Separating classes of groups by first–order sentences

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Abstract. For various proper inclusions of classes of groups $\mathcal{C} \subset \mathcal{D}$, we obtain a group $H \in \mathcal{D}$ and a first–order sentence φ such that $H \models \varphi$ but no $G \in \mathcal{C}$ satisfies φ. The classes we consider include the finite, finitely presented, finitely generated with and without solvable word problem, and all countable groups. For one separation, we give an example of a f.g. group, namely $\mathbb{Z}_p \wr \mathbb{Z}$ for some prime p, which is the only f.g. group satisfying an appropriate first–order sentence. A further example of such a group, the free step-2 nilpotent group of rank 2, is used to show that true arithmetic $\mathrm{Th}(\mathbb{N}, +, \times)$ can be interpreted in the theory of the class of finitely presented groups and other classes of f.g. groups.

1. Introduction

To what extent are properties of a group expressible by a sentence in the first–order language for groups? While some properties, like being abelian or nilpotent of step 2, are first–order by their very definition, most properties cannot even be described by an axiom system (including noneffective ones). Examples of properties of the second kind are: being a torsion group, being finitely generated and being free. Throughout this paper we restrict our attention to countable groups. The latter three properties are easily seen to be non-axiomatizable (even within the countable groups) using the compactness theorem from mathematical logic. However, the following problem could still have a solution: given a "natural" class of groups $\mathcal{C}$ and a *particular* group $H \notin \mathcal{C}$, find a first–order sentence φ such that $H \models \varphi$, but no group in $\mathcal{C}$ satisfies φ. (If this is possible for each countable $H \notin \mathcal{C}$, then $\mathcal{C}$ is axiomatizable within the countable groups.) In our results, H will be a member of a larger natural class of groups $\mathcal{D}$, so the existence of φ implies that first–order logic can distinguish between $\mathcal{C}$ and $\mathcal{D}$, i.e. $\mathrm{Th}(\mathcal{D})$ is a proper subtheory of $\mathrm{Th}(\mathcal{C})$. (Recall that the *theory* $\mathrm{Th}(\mathcal{E})$ of a class $\mathcal{E}$ of groups is the set of first–order sentences which hold in all the members of $\mathcal{E}$.) We illustrate this by giving an example of such a separation, where $\mathcal{C}$ is the class of finite groups, and $\mathcal{D}$ is the class of finitely presented (f.p.) groups with solvable word problem.

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PROPOSITION 1.1. *There is an existential sentence φ_1 which holds in a finitely presented group H_1 with solvable word problem, but fails in all finite groups.*

Proof. Higman (see [**17**, Prop. I.6]) found an example of an infinite f.p. group which has no nontrivial finite quotients: let $H_1 = \langle a_0, a_1, a_2, a_3 \mid r_0, r_1, r_2, r_3\rangle$, where $r_i = a_i^{a_{i+1}} a_i^{-2}$ $(i = 0, 1, 2, 3)$, and $i + 1$ is computed modulo 4. It is easy to show that H has solvable word problem. Let φ_1 be the existential sentence

$$\exists x_0 \ldots \exists x_3 (x_0 \neq 1 \ \& \ \textstyle\bigwedge_{i=0,1,2,3} r_i(x_0, \ldots, x_3) = 1).$$

Then $H \models \varphi_0$. If $G \models \varphi_0$, choose witnesses $b_0, \ldots, b_3$ for φ_0 in G (so that $b_0 \neq 1$). Then $a_i \mapsto b_i$ describes a nontrivial homomorphism $H \to G$, whence G embeds a nontrivial quotient of H. Thus G is infinite. $\diamondsuit$

In this article we show such separation results for most natural classes of countable groups which have been considered in group theoretical investigations related to logic. This is part of a more general endeavor to understand the expressive power of first–order logic in group theory. Here are some results in this direction. Let us say a f.g. group H is *quasi-axiomatizable* if, whenever G is a *f.g.* group which has the same theory as H, then $G \cong H$. The restriction to finitely generated G is essential, since the theory of any infinite n–generated group H has a countable model not isomorphic to H (i.e., the theory of H is not ω–categorical). This follows from the the fact the theory has infinitely many $n + 1$–types, using a basic result of model theory [**5**, Thm 7.3.1],

It can be verified that all f.g. abelian groups are quasi-axiomatizable. Moreover, Szmielew [**18**] characterized pairs of (not necessarily f.g.) abelian groups which have the same theory. Oger [**13**] proved that two f.g. nilpotent groups G, H have the same theory iff $G \times \mathbb{Z} \cong H \times \mathbb{Z}$. Earlier, Hirshon [**3**] had considered the question for which groups A $\mathbb{Z}$ can be cancelled from a direct product $A \times \mathbb{Z}$ in the sense that if $A \times \mathbb{Z} \cong B \times \mathbb{Z}$, then $A \cong B$. He proved that this is the case for any group A which is f.g. torsion free step-2 nilpotent, but gave a counterexample where A is f.g. torsion free step-3 nilpotent. Then, by Oger's result, torsion free step-2 nilpotent groups are quasi-axiomatizable, but there exists a f.g. torsion free step-3 nilpotent group which is not.

Kharlampovich and Myasnikov [**9**] proved that all nonabelian free groups have the same first–order theory, thereby answering a long–open question of Tarski. (Thus, f.g. free groups are not quasi-axiomatizable). While these results, especially the last one, expose a weakness in the expressiveness of first–order logic, our separation and quasi-axiomatizability results go in the other direction.

We introduce the classes to be separated. Recall that a group is recursively presented if it has a presentation $\langle x_0, x_1, \ldots | R\rangle$ where R is a recursively enumerable set of relators (and $x_1, x_2, \ldots$ is a finite or infinite list of generators). The *word problem* for such presentation is the normal closure of R in $F(x_0, x_1, \ldots)$, which is also recursively enumerable. If the set of generators is finite, then the word problem is independent of the particular presentation (up to recursive isomorphism).

In the following list, for $0 \leq i < j \leq 5$ a class included under (i) is a proper subclass of a class under (j). Moreover the classes $\mathcal{C}_{2a}, \mathcal{C}_{2b}$ are incomparable.

LIST 1.2.

0. $\mathcal{C}_0$ = *finite groups*
1. $\mathcal{C}_1$ = *finitely presented groups with solvable word problem*

2. $\mathcal{C}_{2a}$ = *finitely presented groups*
   $\mathcal{C}_{2b}$ = *finitely generated groups with solvable word problem*
3. $\mathcal{C}_3$ = *f.g. and recursively presented groups*
4. $\mathcal{C}_4$ = *finitely generated groups*
5. $\mathcal{C}_5$ = *countable groups.*

For each class $\mathcal{C}_x$, $x \neq 3$, we will explicitly describe a sentence φ_x and a group $H_x \models \varphi_x$ such that no group in a class from List 1.2 not containing $\mathcal{C}_x$ satisfies φ_x. The groups H_1, H_{2b} and H_5 are natural groups. H_1 was already obtained in Proposition 1.1. H_{2a} is a group described by Miller [**6**]. His group is f.p. and none of its nontrivial quotients have a solvable word problem.
For H_{2b} we use the restricted wreath product $H = \mathbb{Z}_p \wr \mathbb{Z}$, where p is a prime. We follow a mostly algebraic approach, but we use the notion of first–order definability with parameters.
We obtain H_4 using some effective model theory. H_4 is a f.g. group encoding via equations with constants a graph which enjoys a first–order property (in the language of graphs without equality) shared by no recursive graph.
H_5 is Hall's universal locally finite group [**2**], the unique countable locally finite group which embeds every finite group and has the further property that any two isomorphic finite subgroups are conjugate. The proof makes use of definability with parameters of finite subsets in this group.
When we consider $\mathcal{C}_{2b}$, we give in fact a sentence φ_{2b} which, together with the (non–first order) information that the group is f.g., describes H_{2b} completely! Thus, $H = H_{2b}$ has the following property.

Definition 1.3. *A f.g. group H is* quasi-finitely axiomatizable *if there is a first–order sentence φ such that $H \models \varphi$, and whenever G is a f.g. group such that $G \models \varphi$, then $G \cong H$.*

This refines the concept of quasi-axiomatizability introduced above. Clearly, all finite groups are quasi-finitely axiomatizable. To our knowledge, H_{2b} is the first known example of an infinite quasi-finitely axiomatizable group. In Section 5, we provide a further example, the free step-2 nilpotent group of rank 2, which coincides with the subgroup $\mathrm{UT}_3^3(\mathbb{Z})$ of $\mathrm{GL}_3(\mathbb{Z})$ consisting of upper triangular matrices with 1's on the main diagonal.
No infinite finitely generated abelian group A is quasi-finitely axiomatizable. To see this, in the terminology of Hodges [**5**, Thm A.2.7.], each sentence φ is equivalent (over the axioms of abelian groups) to a Boolean combination of Szmielew invariant sentences ψ_i which involve primes p_i. Let q be a prime greater than those primes. Then $A \models \psi_i \Leftrightarrow A \times \mathbb{Z}_q \models \psi_i$ for each i, but $A \ncong A \times \mathbb{Z}_q$.
Recall that one way to measure the complexity of formulas in first order logic is to look at the number of quantifier alternations. A Σ_{n+1}–formula has the form $\exists \ldots \exists \forall \ldots \forall \ldots \psi$, with n alternations of quantifiers and ψ quantifier free. For Π_{n+1}–formulas, exchange $\exists, \forall$ in the definition. Unless otherwise stated, we work in the first–order language for groups with symbols for multiplication and inverse and a constant symbol 1 for the neutral element. The sentences φ_x used to separate the classes in List 1.2 up to and including $\mathcal{C}_4$ are all in Σ_3. Thus we have proved that the theories of those classes have distinct Π_3–fragments.
The computational complexity of theories in the language of group theory is compared via many–one reductions: $T \leq_m S$ if there is a computable function F on

sentences such that $\varphi \in T \Leftrightarrow F(\varphi) \in S$. We work towards classifying the computational complexity of the theories $\mathrm{Th}(\mathcal{C})$, where $\mathcal{C}$ is a class from List 1.2. All theories are known to be undecidable, as a consequence of results in [**11**]. An obvious upper bound for theories of the classes under items (1)–(3) is the complexity of true arithmetic $\mathrm{Th}(\mathbb{N}, +, \times)$. Using the fact that $\mathrm{UT}_3^3(\mathbb{Z})$ is quasi-finitely axiomatizable and encodes a copy of the ring of integers, we will show in Section 5 that this upper bound is the actual complexity.
$\mathrm{Th}(\mathcal{C}_0)$ is co–recursively enumerable, and $\mathrm{Th}(\mathcal{C}_5)$ is recursively enumerable. From the usual undecidability proofs, one can derive that the complexity of the first theory is the same as the complexity of the complement of halting problem from recursion theory, and the complexity of the second theory is the same as the one of the halting problem.
We conclude the introduction by describing a further relatively easy example, the group H_{2a}. The argument generalizes the one used in the proof of Proposition 1.1.

PROPOSITION 1.4. *There is an existential sentence φ_{2a} which holds in some finitely presented group $M = H_{2a}$, but fails in all groups which have a presentation with a solvable word problem.*

Proof. Using the Higman Embedding Theorem, Miller [**6**] constructed a finite presentation $\langle a_0, \dots, a_n | r_0, \dots, r_k \rangle$ of a group M such that every nontrivial quotient of M has an unsolvable word problem. Let φ_{2a} be the sentence

$$\exists x_0 \dots \exists x_n (x_0 \neq 1 \ \& \ \textstyle\bigwedge_{0 \leq i \leq k} r_i(x_0, \dots, x_n) = 1).$$

Then $M \models \varphi_1$. If G is a group such that $G \models \varphi_1$, then, by the same argument as in the proof of Proposition 1.1, G embeds a nontrivial quotient of M. Therefore any presentation of G has an unsolvable word problem. ♢

We write a^b for $b^{-1}ab$. The commutator $[a, b]$ is defined to be $a^{-1}b^{-1}ab$.

2. F.g. with solvable word problem versus f.p.

In this section we describe H_{2b} and φ_{2b}, which depend on a fixed prime p. The restricted wreath product $\mathbb{Z}_p \wr \mathbb{Z}$ has the presentation

$$H = H_{2b} = \langle a, d \mid a^p, \ [v_r, v_s] \ (r, s \in \mathbb{Z}, r < s) \rangle, \tag{1}$$

where $v_r = a^{d^r}$. The following is well-known.

PROPOSITION 2.1. *$\mathbb{Z}_p \wr \mathbb{Z}$ is not finitely presented.*

Proof. If H is f.p., then the relators in (1) are in the normal closure of a finite subset. However, if $m, n \in \mathbb{Z}$ and $m < n$, then $[v_m, v_n]$ is not in the normal closure of

$$\{a^p, \ [v_r, v_s] \ (m \leq r < s \leq n)\}. \tag{2}$$

For, after conjugating this set of relators with a power of d, we can assume hat $m = 0$. Let $G = \mathbb{Z}_p^{n+1}$ be generated by $\{a_0, \dots, a_n\}$, and consider the HNN extension $\widetilde{G} = \langle G, d \mid a_i^d = a_{i+1} \ (0 \leq i < n) \rangle$. Then the map $F(a, d) \to \widetilde{G}$ given by $a \mapsto a_0, d \mapsto d$ sends all relators in (2) to 1, but, by Britton's Lemma (see [**16**, Lemma 12.4]), it maps $[v_0, v_n]$ to a non-identity element. ♢

We need a simple lemma. Recall that for groups G, B, E one writes $G = B \rtimes E$ if $BE = G$, $B \lhd G$ and $E \cap B = \{1\}$.

LEMMA 2.2. *If B, E are abelian and $G = B \rtimes E$, then $G' = [B, E]$.*

Proof. Suppose $u, v \in B$ and $x, y \in E$. Then $[ux, by] = [u^x, y][x, v^y]$ by the usual commutator rules and since E, B are abelian. Both commutators on the right hand side are in $[B, E]$. ♢

THEOREM 2.3. *Let $H = H_{2b} = \mathbb{Z}_p \wr \mathbb{Z}$, where p is a prime number. Then H is quasi-finitely axiomatizable via a Σ_3–sentence.*

COROLLARY 2.4. *There is a Σ_3–sentence φ_{2b} which holds in H_{2b}, a finitely generated group with solvable word problem, but fails in all finitely presented groups.* ♢

Proof of Theorem 2.3 We begin by analyzing H. Denote the base group of the wreath product by A. Thus $H = A \rtimes C$, where $A = \bigoplus_{z \in \mathbb{Z}} \mathbb{Z}_p^{(z)}$, $\mathbb{Z}_p^{(z)}$ is a copy of $\mathbb{Z}_p$, $C = \langle d \rangle$ with d of infinite order, and d acts on A by shifting, i.e. $(\mathbb{Z}_p^{(z)})^d = \mathbb{Z}_p^{(z+1)}$. Let a be a generator of $\mathbb{Z}_p^{(0)}$.
In the following, u, v denote elements of A and x, y elements of C. We use additive notation in A. In particular, $[u, x] = u^x - u$. The first statement of the following Lemma is a special case of [**15**, Lemma 2].

LEMMA 2.5.
(i) *The set of commutators of H forms a subgroup. (In fact, $H' = \{[u, d] : u \in A\}$. In particular $H' \leq A$ and thus, H is metabelian and each element of H' has order p.)*
(ii) *The element a has order p. A is the (internal) direct product of H' and $\langle a \rangle$. In particular, $|A : H'| = p$.*
(iii) *C is abelian, $H = A \rtimes C$, and $C - \{1\}$ acts on $A - \{1\}$ without fixed points.*
(iv) *$C = C_H(d)$, the centralizer of d in H.*

Proof. (i) By Lemma 2.2, $H' = [A, C]$. Now $[u, d][v, d] = [u + v, d]$ and $[-u, d] = -[u, d]$, so that $\{[u, d] : u \in A\}$ is a subgroup. Moreover, $[u, d^{n+1}] = [u^{d^n} + \ldots + u^d + u, d]$ and $[u, x^{-1}] = [-u^{x^{(-1)}}, x]$ for all $x \in C$, so that this subgroup contains all commutators $[u, x]$.
(ii) We first show $H' \neq A$. Let $\overline{a}$ be a generator of a cyclic group P of order p. The map $f : F(a, d) \to P \times C$ given by $a \mapsto \overline{a}, d \mapsto d$ maps all relators in (1) to 1, and therefore induces a map $\overline{f} : H \to P \times C$. Since the kernel of this map is properly included in A and $P \times C$ is abelian, H' is properly included in A.
Recall that $v_z = a^{d^z}$. Let $w_z = v_z - a = [a, d^z]$, and let $W = \langle \{w_z : z \neq 0\} \rangle$. Then $W + (a) = A$. Since $W \leq H'$, a has order p and H' is a proper subgroup of A, this proves (ii).
(iii) Immediate.
(iv) We only need to show $C_H(d) \subseteq C$. Suppose $[ux, d] = 1$, where $u \in A, x \in C$. Since $[ux, d] = [u, d]^x [x, d] = [u, d]^x$, this implies $[u, d] = 1$ and hence $u = 1$, because d acts fix point free. ♢

We refer to the situation $H = \mathbb{Z}_p \wr \mathbb{Z} = A \rtimes C$ as the *standard case*. We formulate sufficiently many first–order properties of H such that H is the only f.g. group with these properties. The main statements of Lemma 2.5 can be expressed in first-order logic, using a, d as parameters. Let μ be a sentence asserting about a group G that the product of any two commutators is again a commutator, and that the subgroup formed by the commutators is an abelian group all whose non-identity elements have order p. Then G' is definable in G by the existential formula $\exists v, w\ x = [v, w]$. The desired sentence is

$$\varphi_{2b} \ \equiv \ \mu \ \& \ \exists a \exists d \ [\beta_1(a, d) \ \& \ \ldots \ \& \ \beta_4(a, d)], \tag{3}$$

where the formulas β_i will be determined next (think of the *variables* a, d playing the same role as the *parameters* $a, d \in H$ in the standard case). The formulas β_i are boolean combinations of Π_2 formulas, so that φ_{2b} is Σ_3.

(1) $\beta_1(a, d)$ asserts that $a \neq 1$, $a^p = 1$ (so that $\langle a \rangle$ is definable from a) and the subgroups G', $\langle a \rangle$ generate their direct product (i.e, $G' \cap \langle a \rangle = \{1\}$ and $[G', a] = \{1\}$). As in the standard case, write $A = G'\langle a \rangle$. Then A is an abelian group of exponent p, i.e. a $\mathbb{Z}_p$–vector space, and $|A : G'| = p$. Moreover A is definable from a via a Σ_1–formula.
(2) $\beta_2(a, d)$ expresses that $d \neq 1$ and (iii) of Lemma 2.5 holds. Write $C = C_G(d)$. Then C is definable from d by the formula $[x, d] = 1$. The formula $\beta_2(a, d)$ expresses as well that C is abelian, $A \cap C = \{1\}$ and $AC = G$ (note that $A \lhd G$ since $G' \leq A$). We have ensured that $G = A \rtimes C$. Moreover, $\beta_2(a, d)$ asserts that $C - \{1\}$ acts on $A - \{1\}$ without fix points.
(3) $\beta_3(a, d)$ asserts that no element in $C - \{1\}$ has order $< p$.
(4) $\beta_4(a, d)$ asserts that $C/2C \cong \mathbb{Z}_2$, which can be formulated as follows: $\exists x \in C \forall y \in C \ \neg y^2 = x \ \& \ \forall x_0, x_1, x_2 \in C \exists y \in C \bigvee_{0 \leq i < j \leq 2} \ x_i y^2 = x_j$

In the following we assume G is a f.g. group satisfying φ_{2b} via witnesses $a, d \in G$, and we write $A = G'\langle a \rangle$ and $C = C_G(d)$. We refer to this situation as the *general case*.

LEMMA 2.6. *C is infinite cyclic.*

Proof. We first prove that C is torsion free. If $t \in C - 1$ has order r, then all the orbits in A under the action of t have size r, for if some orbit has size $s < r$, then $t^s \in C - \{1\}$ has a fix point.
Let U be the (finite) t-invariant subspace of A generated by a. Then $U \not\subseteq G'$. Since $|A : G'| = p$, this implies $|U : U \cap G'| = |UG' : G'| = p$. Let n be the dimension of U as a $\mathbb{Z}_p$-vector space. Under the action of t on U, $U - \{0\}$ is partitioned into orbits of the same length r, so $r | p^n - 1$. We show $r < p$. We can assume $n > 1$. Since $G' \cap U$ is T–invariant and has dimension $n - 1$, $r | p^{n-1} - 1$. But $p(p^{n-1} - 1) + (p - 1) = p^n - 1$, and therefore$(p^n - 1, p^{n-1} - 1) = (p^{n-1} - 1, p - 1)$, so $r < p$. Since we assume that $G \models \beta_3(a, d)$, this implies $t = 1$, contradiction.
Since G is f.g., so is C. For a f.g. torsion free abelian group C, the rank of C is the dimension of $C/2C$ as a $\mathbb{Z}_2$– vector space. Since $G \models \beta_4(a, d)$ we may conclude that C is infinite cyclic. ♢

In the general case, d may fail to generate C. So choose a generator $c \in C$. Recall that the group ring $\mathbb{Z}_p C$ consists of expressions $P = \sum_{i=r}^{s} \alpha_i c^i$, where $\alpha_i \in \mathbb{Z}_p$ and

$r, s \in \mathbb{Z}, r \leq s$. Because this ring is the ring of fractions of the usual polynomial ring $\mathbb{Z}_p[c]$ by the multiplicative subset $\{c^n : n \geq 0\}$, it is a principal entire ring (see Lang [**10**, Section II.3 and Exercise 4]). The action of C on A turns A into a $\mathbb{Z}_pC$–module (C–module for short), where the scalar multiplication is given by $u^P = \sum_{i=r}^{s} \alpha_i u^{c^i}$. We will show that this C–module is f.g. free. By Lang [**10**, Thm XV.2.2], for modules over principal entire rings, the following suffices (the first part is well–known).

LEMMA 2.7. (i) *A is finitely generated as a C–module.*

(ii) *The C–module A is* torsion free*, i.e. for each nonzero $P \in \mathbb{Z}_pC$ and $u \in A$, $u^P = 0$ implies $u = 0$.*

Proof. (i) Let $b_1, \ldots, b_m$ be a finite generating set for G. Each element of G is of the form uc^z, where $u \in A, z \in \mathbb{Z}$. Then, after adding c to this generating set, we can assume that $b_i \in A$. If $u \in A$, then $u = 1$ in G/A, so that u is a sum of terms of the form $b_i^{c^z}$. Collecting the terms with the same i, we obtain $u = \sum_{i=1}^{m} b_i^{P_i}$ with appropriate $P_i \in \mathbb{Z}_pC$. Thus $b_1, \ldots, b_m$ generate the C–module A.
(ii) Suppose $u \in A - \{0\}, P = \sum_{i=r}^{s} \alpha_i c^i \in \mathbb{Z}_pC - \{0\}$ and $u^P = 0$. Then $r < s$. We can assume that the leading coefficient in P is -1, so that, for some $s \in \mathbb{Z}, r \leq s$, $u^{c^s} = \sum_{j=r}^{s-1} \alpha_j u^{c^j}$, where $\alpha_j \in \mathbb{Z}_p$. But then, by induction over all $w \geq s$, u^{c^w} is in the finite subspace generated by $u^{c^r}, \ldots, u^{c^{s-1}}$, contrary to the fact that each power of c acts fix point free on $A - \{0\}$. ♢
Clearly in the standard case the C–module A is generated by a and hence has rank 1. To establish $G \cong H$, it then suffices to prove that the free C–module A has rank 1 in the general case. Let $A = B_1 \oplus \ldots \oplus B_n$ be a decomposition of A into a direct sum of free C–modules of rank 1. Then $B_iC = B_i \rtimes C \cong H$, so that $|B_i : [B_i, C]| = p$ by Lemma 2.5. By Lemma 2.2, $G' = [A, C]$ so that $G' = \bigoplus_i [B_i, C]$. Thus $|A : G'| = |\bigoplus_i B_i : \bigoplus_i [B_i, C]| > p$ unless $n = 1$. Hence A has rank 1 and $G \cong H$. ♢

3. F.g. versus recursively presented

In this section we obtain H_4 and φ_4. We use notions from effective model theory. H_4 turns out to have a Δ_2^0–copy.

THEOREM 3.1. *There is a Σ_3–sentence φ_4 which holds in some finitely generated group H_4 but fails in all recursively presented groups.*

Proof. We make use of undirected irreflexive graphs, simply called *graphs* here. Usually the structures encountered in this proof have the integers $\mathbb{Z}$ or a finite interval $[0, \ldots, n]$ as their domains. Such a structure in a finite relational language is *recursive* if those relations are computable.
We first describe a sentence α which holds in some graph $(\mathbb{Z}, E)$, but fails in all recursive graphs. Thereafter we encode $(\mathbb{Z}, E)$ in a f.g. group H_4, using for the coding only equations with constants in H_4, but coding both the edge relation and its complement. The sentence φ_4 expresses that, via some list of constants, a graph satisfying α is coded. This sentence holds in H_4, but fails in all recursively presented groups. It will matter that the sentence α does not contain the equality symbol.

LEMMA 3.2. *There is a Σ_3–sentence α in the language of graphs without equality which holds in some graph $(\mathbb{Z}, E)$ but fails in all recursive graphs.*

Proof. We obtain α in two steps.
1. Let τ be the signature consisting of a ternary relation symbol R and an equality symbol $\approx$. We obtain a τ–structure satisfying a Σ_3–sentence β which fails in each recursive τ–structure. We could work with Tennenbaum's Theorem (see [**8**]). However, we prefer to reconsider Miller's group M from Proposition 1.4, now viewed as a τ–structure, where R is interpreted as the graph of the group operation (and $\approx$ is the usual equality). Sentences in the restricted first–order language with only one binary function symbol for the group operation can directly be translated into sentences in the first–order language over τ. For instance, associativity becomes

$$\forall x \forall y \forall z \forall w \exists u (Rxyu \ \& \ Ruzw) \leftrightarrow \exists u (Ryzu \ \& \ Rxuw),$$

and the existence of neutral element and right inverse is now

$$\exists z [\forall u Ruzu \ \& \ \forall x \exists y Rxyz].$$

Clearly the function symbol for inverse and the constant symbol 1 can be eliminated from a formula in the full group language. Then, the sentence φ_{2a} from Proposition 1.4 can be expressed in $L(\tau)$ and becomes a Σ_2-sentence $\widetilde{\varphi}_{2a}$.
Let $\beta \in L(\tau)$ be the Σ_3–sentence which is the conjunction of $\forall x \forall y \exists^{=1} z \ Rxyz$ (R is the graph of a binary function), the group axioms as sentences in $L(\tau)$ and $\widetilde{\varphi}_{2a}$. Then $M \models \beta$, but no recursive structure $(\mathbb{Z}, R)$ is a model of β: otherwise it would be a group G, and we would obtain a presentation of G with computable word problem, contrary to Proposition 1.4.
2. We use the coding by Σ_1–formulas of a structure $\mathbf{A}$ in a finite relational language with equality in a graph (V, E) given in the proof of Nies [**12**, Thm 4.2]. (There, the coding is used to show that the Σ_2–theory of the class of finite undirected graphs in the language without equality is undecidable.) Here, the finite relational language is $L(\tau)$. The coding in [**12**] uses Σ_1 formulas in the language $L^-(E)$ with a binary relation symbol E, but *without* equality. The domain of $\mathbf{A}$ is represented by all elements satisfying a formula $\mathrm{Cyc}_{3,4}(x)$ (saying that x is in the intersection of a 3-cycle and a 4-cycle), which we will denote here by $\mathrm{Cyc}(x)$. For each n–ary relation symbol S in τ (i.e. R and $\approx$), there are Σ_1–formulas $\varphi_S(x_1, \ldots, x_n)$ and $\varphi_{\neg S}(x_1, \ldots x_n)$ intended to code the relation $S^{\mathbf{A}}$ and its complement on $\{x : \mathrm{Cyc}(x)\}$. A "correctness condition" γ expresses that

- the formulas φ_S, $\varphi_{\neg S}$ define complementary relations on the nonempty set $\{x : \mathrm{Cyc}(x)\}$, for $S \in \{R, \approx\}$
- $\varphi_\approx$ defines an equivalence relation compatible with the relation defined by φ_R

If $Q = (\mathbb{Z}, E) \models \gamma$, then a τ–structure is encoded on $D/_P$, where

$$D = \{a : Q \models \mathrm{Cyc}(a)\} \text{ and } P = \{\langle a, b\rangle : a, b \in D \ \& \ Q \models \varphi_\approx(a, b)\}.$$

Moreover, if Q is recursive, then this structure has a recursive copy: since all the coding formulas are Σ_1, all the relevant sets and relations are recursively enumerable. Let X be $\mathbb{Z}$ or an initial segment of $\mathbb{N}$, and choose a computable $f : X \to D$ whose image is a system of representatives for P. Since $Q \models \gamma$, the preimage under f of $\{\langle a, b\rangle : a, b \in D \ \& \ Q \models \varphi_R(a, b)\}$ is computable.
Now let α be the conjunction of γ and the sentence saying that the τ–structure encoded on $D/_P$ via φ_R satisfies β. Let $Q = (\mathbb{Z}, E)$ encode M, viewed as a τ–structure. Then $Q \models \alpha$, but no recursive graph satisfies β. Moreover, translating

β the same way as in [**12**, Thm 4.2], i.e. using phi_S or $\neg\varphi_{\neg S}$ when appropriate to replace occurences of Sxy, α is Σ_3. $\diamondsuit$

We code an arbitrary undirected graph $Q = (\mathbb{Z}, E)$ into an appropriate group L_Q, using for the coding only equations with constants in L_Q. The group will be an HNN–extension of $F(a, b, c, d)$ by two stable letters r, s.
We think of a^z as representing $z \in \mathbb{Z}$. Let $v_z = b^{a^z}$ $(z \in \mathbb{Z})$, and write

$$t(u, v) = ucvducv.$$

For distinct integers $x, y \in \mathbb{Z}$, let $w_{x,y} = t(v_x, v_y)$. Then the words $w_{x,y}$ freely generate a subgroup of F, since d never cancels in a product of two distinct such words or there inverses. The word $w_{x,y}$ will be used to represent the ordered pair a_x, a_y. Consider the HNN-extension

$$L_Q = \langle F, r, s | w_{x,y}^r = w_{x,y} \text{ (if } Exy); w_{x,y}^s = w_{x,y} \text{ (if } \overline{E}xy)\rangle,$$

where $\overline{E}$ is the complement of the edge relation. Let $\mathbf{c} = (a, b, c, d, r, s)$. We intend to use the equations

$$\varphi_U(u; \mathbf{c}) \equiv [u, a] = e,\ \varphi_E(u_0, u_1; \mathbf{c}) \equiv t(u_0, u_1)^r = t(u_0, u_1), \text{ and}$$
$$\varphi_{\overline{E}}(u_0, u_1; \mathbf{c}) \equiv t(u_0, u_1)^s = t(u_0, u_1)$$

to code the domain of Q, E, and the complement of E, respectively (the last will ensure that Q is recursive if L_Q is recursively presented).
We need to show that we did not extend the centralizer of a when passing from $F(a, b, c, d)$ to L_Q, so that φ_U encodes the intended domain.

LEMMA 3.3. *The centralizer of a in L_Q is $\langle a\rangle$.*

Proof. Let U_E be the subgroup generated by the set $\{w_{x,y} : Exy\}$, and let $U_{\overline{E}}$ be the subgroup generated by $\{w_{x,y} : \overline{E}xy\}$. By a special case of Britton's Lemma (see [**16**, Lemma 12.6]), if a word in the generating symbols for L_Q represents 1 in L_Q, then it contains a subword $t^{-\epsilon}ut^{\epsilon}$, $u \in U_E \cup U_{\overline{E}}, \epsilon \in \{1, -1\}, t \in \{r, s\}$, called a *pinch* (the words might fail to be reduced as written). Suppose that in L_Q, $a^{-1}w^{-1}aw = e$, where w is a word in the generators for L_Q. We can assume w does not contain a pinch. Then there must be a pinch "across" the displayed occurrence of a. So w has the spelling $pt^{\epsilon}v$ and w^{-1} has the spelling $ut^{-\epsilon}q$ where q, p are words in $F(a, b, c, d)$ and $t \in \{r, s\}$ (so that the pinch is $t^{-\epsilon}qapt^{\epsilon}$). Then $q^{-1} = p$. But in $F(a, b, c, d)$, no conjugate of a is in U_E or $U_{\overline{E}}$, since the sum of all exponents of occurrences of a in a word describing an element of these subgroups is 0. $\diamondsuit$
For any list of constants $\mathbf{c} = (a, b, c, d, r, s)$ in a group G, a graph $(C_G(a), E)$ is encoded via the formulas φ_U, φ_E, where $C_G(a)$ is the centralizer $\{u : \varphi_U(u, \mathbf{c})\}$ and $E = \{u, v : u, v \in C_G(a)\ \&\ \varphi_E(u, v; \mathbf{c})\}$. Let $\delta(\mathbf{c})$ be the formula expressing that on $C_G(a)$, φ_E and $\varphi_{\overline{E}}$ define complements. Let φ_4 be a sentence expressing that, for some list $\mathbf{c} = (a, b, c, d, r, s)$, $\delta(\mathbf{c})$ holds and the graph encoded on the centralizer of a satisfies the sentence α. Since the encoding is via equations, φ_4 is Σ_3.
First let $Q = (\mathbb{Z}, E)$ be the graph described above which encodes Miller's group M, and let $H_4 = L_Q$. Then $L_Q \models \varphi_4$. For choose $\mathbf{c}$ as above. By Lemma 3.3, $L_Q \models \varphi_U(u, \mathbf{c}) \Leftrightarrow u \in \langle a\rangle$. Then, by the definition of L_Q, for $x, y \in \mathbb{Z}$, $(x, y) \in E \Leftrightarrow L_Q \models \varphi_E(v_x, v_y; \mathbf{c})$. So the encoded graph is a copy of Q.

Now assume for a contradiction that G is a group with a recursive presentation $\langle F|R\rangle$, F an effectively given countable free group, and $G \models \varphi_4$ via a list of witnesses $\mathbf{c} = (a, b, c, d, r, s)$. Under the natural epimorphism $p : F \to G$, each relation on G has a preimage on F. Moreover, if the relation is defined by an equation with constants in G, then the preimage is recursively enumerable. Thus, taking the preimage of the centralizer $C_G(a)$, $\{\langle u, v\rangle : \varphi_E(u, v; \mathbf{c})\}$ and $\{\langle u, v\rangle : \varphi_{\overline{E}}(u, v; \mathbf{c})\}$, we obtain a graph $(\widehat{C}, \widehat{E})$ of recursively enumerable relations on F such that also the complement $\widehat{C} \times \widehat{C} - \widehat{E}$ is recursively enumerable. Since α does not contain the equality symbol, this graph satisfies α. Pick a 1-1 computable map $f : \mathbb{Z} \to \widehat{C}$, and let $E = \{\langle x, y\rangle : x, y \in \mathbb{Z} \ \& \ \langle f(x), f(y)\rangle \in \widehat{E}\}$. Then $(\mathbb{Z}, E)$ is a recursive graph satisfying α, contradiction. $\diamondsuit$

4. Countable groups vs. f.g. groups

We describe H_5 and φ_5. Philip Hall [**2**] proved there is a up to isomorphism unique countable locally finite group $H = H_5$ which embeds every finite group and has the further property that any two isomorphic finite subgroups are conjugate. This group has a saturation property: if $G_1 \leq G_2$ are finite groups, then an embedding of G_1 into H extends to an embedding of G_2 into H. The natural construction of H gives a presentation with solvable word problem.

THEOREM 4.1. *There is a first–order sentence φ_5 which holds in Hall's universal locally finite group H but fails in all f.g. groups.*

We employ results about H from Hodges [**4**] to prove a lemma which enables us to quantify over finite subsets of H in the first–order language of groups.

LEMMA 4.2. *There is $k \in \mathbb{N}$ and a formula $\gamma(x; b, \overline{q})$ in the first–order language of groups such that, when $b, \overline{q}$ ranges over all $1 + k$–tuples in H,*

$$S_{b,\overline{q}} = \{h \in H : H \models \gamma(h; b, \overline{q})\}$$

ranges precisely over the nonempty finite subsets of H.

Proof. Firstly, Hodges ([**4**, Lemma 8] and the remark thereafter) uses the saturatedness property of H to prove that the cyclic subgroup $\langle b\rangle$ generated by an element b is definable in a uniform way:

$$\langle b\rangle = \{x : \forall y([y, b] = 1 \ \Rightarrow \ [x, y] = 1)\}. \tag{4}$$

Secondly, as a consequence of Hodges [**4**, Theorem 1], there is a word $t(x, \overline{y}) \in F(x, \overline{y})$, $\overline{y}$ a k–tuple of variables, such that, for each finite group A and each map $\beta : A \to A$, there is a finite group B, $A \leq B$, and $\overline{q} \in B^k$ such that

$$\forall a \in A \ \beta(a) = t(a, \overline{q}). \tag{5}$$

Now suppose $\{g_0, \ldots, g_{n-1}\}$ is a finite subset of H. Pick an element b of order n in H and let A be the (finite) subgroup generated by $\{g_0, \ldots, g_{n-1}, b\}$. Let $\beta : A \to A$ be the function mapping b^i to g_i $(0 \leq i < n)$, and mapping all other elements of A to g_0. Choose a finite group $B \geq A$ and $\overline{q} \in B^k$ such that (5) holds. By the saturation property of H, we can assume that B is a subgroup of H. Then, for each $g \in H$, $g \in \{g_0, \ldots, g_{n-1}\}$ iff there is some $c \in \langle b\rangle$ such that $t(c, \overline{q}) = g$.

Thus, by (4), as our formula $\gamma(h; b, \overline{q})$ we may choose $\exists x\{\forall y([y, b] = 1 \Rightarrow [x, y] = 1)\ \&\ h = t(x, \overline{q})\}$. Conversely, since each set $\langle b\rangle$ is finite and nonempty, all the sets $\{h \in H : H \models \gamma(h; b, \overline{q})\}$ are finite and nonempty. ♢

Proof of Theorem 4.1. Let S, S_1, S_2, U range over subsets of a group of the form $\{x : \gamma(x; b, \overline{q})\}$. We may replace quantification over such subsets by quantification over parameter lists $b, \overline{q}$, so that there is a first–order sentence φ_5 expressing the following.

$$\begin{aligned} &\forall x \exists S\ S = \{x\}\ \& \\ &\forall S_1 \forall x \exists S_2\ S_2 = S_1 \cup \{x\}\ \& \\ &\forall S \exists U \exists x[S \subseteq U\ \&\ U \text{ is subgroup } \&\ x \notin U]. \end{aligned} \tag{6}$$

Then $H \models \varphi_5$ by Lemma 4.2 and the fact that H is infinite and locally finite. Now suppose φ_5 holds in an arbitrary group G. By the first two clauses in (6), the sets $\{g \in G : G \models \gamma(h; b, \overline{q})\}$, where $b, \overline{q}$ now ranges through G^{1+k}, *include* all nonempty finite subsets of G. Then, by the third clause, G is not finitely generated. ♢

5. $\mathbf{UT}_3^3(\mathbb{Z})$ is quasi-finitely axiomatizable

In this Section we give a further example of a quasi-finitely axiomatizable group, namely $\mathrm{UT}_3^3(\mathbb{Z})$. We use the result to establish that $\mathrm{UT}_3^3(\mathbb{Z})$ is a prime model, and to settle the complexity of various theories of classes of groups.

We use the terminology of [**7**]. By Exercise 16.1.3 the free step-2 nilpotent group with generators a, b is isomorphic to $U = \mathrm{UT}_3^3(\mathbb{Z})$, where a, b correspond to the transvections $t_{23}(1), t_{12}(1)$, respectively. The center $C(U)$ is an infinite cyclic group generated by $c = [b, a] = t_{13}(1)$. Then $C(U)$ coincides with the set of commutators: $U' \subseteq C(U)$ since the group is nilpotent of class 2, but also $c^z = [b^z, a]$ for each $z \in \mathbb{Z}$.

Mal'cev [**11**] introduced an existential formula $\mu(x, y, z; a, b)$, using the generators a, b as parameters, which defines the graph of binary operation $M_{a,b}$ on $C(U)$ in a way that

$$(C(U), \circ, M_{a,b}) \cong (\mathbb{Z}, +, \times) \tag{7}$$

(here $\circ$ is the group operation in U). The formula is $\mu(x, y, z; a, b) \equiv \exists u, v\{[u, b] = [v, a] = 1\ \&\ x = [a, u]\ \&\ y = [b, v]\ \&\ z = [u, v]\}$.

THEOREM 5.1. *$UT_3^3(\mathbb{Z})$, or equivalently the free step-2 nilpotent group of rank 2, is quasi-finitely axiomatizable.*

Proof. We give an axiom system consisting of sentences $\alpha_1, \ldots, \alpha_4$. Let α_1 express that the group is step-2 nilpotent and that its center equals the set of commutators. The main problem is to find a single sentence α_2 which holds in U and implies that the center is torsion free. Our sentence implies in fact that the center is linearly orderable as a group. Recall Lagrange's Theorem: an integer is nonnegative iff it is the sum of four squares of integers. We use this fact in the definition of $P_{r,s}$ below. Since multiplication is definable in $C(U)$ with parameters a, b, $U \models \gamma(a, b)$, where $\gamma(r, s)$ is the formula expressing

- $\mu(x, y, z; r, s)$ defines a binary operation $M_{r,s}$

- if $P_{r,s} = \{u : \exists v_1 \dots \exists v_4 \ u = M_{r,s}(v_1, v_1) \circ \dots \circ M_{r,s}(v_4, v_4)\}$, then $x \leq y \leftrightarrow y - x \in P_{r,s}$ defines a linear order which turns $C(G)$ into an ordered abelian group with $[r, s]$ being the least non-negative element.

Let α_2 be $\exists r \exists s \ \gamma(r, s)$. Whenever $G \models \alpha_2$, then $C(G)$ is linearly orderable and hence torsion free.
Recall that a f.g. nilpotent group with torsion free center is itself torsion free (see [**7**, Section 16]), and that every subgroup of a f.g. nilpotent group is f.g. Now, since $C(G)$ is f.g., there is a sentence α_3 expressing that $C(G)/2C(G) \cong \mathbb{Z}_2$ (similar to β_4 in Section 2).
Next, $G_{ab} = G/C(G)$ is torsion free: if $u \notin C(G)$, pick $v \in G - \{1\}$ such that $[u, v] \neq 1$. Then $[u^n, v] = [u, v]^n \neq 1$, so that $u^n \notin C(G)$. Let the sentence α_4 express that $G_{ab}/2G_{ab}$ has 4 elements. Then α_4 implies that G_{ab} has rank 2.
We need to verify that a f.g. group G satisfying the axiom system is isomorphic to H. We first show that G is generated by any two elements c, d such that $G \models \gamma(c, d)$. Since $G \models \alpha_3$, Z is infinite cyclic, hence $C = C(G) = \langle [c, d] \rangle$. It suffices to show that $\langle Cc, Cd \rangle = G_{ab}$.
Since G_{ab} has rank 2, we can choose $g, h \in G$ such that $\langle Cg, Ch \rangle = G_{ab}$. Then there are $x, y, z, w \in \mathbb{Z}$ and $u, v \in C(G)$ such that $c = ug^x h^y, d = vg^z b^w$. Hence $[c, d] = [g^x h^y, g^z b^w] = [g, h]^{xw-yz}$. But also $[c, d]^r = [g, h]$ for some $r \in \mathbb{Z}$. Since $C(G)$ is torsion free, the determinant $xw - yz$ is 1 or -1. Thus Cc, Cd generate G_{ab}.
Since a, b generate U freely as a step-2 nilpotent group, there is an onto homomorphism $h : U \to G$ mapping a to c and b to d. It remains to be shown that h is 1-1. Since $h([a, b]) = [c, d]$, h induces an isomorphism $C(U) \to C(G)$. Since h induces an isomorphism $U_{ab} \to G_{ab}$ as well, h is itself an isomorphism. ♢

Let $\alpha(r, s)$ be the formula α_1 & $\gamma(r, s)$ & α_3 & α_4. The proof shows in fact that if G is f.g. and $G \models \gamma(c, d)$, then G is free step-2 nilpotent with c, d as free generators. In particular, all such pairs of generators are automorphic. We use this to derive a model theoretic fact on $\mathrm{UT}_3^3(\mathbb{Z})$. A structure $\mathbf{A}$ in a countable language is a *prime model* of a theory T if it is the least models of T, in the sense that $\mathbf{A}$ is elementarily embedded into any other model. For instance, $(\mathbb{N}, +, \times)$ is a prime model of $\mathrm{Th}(\mathbb{N}, +, \times)$. Prime models are uniquely determined by their theory. The following is common knowledge.

PROPOSITION 5.2. *Let $\mathbf{A}$ be a countable structure. Then $\mathbf{A}$ is a prime model of its theory iff for each n, each orbit under the action of $Aut(\mathbf{A})$ on $\mathbf{A}^n$ is first–order definable without parameters.*

COROLLARY 5.3. *$UT_3^3(\mathbb{Z})$ is the prime model of its theory.*

Proof. Suppose $(t_1(a, b), \dots, t_n(a, b))$ is a tuple in an orbit of U^n. Then the following formula with free variables $z_1, \dots, z_n$ defines the orbit: $\exists r \exists s \ [\gamma(r, s)$ & $z_1 = t_1(r, s)$ & $\dots$ & $z_n = t_n(r, s)]$. ♢

THEOREM 5.4. *Suppose $\mathcal{C}$ is a class of f.g. groups containing $UT_3^3(\mathbb{Z})$. Then $Th(\mathbb{N}, +, \times)$ is many–one reducible to $\mathrm{Th}(\mathcal{C})$.*

Proof. Given a sentence ψ in the language of number theory, let $F(\psi)$ be the sentence

$$\forall r \forall s [\alpha(r, s) \ \Rightarrow \ \widetilde{\psi}],$$

where $\widetilde{\psi}$ is obtained from ψ by letting the quantifiers run over the set $P_{r,s}$ defined above, and replacing $+$ by the group operation $\circ$ and $\times$ by $M_{r,s}$. Then F is computable, and $\psi \in \mathrm{Th}(\mathbb{N}, +, \times) \Leftrightarrow F(\psi) \in \mathrm{Th}(\mathcal{C})$. $\diamondsuit$

Let $(P_e)_{e\in\mathbb{N}}$ be a list of all presentations of the form $\langle x_0, x_1, \ldots | R\rangle$, where R is recursively enumerable. A class $\mathcal{C}$ of groups is *arithmetically recognizable* if there is an arithmetical set $S \subseteq \mathbb{N}$ such that $\mathcal{C}$ coincides with the class of groups given by a presentation P_e, $e \in S$. It can be checked that $\mathrm{Th}(\mathcal{C})$ is many–one reducible to true arithmetic $\mathrm{Th}(\mathbb{N}, +, \times)$ for such a class $\mathcal{C}$. We are now able to classify the complexity of various theories, notably the theory of the class of f.p. groups.

COROLLARY 5.5. *Suppose $\mathcal{C}$ is an arithmetically recognizable class of f.g. groups and $UT_3^3(\mathbb{Z}) \in \mathcal{C}$. Then $Th(\mathbb{N}, +, \times)$ is many–one equivalent to* $\mathrm{Th}(\mathcal{C})$.

Examples of such classes include: $\{\mathrm{UT}_3^3(\mathbb{Z})\}$, the classes $\mathcal{C}_1$ through $\mathcal{C}_3$ from List 1.2, the f.g. nilpotent groups, the f.g. step-c nilpotent groups ($c \geq 2$), and the f.g. metabelian groups. (The last class is arithmetically recognizable since the word problem for f.g. metabelian groups is solvable [**1**, Thm 2.1].)

In recent work, Morozov and the author proved that the theory of the class $\mathcal{C}_4$ of f.g. groups is Π_1^1–complete. They also prove that the word problem of a quasi-finitely axiomatizable group is hyperarithmetical, and give an example of such a group at each level of the hyperarithmetical hierarchy. Finally, they show that $\mathrm{Th}(\mathbb{Z}_p \wr \mathbb{Z})$ interprets true arithmetic.

In order to find more natural examples of quasi-finitely axiomatizable groups, it would be interesting to settle the generalizations of Theorem 5.1 to other free nilpotent groups and other groups $\mathrm{UT}_n^n(\mathbb{Z})$, for $n \geq 4$. A further good candidate for a quasi-finitely axiomatizable group is Ivanov's group for prime p (see [**14**, Thm 41.2]), an infinite 2-generated group of exponent p with exactly p conjugacy classes.

References

[1] G. Baumslag, F. Cannonito, and D. Robinson. The algorithmic theory of f.g. metabelian groups. *Trans. Amer. Math. Soc.*, 344(2):629–648, 1994.

[2] P. Hall. Some constructions for locally finite groups. *J. London Math. Soc.*, 34:305–319, 1959.

[3] R. Hirshon. Some cancellation theorems with applications to nilpotent groups. *J. Austral. Math. Soc (series A)*, 23:147–165, 1977.

[4] Wilfrid Hodges. Finite extensions of finite groups. In G. Müller and M. Richter, editors, *Models and sets*, pages 193–206. Springer–Verlag, 1984.

[5] Wilfrid Hodges. *Model Theory*. Enzyklopedia of Mathematics. Cambridge University Press, Cambridge, 1993.

[6] C.F. Miller III. The word problem in quotients of a group. In J.N. Crossley, editor, *Aspects of effective algebra*, pages 209–210. Upside down a book company, 1981.

[7] M. Kargapolov and J. Merzljakov. *Fundamentals of the theory of groups*. Springer–Verlag, 1979.

[8] Richard Kaye. *Models of Peano arithmetic*, volume 15 of *Oxford Logic Guides*. The Clarendon Press Oxford University Press, New York, 1991. Oxford Science Publications.

[9] O. Kharlampovich and A. Myasnikov. Tarski's problem about the elementary theory of free groups has a positive solution. *Electronic research announcements of the AMS*, 4:101–108, 1998.

[10] S. Lang. *Algebra*. Addison-Wesley, 1965.

[11] A. Mal'cev. On a correspondence between rings and groups. *Amer. Math. Soc. Translations*, 45:221–231, 1965.

[12] A Nies. Undecidable fragments of elementary theories. *Algebra Universalis*, 35:8–33, 1996.

[13] F. Oger. Cancellation and elementary equivalence of finitely generated finite–by–nilpotent groups. *J. London Math. Soc.*, 30:293–299, 1991.
[14] A. Yu. Olshanskii. *The geometry of defining relations in groups*. Kluwer, 1991.
[15] A.H. Rhemtulla. Commutators of certain finitely generated soluble groups. *Canad. J. Math.*, 21:1160–1164, 1969.
[16] J.J. Rotman. *The theory of Groups, 2nd ed.* Allyn and Bacon, 1973.
[17] J.P. Serre. *Trees*. Springer–Verlag, 1980.
[18] W. Szmielew. Elementary properties of abelian groups. *Fund. Math.*, 41:203–71, 1955.

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