SCOTT ANALYSIS OF POLISH SPACES

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Abstract. We study the usual notion of Scott rank for Polish metric space. The language consists of distance relations such as, for each rational $q > 0$, a relation $R_{<q}(x, y)$ saying that the distance of x and y is less q. We show that compact spaces have low Scott rank, and that there are spaces of arbitrarily high countable Scott rank.

1. Introduction

We view a metric space (X, d) as a structure for the signature

$$\{R_{<q}, R_{>q} \colon\ q \in \mathbb{Q}^+\},$$

where $R_{<q}$ and $R_{>q}$ are binary relation symbols. The intended meaning of $R_{<q}xy$ is that $d(x, y) < q$. The intended meaning of $R_{>q}xy$ is that $d(x, y) > q$. Clearly, isomorphism is isometry.

We ask to which extent do Polish metric spaces behave like countable structures. Here is one aspect where they do: if $X_0 \cong_p X_1$ then $X_0 \cong X_1$. Here $\cong_p$ denotes partial isomorphism: that there is a non-empty system of back-and-forth relations (see Barwise's article [1]).

For a tuple $\overline{x} \in X^n$ consider the $n \times n$ distance matrix

$$D_n(\overline{x}) = d(x_i, x_j)_{i,j<n}.$$

We often view this matrix as a tuple in $\mathbb{R}^{n^2}$ with the max norm $\|.\|_{\max}$. Note that for any matrix $A \in \mathbb{Q}^{n^2}$ and any positive rational p, there is a quantifier free positive first-order formula $\phi_{A,n,p}(\overline{x})$ in the signature above expressing that $\|D_n(\overline{x}) - A\|_{\max} < p$.

For the reader's convenience, we recall the definitions of α-equivalence and Scott rank of a structure.

Definition 1.1. Let M be an $\mathcal{L}$-structure. We define inductively what it means for finite tuples of same length $\bar{a}, \bar{b}$ from M to be α-equivalent, denoted by $\bar{a} \equiv_\alpha \bar{b}$.

- $\bar{a} \equiv_0 \bar{b}$ if and only if the quantifier-free types of the tuples are the same.
- For a limit ordinal α, $\bar{a} \equiv_\alpha \bar{b}$ if and only if $\bar{a} \equiv_\beta \bar{b}$ for all $\beta < \alpha$.
- $\bar{a} \equiv_{\alpha+1} \bar{b}$ if and only if both of the following hold:
  - For all $x \in M$, there is some $y \in M$ such that $\bar{a}x \equiv_\alpha \bar{b}y$
  - For all $y \in M$, there is some $x \in M$ such that $\bar{a}x \equiv_\alpha \bar{b}y$

The *Scott rank* $\mathrm{sr}(M)$ of a structure M is defined as the smallest α such that $\equiv_\alpha$ implies $\equiv_{\alpha+1}$ for all tuples of that structure. We remark that always $\mathrm{sr}(M) < |M|^+$.

Using a back and forth argument, one shows that a countable structure has Scott rank 0 iff it is ultrahomogeneous.

A metric space X is called ultrahomogeneous if any isometry between finite subsets of X can be extended to an automorphism of X.

Fact 1.2. *A Polish space has Scott rank* 0 *iff it is ultrahomogeneous.*

The nontrivial left-to right direction is proved via a back and forth argument where the $\forall$ player takes points from the countable dense set. The resulting partial isometry can be extended to a full isometry by completeness.

Question 1.3. *(a) Does every Polish metric space have countable Scott rank?*

(b) Can it in fact be described within the class of Polish spaces by an $L_{\omega_1,\omega}$ *sentence?*

Note that (b) is stronger than (a) because even with countable Scott rank, the canonical Scott sentence (see e.g. Barwise [1] again) contains continuum size conjunctions/disjunctions.

We note that (a) has been answered in the affirmative by Michal Ducha, December 2013.

By analogy with countable models, some evidence for the truth of (b) would be obtained as follows. By Gao-Kechris, isometry of Polish spaces (suitably encoded by a real) is $\leq_B$ the orbit equivalence relation E_I obtained by the action of $Iso(\mathbb{U})$ on the Effros Space $F(\mathbb{U})$. Each orbit is Borel (Luzin-Nardzewsky). So the isometry class of a Polish space is Borel. An $L_{\omega_1,\omega}$ description of the space would provide a direct argument for this.

Can we show directly that each isometry class is Borel?

Note that the Scott rank is less than the least hyperprojective α; that is, the least α such that $L_\alpha(\mathbb{R})$ is a model of Kripke-Platek set theory. This is in fact true for Borel structures.

2. Compact metric spaces have Scott rank at most ω.

Let X be a compact metric space. Then for each n, the set $D_n(X) \subseteq \mathbb{R}^n$ is compact. Gromov [3] showed that the space X is described up to isometry by this sequence of compact sets $D_n(X)_{n\in\mathbb{N}}$ (see [2, proof of 14.2.1], but note that our D_n is denoted M_{n-1} there). This shows that isometry of compact spaces is smooth, i.e., Borel reducible to the identity on $\mathbb{R}$.

Theorem 2.1. *Let* X, Y *be compact metric spaces. Suppose that tuples* $\widetilde{a} \in X^p, \widetilde{b} \in Y^p$ *satisfy the same existential positive formulas. Then there is an isometry from* X *to* Y *mapping* $\widetilde{a}$ *to* $\widetilde{b}$*. In particular, each compact metric space* (X, d) *is* $\exists$*-homogeneous.*

Proof. Recall that any isometric self-embedding of a compact metric space is onto (see [2, proof of 14.2.1]). So by the symmetry it suffices to find an isometric embedding of X into Y mapping $\widetilde{a}$ to $\widetilde{b}$.

The following slightly extends the above-mentioned result of Gromov (see [2, Exercise 14.2.3]).

Lemma 2.2. *Suppose that for any* $\epsilon > 0$*, for any* n *and tuple* $\overline{x} \in X^{\overline{n}}$ *there is a tuple* $\overline{y} \in Y^{\overline{n}}$ *such that*

$$\left\| D(\widetilde{a}, \overline{x}) - D(\widetilde{b}, \overline{y}) \right\|_{max} < \epsilon.$$

Then there is an an isometric embedding of X to Y mapping $\widetilde{a}$ to $\widetilde{b}$.

Proof. For each k let $\overline{y}_k$ be a tuple such that $\left\| D(\widetilde{a}, \overline{x}) - D(\widetilde{b}, \overline{y}) \right\|_{max} < 1/k$. Let $\overline{y}$ be the limit of a convergent subsequence of $(\overline{y}_k)_{k\in\mathbb{N}}$. This shows that for each $\overline{x}$ there is $\overline{y}$ such that $D(\widetilde{a}, \overline{x}) = D(\widetilde{b}, \overline{y})$.

We can now proceed almost exactly as in [2, proof of 14.2.1]. Let $(x_i)_{i\in\mathbb{N}}$ be a dense sequence in X. For each $n \in \mathbb{N}$ there are $y_0^n, \dots, y_n^n \in Y$ such that

$$D(\widetilde{a}, x_0, \dots, x_n) = D(\widetilde{b}, y_0^n, \dots, y_n^n).$$

There is $A_0 \subset \mathbb{N}$ with $0 \notin A_0$ such that $z_0 := \lim_{n\in A_0} y_0^n$ exists. There is $A_1 \subset A_0$ with $\min A_0 \notin A_1$ such that $z_1 := \lim_{n\in A_1} y_1^n$ exists. Proceeding that way we obtain a sequence of points $z_n \in Y$ and a descending sequence of sets $A_0 \supset A_1 \supset A_2 \supset \dots$ with $\min A_k \notin A_{k+1}$ such that $z_k = \lim_{n\in A_k} y_k^n$.

Let $A = \{\min A_k \colon k \in \mathbb{N}\}$. Then $A \setminus A_k$ is finite for each k, so $z_k = \lim_{n\in A} y_k^n$ for each k. One uses this to show that $d(x_i, x_j) = d(z_i, z_j)$ for each i, j, and $d(a_r, x_i) = d(b_r, z_i)$ for each r, j. Hence the map $x_i \mapsto z_i$ can be extended to the required isometric embedding of X into Y $\square$

It now suffices to show that if $\widetilde{a} \in X^p, \widetilde{b} \in Y^p$ satisfy the same existential positive formulas, then the hypothesis of the lemma is satisfied. Recall that the formula $\phi_{A,n,p}(\overline{x})$ expresses that $\|D_n(\overline{x}) - A\|_{\max} < p$. Given $\overline{x} \in X^n$ choose a rational $(k+n) \times (k+n)$ matrix A such that

$$\|D(\widetilde{a}, \overline{x}) - A\|_{\max} < p = \epsilon/2.$$

Thus $\exists \overline{x}\, \phi_{A,n+k,p}(\widetilde{a}, \overline{x})$ holds in X. Hence there is $\overline{y} \in Y^n$ such that $\phi_{A,n+k,p}(\widetilde{b}, \overline{y})$ holds in Y. This implies $\left\| D(\widetilde{a}, \overline{x}) - D(\widetilde{b}, \overline{y}) \right\|_{max} < \epsilon$ as required. $\square$

The case $\widetilde{a} = \widetilde{b} = \emptyset$ yields:

Corollary 2.3. *Within the class of compact metric spaces, each member is uniquely described by its existential positive first-order theory.*

Note that a complete metric space is compact iff it is totally bounded, namely, satisfies the $L_{\omega_1,\omega}$ sentence

$$\bigwedge_{q\in\mathbb{Q}^+} \bigvee_{n\in\mathbb{N}} \exists x_0 \dots x_{n-1} \forall y \bigvee_{i<n} d(x_i, y) < q.$$

Thus each compact spaces can be described by an $L_{\omega_1,\omega}$ sentence within the class of Polish metric spaces by.

Corollary 2.4. *The Scott rank of any compact metric space is at most ω.*

Note that ω is an artifact of our particular definition of Scott rank. If we allowed extensions by an arbitrary finite number of elements, the rank would be 1 instead.

3. Countable Polish spaces can have arbitrary countable Scott rank

Countable Polish spaces must have countable Scott rank. In this section, we show that arbitrary large countable ordinal ranks are possible in countable spaces. The examples we construct will even be discrete spaces.

We will inductively construct sub-trees of $\omega^{<\omega}$ with growing Scott ranks. Note that we can view those trees as model theoretic structures, for example in the language of a unary function f with the semantics "$f(a) = b$ if and only if b is the immediate predecessor of a". Also, those trees can be regarded as metric spaces with the metric induced by the usual one on $\omega^{<\omega}$. It is easy to see that the relations $\equiv_\alpha$ (and hence the notion of Scott rank) is the same in both the tree structure and the metric space structure.

We denote finite sequences from $\omega^{<\omega}$ by $\langle n_1, n_2, \ldots, n_k\rangle$ and the concatenation of two sequences s, t by $s^\frown t$. Now we proceed to the definition of $T_n^\alpha \subset \omega^{<\omega}$ ($n \leq \omega$, $\alpha < \omega_1$) by induction over α.

Definition 3.1. For any $n \leq \omega$, let T_n^0 be the collection of the empty sequence and the sequences $\langle a\rangle$ for all $a < n$.

Now suppose we have defined all the T_n^α for some $\alpha < \omega_1$. Let $T_n^{\alpha+1}$ be the collection of

- the empty sequence
- all sequences of the form $\langle 2a\rangle^\frown s_a$ where $a < \omega$ and $s_a \in T_a^\alpha$
- for any $a < n$, all sequences of the form $\langle 2a+1\rangle^\frown s$ with $s \in T_\omega^\alpha$

Finally, suppose α is a countable limit ordinal and that the T_n^β are already defined for all $n \leq \omega$ and $\beta < \alpha$. Fix any bijection $b : \alpha \times \omega \to \omega$ and define T_n^α as the collection of

- the empty sequence
- all sequences of the form $\langle a\rangle^\frown s_a$ where $a = b(c, d)$ for some $c < \alpha$ and $d \leq n$ and $s_a \in T_\omega^c$

Informally speaking, at each stage of the construction we glue infinitely many of the already defined trees into a new one. $T_n^{\alpha+1}$ contains all T_k^α for $k < \omega$ exactly once and T_ω^α exactly n times. At limit stages, we amalgamate $n + 1$ copies of each T_ω^β ($\beta < \alpha$).

Inductively, we see that none of the constructed trees has an infinite path and that all of them, seen as metric spaces, are discrete and complete (thus they are Polish). The T_m^0 are homogeneous, so they have Scott rank 0, and we can verify inductively that T_n^α has Scott rank $\alpha \cdot \omega$ for all $\alpha < \omega_1$ and all $n \leq \omega$.

References

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[3] Misha Gromov. *Metric structures for Riemannian and non-Riemannian spaces*. Modern Birkhäuser Classics. Birkhäuser Boston Inc., Boston, MA, english edition, 2007. Based on the 1981 French original, With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.