof. Define M as follows. If n enters $\varnothing'$ at stage s, push 2^{-n} capital up to some string of length s that looks DNC_2 at stage s.

Let N be a martingale majorizing M.

Case 1. N has a DNC_2 atom. A martingale computes all of its atoms, so in this case, N has PA degree.

Case 2. N has no DNC_2 atoms. Then for each n there is a stage $f(n) = s$ such that all strings of length s that still look DNC_2 at stage s have value less than 2^{s-n} (hence less than 2^{-n} of the initial capital). By construction, f dominates the settling time function for $\varnothing'$. So N has PA degree. ☐

2.3. Jordan decomposition on the rationals.
Let us say that

$$f \leqslant_{\texttt{slope}} g :\Leftrightarrow \forall x < y\, [f(y) - f(x) \leqslant g(y) - g(x)].$$

That is, the slopes of g are at least as large as the slopes of f. Clearly, this is equivalent to saying that $h := g - f$ is nondecreasing. Thus, the problem of finding a Jordan decomposition of a function f of bounded variation is equivalent to finding a nondecreasing function g with $f \leqslant_{\texttt{slope}} g$. This was already pointed out in [**?**].

We only require that g, h are defined in $I_{\mathbb{Q}} := \mathbb{Q} \cap [0,1]$. An $\mathbb{R}$ -valued function g defined on $I_{\mathbb{Q}}$ is given by a path Z_f through a binary tree. Let $\langle p_n, q_n\rangle$ be a list of all pairs of rationals $\langle p, q\rangle$ with $0 \leqslant p \leqslant 1$. We let $Z_f(2n) = 1$ iff $g(p_n) < q_n$. We let $Z_f(2n+1) = 1$ iff $g(p_n) > q_n$. We often identify f and Z_f. It is clear that the nondecreasing functions form a Π^0_1 class.

Given f, via the encoding above, the functions g defined on $I_{\mathbb{Q}}$ with $f \leqslant_{\texttt{slope}} g$ form a nonempty $\Pi^0_1(f)$ class.

Theorem 2.3. *There is a computable function f on $[0,1]$ such that each function $g\colon I_{\mathbb{Q}} \to \mathbb{R}$ with $f \leqslant_{\texttt{slope}} g$ has PA degree.*

Proof. Let $\mathcal{P} \subseteq 2^\omega$ be a nonempty Π^0_1 class of sets of PA degree, such as the (binary encoded) completions of Peano arithmetic. As usual $\mathcal{P}_s$ is a clopen set computable from s approximating $\mathcal{P}$ at stage s. So $\mathcal{P} = \bigcap_s \mathcal{P}_s$. By standard methods there is a computable prefix-free sequence $\langle\sigma_s\rangle_{s\in\omega}$ of strings of length s such that $[\sigma_s] \cap \mathcal{P}_s \neq \varnothing$ for each n.

Given $\sigma \in 2^{<\omega}$, let $I_\sigma = [0.\sigma, 0.\sigma + 2^{-|\sigma|}]$ be the corresponding closed subinterval of $[0,1]$. By stage s we determine f up to a precision of 2^{-s}. Suppose n enters $\varnothing'$

at stage s. Let $\sigma = \sigma_s$. We define f on I_σ to be a sawtooth function of height 2^{-s} with 2^{s-n} many teeth.

Now suppose $g\colon I_{\mathbb{Q}} \to \mathbb{R}$ is a function such that $f \leqslant_{\texttt{slope}} g$. As before for $x \in [0,1]$ let $\widehat{g}(x) = \sup\{g(q) \mid q \leqslant x \wedge q \in I_{\mathbb{Q}}\}$.

Case 1. $\widehat{g}$ is discontinuous at the real $y = 0.Y$ for some $Y \in \mathcal{P}$. Then $Y \leqslant_T g$, so g is of PA degree. (To see this, fix rational r with $\widehat{g}(y) < r < g^+(y)$. Then $p < y \leftrightarrow g(p) < r$, and $p > y \leftrightarrow g(p) > r$.)

Case 2. Otherwise. Then $\varnothing' \leqslant_T g$: given n, using g compute stage s such that for each σ of length s with $[\sigma] \cap \mathcal{P}_s \neq \varnothing$, we have $g(\max I_\sigma) - g(\min I_\sigma) < 2^{-n}$. This s exists by case assumption using compactness of Cantor space. Then as before we have $n \in \varnothing' \leftrightarrow n \in \varnothing'_s$. □

3. A K-compression functions can fail to have PA degree

Let $K\colon \omega \to \omega$ be prefix-free Kolmogorov complexity on natural numbers. We show that the computably bounded Π^0_1 class of K -compression functions has incomplete Muchnik degree.

For $f \in \omega^{\leqslant\omega}$, let

$$\mathrm{wt}(f) = \textstyle\sum_{n \in \mathrm{dom}\, f} 2^{-f(n)}.$$

We say that $f \in \omega^\omega$ has *finite weight* if $\mathrm{wt}(f) < \infty$. If $P \subseteq \omega^\omega$ and $q \in \mathbb{Q}$, then we let $P_{\leqslant q} = \{f \in P\colon \mathrm{wt}(f) \leqslant q\}$. Define $P_{<q}$ similarly. Note that if P is a (computably bounded) Π^0_1 class, then $P_{\leqslant q}$ is too. (This is not usually true of $P_{<q}$.)

Let $P^K = \{f \in \mathrm{id}^\omega\colon f \leqslant K\}$. Note that P^K is a computably bounded Π^0_1 class. We claim that P^K contains a finite weight function; this follows easily from the fact that $\sum_{n\in\omega} 2^{-K(n)} < 1$. In particular, let $K^*(n) = \min\{K(n), n\}$. Then $\mathrm{wt}(K^*) \leqslant \mathrm{wt}(K) + \sum_{n\in\omega} 2^{-n} < 3$ and $K^* \in P^K$. Therefore, P^K_3 is a computably bounded and nonempty Π^0_1 class containing only bounded weight K-bounded functions. This proves:

Proposition 3.1. *Every PA degree computes a K-bounded function of finite weight.*

[[But in fact we'll show the covering properties imply this too]] This into INTRO

Our goal is to prove that it is *strictly easier* to compute K-bounded functions of finite weight.

Theorem 3.2. *There is a K-bounded function $f\colon \omega \to \omega$ of finite weight that does not have PA degree.*

We build f using a forcing argument.

The *forcing conditions* are triples of the form (σ, P, q) where

- $\sigma \in \mathrm{id}^{<\omega}$;
- $P \subseteq P^K$ is a Π^0_1 class such that:
  - Every $h \in P$ extends σ;
  - If $h \in P$, $g \leqslant h$ and g extends σ, then $g \in P$;
- $q \in \mathbb{Q}$ and $P_{\leqslant q} \neq \varnothing$.

The condition (σ, P, q) should be thought of as saying that $f \in P_{\leqslant q}$. We say that (τ, R, s) *extends* (σ, P, q) if $\sigma \preccurlyeq \tau$, $R \subseteq P$ and $s \leqslant q$. Note that $(\langle\rangle, P^K, 3)$ is a condition, so the set of conditions is nonempty.

For a filter G of forcing conditions, we let

$$f_G = \bigcup \sigma \ \llbracket (\sigma, P, q) \in G \text{ for some } P \text{ and } q \rrbracket.$$

Then $f_G \in \mathrm{id}^{\leqslant\omega}$. If (σ, P, q) is a condition, then we can find τ properly extending σ such that (τ, P, q) is also a condition (take τ to be an initial segment of a function witnessing that $P_{\leqslant q}$ is nonempty). This shows that if G is only mildly generic, then $f_G \in \mathrm{id}^{\omega}$.

Lemma 3.3. *Suppose that $(\sigma, P, q) \in G$. Then $f_G \in P_{\leqslant q}$.*

Proof. Let $\tau \prec f_G$. There is some condition $(\tau, Q, s) \in G$, and by extending the condition (and τ) if necessary we may assume that the condition (τ, Q, s) extends the condition (σ, P, q). Since $Q_{\leqslant s}$ is nonempty and $Q_{\leqslant s} \subseteq P_{\leqslant q}$, we have that $[\tau] \cap P_{\leqslant q}$ is nonempty. Since this is true for all $\tau \prec f_G$ and $P_{\leqslant q}$ is closed, we see that $f_G \in P_{\leqslant q}$. □

Since $P \subseteq P^K$ for any condition (σ, P, q), we conclude that f_G is K-bounded. Lemma 3.3 also implies that $\mathrm{wt}(f_G)$ is finite.

There is not much difference between $P_{\leqslant q}$ and $P_{<q}$.

Lemma 3.4. *Let (σ, P, q) be a condition. Then $P_{<q}$ is nonempty.*

Proof. Suppose not. Then $P_{\leqslant q}$ is nonempty and every element of $P_{\leqslant q}$ has weight exactly q, i.e. $P_{\leqslant q} = P_{=q}$. This gives us an algorithm for computing $\varnothing'$. Note that if m enters $\varnothing'$ at stage s then $K(s) \leqslant^+ m$. Hence it suffices, given any $m < \omega$ to find some $n < \omega$ such that $K(x) \geqslant m$ for all $x \geqslant n$. This we can do.

Let T be a computable subtree of $\mathrm{id}^{<\omega}$ such that $[T] = P_{=q}$. For $r < q$, let O_r be the set of finite strings $\sigma \in \mathrm{id}^{<\omega}$ with $\mathrm{wt}(\sigma) > r$; so $\mathrm{id}^{\omega}_{>r}$ is the open subset of id^{ω} generated by O_r. (Note however that it is possible that $[\sigma] \subset \mathrm{id}^{\omega}_{>r}$ for strings $\sigma \notin O_r$; $[\sigma] \subset \mathrm{id}^{\omega}_{>r}$ if and only if $\mathrm{wt}(\sigma) + 2^{-|\sigma|+1} > r$.) Let T_n be the set of strings on T of length n. Since $P_{=q} \subset \mathrm{id}_{>r}$ and id^{ω} is compact, for every $r < q$ there is some $n < \omega$ such that $T_n \subset O_r$; such n can be of course found effectively from r. If $T_n \subset O_{>q-2^{-m}}$ then $K(x) > m$ for all $x \geqslant n$. For we know that there is some $\sigma \in T_n$ which is extendible ($[\sigma] \cap P_{=q} \neq \varnothing$); if $h \in [\sigma] \cap P_{=q}$, $x \geqslant |\sigma|$ and $\mathrm{wt}(h) - \mathrm{wt}(\sigma) < 2^{-m}$ then $h(x) > m$. Since (σ, P, q) is a condition, we know that $h \leqslant K$. □

Remark 3.5. Thus, if (ρ, R, t) is a condition then there is some $t' < t$ such that (ρ, R, t) is a condition as well. Thus, by genericity, if $(\sigma, P, q) \in G$, then there is some $q' < q$ such that $(\sigma, P, q') \in G$. By Lemma 3.3, $f_G \in P_{\leqslant q'}$, and so $f_G \in P_{<q}$.

It remains to show that f_G does not have PA degree. This will follow from genericity (and Lemma 3.3), once we show that for any Turing functional Γ, the collection of conditions

$$E_\Gamma = \{(\sigma, P, q) \,:\, \text{for all } h \in P_{\leqslant q}, \Gamma(h) \notin \mathrm{DNC}_2\}$$

is dense below the base condition $(\langle\rangle, P^K, 3)$.

Let (σ, P, q) be a condition. We want to find an extension of this condition in E_Γ. First, we find an extension (σ^*, P^*, q) of (σ, P, q) and some rational $\varepsilon > 0$ such that letting $r = \mathrm{wt}(\sigma^*)$ we have:

- $r + 3\varepsilon < q$; and
- $P^*_{<r+\varepsilon}$ is nonempty.

Note however that even though this means that $(\sigma^*, P^*, r+\varepsilon)$ is a condition, we will find, in E_Γ, an extension of (σ^*, P^*, q) rather than $(\sigma^*, P^*, r+\varepsilon)$.

Finding σ^*, P^* and ε is easy. By Lemma 3.4, let $h^* \in P_{<q}$. Pick ε small enough that $\mathrm{wt}(h^*) + 3\varepsilon < q$. Take $\sigma^* \prec h^*$ such that $\mathrm{wt}(h^*) - \mathrm{wt}(\sigma^*) < \varepsilon$. Let $P^* = P \cap [\sigma^*]$.

We now define a partial computable process which may either output 0 or 1. The output of this process will be $J(e)$ for some e, and by the recursion theorem, we may assume we know e in the definition of this process. Consider the Π^0_1 class Q obtained from P^* by removing not only all the strings τ of weight below $r + 2\varepsilon$ for which $\Gamma(\tau, e)\downarrow$, but also all strings dominating such strings σ:

$$Q = \{h \in P^* : \neg(\exists \tau \succcurlyeq \sigma^*)[\, \tau \leqslant h \text{ and } \mathrm{wt}(\tau) < r + 2\varepsilon \text{ and } \Gamma(\tau, e)\downarrow\,]\}.$$

The point is of course that if $h \in Q$, $g \leqslant h$ and $\sigma^* \prec g$ then $g \in Q$. Note, however, that Q is not quite the same as the class obtained by removing h for which there is some $g \in P^*$, $g \leqslant h$, $\mathrm{wt}(g) \leqslant r + 2\varepsilon$ and $\Gamma(g, e)\downarrow$. The class Q is smaller, since τ may not be extended to some h-dominated g of weight at most $r + 2\varepsilon$.

If $Q_{\leqslant r+2\varepsilon} \neq \varnothing$ then our computable process does not terminate. Suppose that $Q_{\leqslant r+2\varepsilon} = \varnothing$. This of course is eventually effectively recognised, as $Q_{\leqslant r+2\varepsilon}$ is a Π^0_1 class, effectively obtained from e. By compactness, we can find some $n < \omega$ and a finite subset C of id^n such that:

(1) Every $\sigma \in C$ extends σ^* and $\mathrm{wt}(\sigma) < r + 2\varepsilon$.
(2) For every $\sigma \in C$ there is some $\tau \leqslant \sigma$ (in particular $|\tau| \leqslant |\sigma|$) such that $\mathrm{wt}(\tau) < r + 2\varepsilon$, $\tau \succcurlyeq \sigma^*$ and $\Gamma(\tau, e)\downarrow$.
(3) If $\sigma \in C$, $\sigma' \leqslant \sigma$ and $\mathrm{wt}(\sigma') < r + 2\varepsilon$ then $\sigma' \in C$.
(4) There is some $\sigma \in C$ such that $[\sigma] \cap P^*_{<r+\varepsilon} \neq \varnothing$.

Let $\hat{C}$ be the set of strings in C whose weight is smaller than $r + \varepsilon$. Condition (4) says that $\hat{C}$ is nonempty, indeed some $\sigma \in \hat{C}$ is extendible in $P^*_{<r+\varepsilon}$. Now an important point is that if $\sigma, \sigma' \in \hat{C}$ then the pointwise minimum $\min \sigma, \sigma'$ is in C, as both σ and σ' extend σ^* and so $\mathrm{wt}(\sigma) - \mathrm{wt}(\sigma^*) < \varepsilon$, and similarly for σ'. For any pair σ, σ' of strings from $\hat{C}$ find some $\tau \leqslant \min(\sigma, \sigma')$ of weight less than $r + 2\varepsilon$, extending σ^*, such that $\Gamma(\tau, e)\downarrow$; let $c(\{\sigma, \sigma'\}) = \Gamma(\tau, e)$ (which we assume is either 0 or 1).

There is some colour $i \in \{0, 1\}$ such that for all $\sigma \in \hat{C}$ there is some $\sigma' \in \hat{C}$ such that $c(\{\sigma, \sigma'\}) = i$ (if this fails say for 0, then a single σ' witnesses it for 1). This colour is the output of the computable process just described.

We now describe the extension of (σ^*, P^*, q) in E_Γ. Of course, there are two cases. If $Q_{\leqslant r+2\varepsilon}$ is nonempty, then $(\sigma^*, Q, r+2\varepsilon)$ is a condition, and $\Gamma(h, e)\uparrow$ for all $h \in Q_{\leqslant r+2\varepsilon}$. We assume, then, that $Q_{\leqslant r+2\varepsilon}$ is empty. Let i be the outcome of the computable process described above. Let $\sigma \in \hat{C}$ be "correct", i.e., $[\sigma] \cap P^*_{<r+\varepsilon} \neq \varnothing$; fix some $h^* \in P^*$ with $\mathrm{wt}(h^*) < r + \varepsilon$ and $\sigma \prec h$. There is some $\sigma' \in \hat{C}$ such that $c(\{\sigma, \sigma'\}) = i$; so let $\tau \leqslant \sigma, \sigma'$ extend σ^* with $\mathrm{wt}(\tau) < r + 2\varepsilon$ and $\Gamma(\tau, e) = i$. Let $R = P^* \cap [\tau]$. We claim that $R_{\leqslant r+3\varepsilon}$ is nonempty. For we can let $g = \tau\hat{\ } h^* \restriction_{[|\tau|, \infty)}$. Note that $g \leqslant h^*$, so $g \in P^*$. And

$$\mathrm{wt}(g) = \mathrm{wt}(\tau) + (\mathrm{wt}(h^*) - \mathrm{wt}(\sigma)) \leqslant \mathrm{wt}(\tau) + (\mathrm{wt}(h^*) - \mathrm{wt}(\sigma^*)) < (r + 2\varepsilon) + \varepsilon.$$

Thus $(\tau, R, r + 3\varepsilon)$ is a condition and it extends (σ^*, P^*, q) since $r + 3\varepsilon \leqslant q$. Every $h \in R$ extends τ and so $\Gamma(h, e) = i = J(e)$, and so $\Gamma(h) \notin \mathrm{DNC}_2$.

4. Discrete measure property

rewrite in terms of discrete measures

For a sequence $\bar{A} = \langle A_n \rangle$ of subsets of ω, we let $\mathrm{wt}(\bar{A}) = \sum_n 2^{-n}|A_n|$. I.e., every element of A_n receives weight 2^{-n}. Write $\bar{A} \subseteq \bar{B}$ to mean $A_n \subseteq B_n$ for all n; we say that $\bar{B}$ *covers* $\bar{A}$. We say that an oracle D has the *discrete covering property* if every uniformly c.e. sequence of finite weight is covered by some D-uniformly computable sequence of finite weight.

Proposition 4.1. *An oracle D computes a K-compression function if and only if it has the discrete covering property.*

Proof. In one direction, let A_n be the set of strings whose prefix-free complexity is at most n. If $\bar{B}$ covers $\bar{A}$ then we can let $f(\sigma)$ be the least n such that $\sigma \in B_n$. □

Proposition 4.2. *Let D be an oracle which has the discrete covering property. For every order function h, D computes an h-DNR function.*

Proof. MISSING □

5. The continuous covering property

For an open subset U of 2^ω, let A_U be the set of strings σ such that $[\sigma] \subseteq U$. We say that an oracle D has the *continuous covering property* if for every c.e. open subset U of 2^ω of measure smaller than 1, there is an open subset V of 2^ω containing U such that $\lambda(V) < 1$ and A_V is computable from D. Equivalently, if for any computable tree T such that $\lambda([T]) > 0$, there is a D-computable tree S with no dead ends such that $S \subseteq T$ and $\lambda([S]) > 0$.

Proposition 5.1. *Every oracle which has the continuous covering property also has the discrete covering property.*

Proof. Let $\langle C_{n,k} \rangle$ be an array of clopen sets (given canonically) which is independent, such that $\lambda(C_{n,k}) = 2^{-n}$ for all k. For an array $\bar{E} = \langle E_n \rangle$ of subsets of ω, let

$$U_{\bar{E}} = \bigcup C_{n,k} \quad [\![k \in E_n]\!].$$

Then $\bar{E}$ has finite weight if and only if $\lambda(U_{\bar{E}}) < 1$. □

Lemma 5.2. *Let $\mathbf{P}$ be the complement of the first component of a standard universal Martin-Löf test. For any tree T with no dead ends, if $\lambda([T]) > 0$, $[T] \subseteq \mathbf{P}$ and further, for all $\sigma \in T$, $\lambda([T] \cap [\sigma]) > 0$, then $\deg_{\mathrm{T}}(T)$ has the continuous covering property.*

Proof. We dually work with effectively open sets. Let U be the first component of a standard universal Martin-Löf test, that is, obtained by gluing together all Martin-Löf tests. So $\lambda(U) < 1$ and for any Martin-Löf test $\langle V_n \rangle$ there is some n such that $V_n \subset U$.

Let T be a tree as described for $\mathbf{P}$. We get an open set W such that

$$\{\sigma \in 2^{<\omega} : [\sigma] \subseteq W\} = \{\sigma \in 2^{<\omega} : \lambda(W|\sigma) = 1\}$$

is T-computable, $\lambda(W) < 1$ and $U \subseteq W$.

Let V be an effectively open set with $\lambda(V) < 1$, given by c.e. antichain of strings which we also call V. Recall that V^{n+1} is the set of concatenations $\sigma\hat{}\tau$ for $\sigma \in V^n$ and $\tau \in V$ (and $V^0 = \{\langle\rangle\}$). The sequence $\langle V^n \rangle$ can be refined to a Martin-Löf test, and so there is some n such that $V^n \subseteq W$. Let n be the least such; $n > 0$ since $\lambda(W) < 1$. Let σ be a string witnessing that $V^{n-1} \nsubseteq W$, i.e. $\sigma \in V^{n-1}$ and $[\sigma] \nsubseteq W$. Now consider $W|\sigma = \{X \in 2^\omega : \sigma\hat{\ }X \in W\}$. Then $[\sigma] \nsubseteq W$ implies that $\lambda(W|\sigma) < 1$; $V^n \subseteq W$ implies that $V \subseteq W|\sigma$; and

$$\{\tau : [\tau] \subseteq W|\sigma\} = \{\tau : [\sigma\hat{\ }\tau] \subseteq W\}$$

is T-computable. □

We wish to separate the continuous covering property from PA. We will work with Π^0_1 classes of trees with no dead ends. Note that there is a 1-1 correspondence between closed subsets of 2^ω and subtrees of $2^{<\omega}$ with no dead ends. For brevity, we let $\mathcal{T}$ be the set of nonempty trees with no dead ends. Coded by characteristic functions, $\mathcal{T}$ itself is an effectively closed subset of Cantor space.

Below we ignore the difference between $[T]$ and T (for $T \in \mathcal{T}$) and write T for both. Note that for $T, S \in \mathcal{T}$, $T \subseteq S$ iff $[T] \subseteq [S]$. The operation of intersection is well defined; for $S, T \in \mathcal{T}$ we let $S \cap T$ be the unique element R of $\mathcal{T}$ such that $[R] = [S] \cap [T]$.

Infinite trees are built up of finite ones. Let $\mathcal{T}_{<\omega}$ be the collection of all nonempty finite subtrees of $2^{<w}$. For $T \in \mathcal{T}$ and $\vartheta \in \mathcal{T}_{<\omega}$ we say that T extends ϑ (and sometimes write $\vartheta \prec T$) if $\vartheta \subset T$ and every $\sigma \in T$ is comparable with a leaf of ϑ. For each $\vartheta \in \mathcal{T}_{<\omega}$ we let $[\vartheta]$ be the collection of $T \in \mathcal{T}$ which extend ϑ. This is a clopen subset of $\mathcal{T}$, and the collection of these sets generates the topology on $\mathcal{T}$. We often restrict ourselves to trees of a fixed height; for $n < \omega$, let $\mathcal{T}_n$ be the set of finite trees all of whose leaves have length n. For $\vartheta \in \mathcal{T}_{<\omega}$ and n greater than the height of ϑ we let $[\vartheta]_n$ be the set of trees $\varpi \in \mathcal{T}_n$ which extend ϑ, again in the sense that each $\tau \in \varpi$ extends some $\sigma \in \vartheta$ and each $\sigma \in \vartheta$ is extended by some $\tau \in \varpi$; we write $\vartheta \preccurlyeq \varpi$. Note that for $\vartheta, \varpi \in \mathcal{T}_{<\omega}$, $\vartheta \preccurlyeq \varpi$ if and only if $[\varpi] \subseteq [\vartheta]$.

We also implicitly use the bijection between $\mathcal{T}_{<\omega}$ and the collection of finite antichains of strings (a tree is mapped to its leaves). For example, for a finite antichain of strings u we let $[u]$ be $[\vartheta]$ where u is the set of leaves of ϑ. A tree $\vartheta \in \mathcal{T}_{<\omega}$ and its set of leaves are both identified with the clopen subset of 2^ω determined by ϑ. Thus for example, for a finite antichain u of strings we let $\lambda(u) = \sum_{\sigma \in u} 2^{-|\sigma|}$. Similarly, for $T \in \mathcal{T}$ and $\tau \in 2^{<\omega}$ we let $T \cap \tau = \{\sigma \in T : \sigma \not\perp \tau\}$.

Fix the Π^0_1 class $\mathbf{P}$ from Lemma 5.2. Our forcing conditions are triples $(u, P, \bar{q})$ such that:

- u is a nonempty finite antichain of strings;
- $P \subseteq \mathcal{T}$ is a Π^0_1 subset of $[u]$ such that:
  - for all $T \in P$ we have $T \subseteq \mathbf{P}$;
  - if $T \in P$, $S \in [u]$ and $S \subseteq T$ then $S \in P$.
- $\bar{q} = \langle q_\sigma \rangle_{\sigma \in u}$ is a sequence of positive rational numbers indexed by u, and

$$P_{\geqslant \bar{q}} = \{T \in P : \text{for all } \sigma \in u, \lambda(T \cap \sigma) \geqslant q_\sigma\}$$

  is nonempty.

If we let P be the set of trees $T \in \mathcal{T}$ such that $T \subseteq \mathbf{P}$ and q be any rational number smaller than $\lambda(\mathbf{P})$, then $(\{\langle\rangle\}, P, \langle q \rangle)$ is a condition, so the set of conditions is nonempty.

A condition $(v, R, \bar{r})$ extends a condition $(u, P, \bar{q})$ if:

(1) $u \preccurlyeq v$;
(2) $R \subseteq P$; and
(3) for all $\sigma \in u$,

$$q_\sigma \leqslant \sum r_\tau \ [\![\tau \in v \ \ \& \ \ \tau \succcurlyeq \sigma]\!].$$

Note that if $u \preccurlyeq v$ then condition (3) is equivalent to $[v]_{\geqslant \bar{r}} \subseteq [u]_{\geqslant \bar{q}}$. In particular, we see that if a condition $(v, R, \bar{r})$ extends a condition $(u, P, \bar{q})$ then $R_{\geqslant \bar{r}} \subseteq P_{\geqslant \bar{q}}$.

Lemma 5.3. *Let $(u, P, \bar{q})$ be a condition. Then*

$$P_{>\bar{q}} = \{T \in P \,:\, \text{for all } \sigma \in u, \lambda(T \cap \sigma) > q_\sigma\}$$

is nonempty.

Proof. Suppose not. Let v be a $\subseteq$-maximal subset of u for which there is some $T \in P_{\geqslant \bar{q}}$ with $\lambda(T \cap \sigma) > q_\sigma$ for all $\sigma \in v$. Let $\varepsilon > 0$ be rational smaller than $\lambda(T \cap \sigma) - q_\sigma$ for all $\sigma \in v$, and let $q'_\sigma = q_\sigma + \varepsilon$ for $\sigma \in v$ and $q'_\sigma = q_\sigma$ for $\sigma \in v - u$. Thus, $P_{\geqslant \bar{q}'}$ is nonempty, and for all $S \in P_{\geqslant \bar{q}'}$, for all $\sigma \in v - u$ we have $\lambda(S \cap \sigma) = q_\sigma$. Choose any $\sigma \in v - u$, and let $Q = \{S \cap \sigma \,:\, S \in P_{\geqslant \bar{q}'}\}$. Let $q = q_\sigma$. So Q is a nonempty Π^0_1 subclass of $\mathcal{T}$ and for all $T \in Q$, $T \subseteq \mathbf{P}$ and $\lambda(T) = q$.

Let $V_n = \varnothing$ if $n \notin \varnothing'$, and otherwise let $V_n = \{\sigma\hat{\ }0^n \,:\, |\sigma| = s\}$ where s is the stage at which n enters $\varnothing'$. Since $\lambda(V_n) \leqslant 2^{-n}$ and $\langle V_n \rangle$ is uniformly c.e., for all sufficiently large n we have $V_n \cap \mathbf{P} = \varnothing$.

Let $m < \omega$. By compactness, we can effectively find some $t < \omega$ and some $C \subseteq \mathcal{T}_t$ such that $Q \subseteq \bigcup_{\vartheta \in C} [\vartheta]$ and such that $q/\lambda(\vartheta) > 1 - 2^{-m}$ for all $\vartheta \in C$. We then claim that provided that m is large enough, $m \in \varnothing'$ if and only if $m \in \varnothing'_t$. For fix some $\vartheta \in C$ such that $[\vartheta] \cap Q \neq \varnothing$, and fix some $T \in [\vartheta] \cap Q$. If m enters $\varnothing'$ at stage $s > t$ then for every leaf σ of ϑ, $\lambda(T|\sigma) \leqslant \lambda(\mathbf{P}|\sigma) \leqslant 1 - 2^{-m}$ and so $q = \lambda(T) \leqslant (1 - 2^{-m})\lambda(\vartheta)$ which is not the case. This algorithm for computing $\varnothing'$ gives the desired contradiction. ◻

For a filter G of forcing conditions, we let T_G be the downward closure of

$$\bigcup u \ [\![(u, P, \bar{q}) \in G \text{ for some } P \text{ and } \bar{q}]\!].$$

We assume from now that G is fairly generic.

Lemma 5.4. *$T_G \in \mathcal{T}$.*

Proof. We need to show that every $\sigma \in T_G$ has a proper extension in T_G.

Let $(u, P, \bar{q})$ be a condition, and let $\sigma \in u$. By Lemma 5.3, let $T^* \in P_{>\bar{q}}$. Let $I = \{i < 2 \,:\, \sigma\hat{\ }i \in T^*\}$ which is nonempty. Let

$$u' = (u - \{\sigma\}) \cup \{\sigma\hat{\ }i \,:\, i \in I\}.$$

Define $\bar{q}'$ by extending $\bar{q}$ but replacing q_σ by $q_{\sigma\hat{\ }i}$ for $i \in I$, so that $q_{\sigma\hat{\ }i} \leqslant \lambda(T^* \cap (\sigma\hat{\ }i))$ and $\sum_{i \in I} q_{\sigma\hat{\ }i} \geqslant q_\sigma$. Then $(u', P \cap [u'], \bar{q}')$ is a condition extending $(u, P, \bar{q})$ and σ has a proper extension in u'. ◻

Lemma 5.5. *Let $(u, P, \bar{q}) \in G$. Then $T \in P_{\geqslant \bar{q}}$.*

Proof. ◻

Let $\Gamma \colon \mathcal{T} \to 2^\omega$ be a Turing functional. Let E_Γ be the set of conditions $(u, P, \bar{q})$ such that $\Gamma(T) \notin \mathrm{DNC}_2$ for all $T \in P_{\geqslant \bar{q}}$. We show that E_Γ is dense.

First we prepare.

Lemma 5.6. *Let $(u, P, \bar{q})$ be a condition. There are $u^* \geqslant u$ and rational numbers $\varepsilon_\tau < 1/2 \cdot 2^{-|\tau|}$ for $\tau \in u^*$ such that letting $P^* = P \cap [u^*]$ and $r_\tau = 2^{-|\tau|}$ we have:*

(1) *For all $\sigma \in u$,*

$$q_\sigma \leqslant \sum (r_\tau - 3\varepsilon_\tau) \ [\![\tau \in u^* \ \& \ \tau \geqslant \sigma]\!];$$

and so $[u^*]_{\geqslant \bar{r} - 3\bar{\varepsilon}} \subseteq [u]_{\geqslant \bar{q}}$; and

(2) *$P^*_{> \bar{r} - \bar{\varepsilon}}$ is nonempty.*

Proof. By Lemma 5.3, let $T \in P_{> \bar{q}}$. Fix some $\sigma \in u$. Take a positive rational number δ_σ such that $5\delta_\sigma < \lambda(T \cap \sigma) - q_\sigma$. Find some finite antichain v_σ of extensions of σ such that $T \cap \sigma \in [v_\sigma]$ and further $\lambda(v_\sigma) - \lambda(T \cap \sigma) < \delta_\sigma$ (where we again identify v_σ with the clopen subset of Cantor space it determines).

Let $w_\sigma = \{\tau \in v_\sigma : \lambda(T|\tau) \leqslant 1/2\}$. We first show that $\lambda(w_\sigma) \leqslant 2\delta_\sigma$. Let $u^*_\sigma = v_\sigma - w_\sigma$. We have

$$\lambda(w_\sigma) + \lambda(u^*_\sigma) = \lambda(v_\sigma) \leqslant \lambda(T \cap \sigma) + \delta_\sigma.$$

Because $T \cap \sigma$ extends v_σ,

$$\lambda(T \cap \sigma) = \lambda(T \cap v_\sigma) = \lambda(T \cap w_\sigma) + \lambda(T \cap u^*_\sigma).$$

The definition of w_σ means that $\lambda(T \cap w_\sigma) \leqslant \lambda(w_\sigma)/2$. Together with $\lambda(T \cap u^*_\sigma) \leqslant \lambda(u^*_\sigma)$, putting everything together yields

$$\lambda(w_\sigma) + \lambda(u^*_\sigma) \leqslant \lambda(w_\sigma)/2 + \lambda(u^*_\sigma) + \delta_\sigma$$

from which we get the desired conclusion. In particular,

$$\lambda(u^*_\sigma) \geqslant \lambda(T \cap u^*_\sigma) \geqslant \lambda(T \cap \sigma) - 2\delta_\sigma > q_\sigma + 3\delta_\sigma, \tag{5.1}$$

so u^*_σ is nonempty.

Let $u^* = \bigcup_{\sigma \in u} u^*_\sigma$ and let $T^* = T \cap u^*$. Note that T^* extends u^*. Because $T^* \subseteq T$, $T^* \in P^* = P \cap [u^*]$.

Again fix $\sigma \in u$. We know that

$$\sum_{\tau \in u^*_\sigma} (r_\tau - \lambda(T^* \cap \tau)) \leqslant \sum_{\tau \in v_\sigma} (r_\tau - \lambda(T \cap \tau)) = \lambda(v_\sigma) - \lambda(T \cap \sigma) < \delta_\sigma.$$

Hence we can choose, for $\tau \in u^*_\sigma$, rational numbers ε_τ just a little greater than $r_\tau - \lambda(T^* \cap \tau)$ (so that $\lambda(T^* \cap \tau) > r_\tau - \varepsilon_\tau$ and so T^* witnesses that $P^*_{\geqslant \bar{r} - \bar{\varepsilon}}$ is nonempty) but such that

$$\sum_{\tau \in u^*_\sigma} \varepsilon_\tau < \delta_\sigma.$$

By Equation (5.1),

$$\sum_{\tau \in u^*_\sigma} r_\tau - 3\varepsilon_\tau > q_\sigma + 3\delta_\sigma - 3\delta_\sigma$$

as required. ∎

Let $S, T \in [u^*]_{> \bar{r} - \bar{\varepsilon}}$. For all $\tau \in u^*$ we have $\lambda(S \cap T \cap \tau) > 2^{-|\tau|} - 2\varepsilon_\tau > 0$. It follows that $S \cap T \in [u^*]_{> \bar{r} - 2\bar{\varepsilon}}$. We can then run the proof from above.

Given e, we let Q be the set of $T \in P^*$ which are not removed by finding some $\vartheta \in \mathcal{T}_{<\omega}$ with $\vartheta \subset T$, $\vartheta \geqslant u^*$, $\lambda(\vartheta \cap \sigma) > r_\sigma - 2\varepsilon_\sigma$ for all $\sigma \in u^*$, and such that $\Gamma(\vartheta, e)\downarrow$. Note that here by $\vartheta \subset T$ we do mean the sets of strings, not the associated closed sets. If $Q_{\geqslant \bar{r} - 2\bar{\varepsilon}}$ is nonempty then $(u^*, Q, \bar{r} - 2\bar{\varepsilon})$ is an extension of $(u, P, \bar{q})$ in E_Γ.

Blah blah blah do the same. For the final step, suppose that $\Gamma(\vartheta, e)\downarrow = i$, with $\vartheta \in \mathcal{T}_{<\omega}$, $\vartheta \in [u^*]_{>\bar{r}-2\bar{\varepsilon}}$ and $S \subset T$ for some $T \in P^*_{>\bar{r}-\bar{\varepsilon}}$. Then $T \cap S \in P^*$ and $T \cap S \in [u^*]_{>\bar{r}-3\bar{\varepsilon}}$.

6. The mystery class: High(CR, MLR)

High(CR,MLR) is the continuous measure property

Proposition 6.1. *Every oracle in* High(CR, MLR) *has the continuous covering property.*

Proof. Let M be a D-computable martingale which succeeds on all non-randoms. Assuming the M-capital of the root is bounded by 1, let V be the set of minimal strings with $M(\sigma) \geqslant 2$. By "failed forcing", V contains a component of the universal ML test. Note that V has the property that $\lambda(V|\sigma) = 1$ implies $[\sigma] \subseteq V$. This makes it universal for continuous coverings. □

If X is High(CR, MLR), we know that all X-computable martingales together succeed on the non-ML-random sequences. But in fact, a previously unpublished proof of Kastermans, Lempp, Miller shows that one X-computable martingale is enough.

Proposition 6.2 (Kastermans, Lempp, Miller (2009))**.** *If $X \in$ High(CR, MLR), then there is an X-computable martingale that succeeds on every non-ML-random.*

Lemma 6.3. *Assume that M is an X-computable martingales and let $r > M(\lambda)$. If there is no non-ML-random on which M is bounded by r, then there is an X-computable martingale that succeeds on every non-ML-random.*

Proof. We may assume, without loss of generality, that r is rational. We use the notation $\sigma[n, \infty]$ to denote the tail of a string $\sigma \in 2^\omega$ with the first $n \in \omega$ bits removed. For example $01001[2, \infty] = 001$. Define an X-computable martingale M^* as follows. We let M^* simulate M until it reaches a string σ_0 such that $M(\sigma_0) > r$. When this happens, M^* sets aside the capital $c_0 = M(\sigma_0) - M(\lambda) > r - M(\lambda)$ and restarts its simulation of M. More precisely, let $n_0 = |\sigma_0|$. Then for $\tau \succcurlyeq \sigma_0$, let $M^*(\tau) = c_0 + M(\tau[n_0, \infty])$. This continues until we reach a string $\sigma_1 \succ \sigma_0$ such that $M(\sigma_1[n_0, \infty]) > r$. If this happens, M^* once again sets aside the spare capital and restarts M. The definition continues in this way, producing a (possibly finite) sequence $\sigma_0 \prec \sigma_1 \prec \sigma_2 \prec \cdots$.

We claim that M^* is the desired X-computable martingale. Let $Z \in 2^\omega$ be non-ML-random. Each time M^* restarts M, it sets aside more than $r - M(\lambda)$ from its working capital. If this happens infinitely often, then it clearly succeeds on Z. But since Z is non-ML-random, no tail of Z is Martin-Löf random, so M^* restarts M infinitely often. □

Proof of Proposition 6.2. By contrapositive, suppose that there is no X-computable martingale that succeeds on every non-ML-random. To show that $X \notin$ High(CR, MLR), we construct a sequence $Z \in 2^\omega$ that is computably random relative to X but not ML-random. Let $\{M_s\}_{s\in\omega}$ be a list of all X-computable martingales. We build a sequence of strings $\sigma_0 \prec \sigma_1 \prec \sigma_2 \prec \cdots$ and a sequence of X-computable martingales $\{M^*_s\}_{s\in\omega}$ such that

$$(\forall n \in (0, |\sigma_s|])\ M^*_s(\sigma_s \upharpoonright n) \leqslant 1 - 2^{-s}$$

and M_s^* is a weighted sum of M_0 through M_{s-1}, for each s. Start by letting M_0^* be the constant 0 martingale and $\sigma_0 = \lambda$. At stage s we have σ_s and M_s^* as above. By the lemma and our assumption, there is a non-ML-random $Y \succ \sigma_s$ such that M_s^* is bounded by $1 - 3 \cdot 2^{-s-2}$. (Note that we are applying the lemma to martingale M defined by $M(\tau) = M_s^*(\sigma_s\tau)$, for all $\tau \in 2^{<\omega}$.) Since Y is not Martin-Löf random, we can find a $\sigma_{s+1} \prec Y$ extending σ_s such that $K(\sigma_{s+1}) \leqslant |\sigma_{s+1}| - s$. Let

$$\varepsilon_s = \frac{2^{-s-2}}{2^{|\sigma_{s+1}|}M_s(\lambda)} \text{ and define } M_{s+1}^* = M_s^* + \varepsilon_s M_s.$$

Note that M_s is bounded by $2^{|\sigma_{s+1}|}M_s(\lambda)$ and M_s^* is bounded by $1 - 3 \cdot 2^{-s-2}$, on every initial segment of σ_{s+1}, so M_{s+1}^* is bounded by $1 - 2^{-s-1}$, as required.

Now let $Z = \bigcup_{s\in\omega} \sigma_s$. We have ensured that Z is not Martin=Löf random. Furthermore, $(\forall n)\ M^*(Z \upharpoonright n) \leqslant 1$, where $M^* = \sum_{s\in\omega} \varepsilon_s M_s$ (which is not X-computable, but we do not need it to be). Therefore, $(\forall n)\ M_s(Z \upharpoonright n) \leqslant 1/\varepsilon_s$ for every s. So Z is computably random relative to X. Hence, $X \notin \mathrm{High}(\mathrm{CR}, \mathrm{MLR})$. □

Corollary 6.4 (Franklin, Stephan and Yu []). *If $X \in \mathrm{High}(\mathrm{CR}, \mathrm{MLR})$, then X computes a Martin-Löf random.*

Proof. Let M be the X-computable martingale that succeeds on every non-ML-random, as guaranteed by the proposition. From M we can compute a $Z \in 2^\omega$ on which M does not succeed. Then $Z \leqslant_T X$ is Martin-Löf random. □

References

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