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FA-presentable groups and rings

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Abstract

We consider structures which are FA-presentable. It is known that an FA-presentable finitely generated group is virtually abelian; we strengthen this result by showing that an arbitrary FA-presentable group is locally virtually abelian. As a consequence, we prove that any FA-presentable ring is locally finite; this is a significant restriction and allows us to say a great deal about the structure of FA-presentable rings. In particular, we show that any FA-presentable ring with identity and no zero divisors is finite.

Keywords: FA-presentable; Automatic; First-order theory; Group; Heisenberg group; Ring; Locally finite

1. Introduction

In this paper we investigate computing in structures. Recall that a *structure* $\mathcal{A}$ is a tuple $(A, R_1, \ldots, R_n)$ where:

- A is a set called the *domain* of $\mathcal{A}$;
- for each i with $1 \leqslant i \leqslant n$, there is an integer $r_i \geqslant 1$ such that R_i is a subset of A^{r_i}; r_i is called the *arity* of R_i.

There are many naturally occurring examples; for instance, a group can be viewed as a structure $(G, \circ, e, ^{-1})$, where $\circ$ has arity 3, e has arity 1, and $^{-1}$ has arity 2, and a ring with identity can

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be viewed as a structure $(R, +, -, \bullet, 0, 1)$, where $+$ and $\bullet$ have arity 3, $-$ has arity 2, and 0 and 1 have arity 1.

If our structures are finite then we have the important issue as to whether we can compute efficiently. Some natural questions link, for example, notions of logical definability with computational complexity. However, in many cases, the structures we deal with are infinite and so we would wish to extend our notions and techniques to classes of (potentially) infinite structures. Of course, it is necessary that each structure has a finite description; once we have this, we can then consider notions of computability.

A natural approach is to take the most general notion, such as the Turing machine (or equivalent) as the model of computation. While this is a natural definition, we would need to look for alternatives if we want to consider feasible computation. Khoussainov and Nerode proposed in [11] a very interesting restriction of the general idea; they introduced *automatic* or *FA-presentable* structures, which are structures whose domain and relations can be checked by finite automata as opposed to Turing machines; we define this notion formally in Section 2. An important point is that FA-presentable structures have nice algorithmic and logical properties and, as a result, have they been the subject of some very interesting research. In particular, such structures can be described by a finite amount of information and are closed under first-order interpretability; moreover, the model checking problem is decidable.

Of course, given any notion of computation in structures, we want to investigate how widely it can be applied. In particular, given a class $\Im$ of structures, it is natural to ask which structures in $\Im$ are FA-presentable. As any finite structure in $\Im$ will be FA-presentable, as far as this issue is concerned we are really only interested in the infinite structures in $\Im$. So far, this type of question has proved to be rather difficult in general, in the sense that, despite appreciable effort, there are not many cases where such a characterization of the FA-presentable structures exists.

One could ask to what extent it is feasible to obtain such a characterization. Examples where we do have complete descriptions include the cases of Boolean algebras [12] and ordinals [3]. For instance, a countably infinite Boolean algebra is FA-presentable if and only if it is isomorphic to B^n for some $n \geqslant 1$, where B is the Boolean algebra of finite or cofinite subsets of $\mathbb{N}$. (Our convention throughout this paper is that $\mathbb{N}$ contains 0.) In both these cases the isomorphism problem is decidable (see [12] for the case of Boolean algebras and [14] for ordinals). On the other hand, it is impossible to characterize the FA-presentable graphs; here the isomorphism problem is Σ_1^1-complete [12]. It seems that, as far as FA-presentability is concerned, some classes are reasonably well behaved, and so there is some possibility of a complete classification, whereas others are wild and there is no reasonable way of characterizing them.

In general, when one is attempting to characterize the FA-presentable structures in a class, there are two directions one can take. One is to find various types of examples; the other (and often harder) line is to establish restrictions on the possible structures that can occur. The main purpose of this paper is to provide some results in the second direction. We will be particularly concerned with FA-presentable groups and rings. As far as groups are concerned, a complete classification in the case of finitely generated groups was given in [19], but the general case remains open. We also have the notion of an *automatic group* in the sense of Epstein et al. [4]. This concept captures a wide class of groups and has been very successful for the development of practical algorithms for solving certain problems in this class (see [7] for example). The notion has been generalized to semigroups (as in [2,9,20]). The considerable success of the theory of automatic groups is another motivation to have a general notion of automatic structures; see also [21,23] in this regard.

It is interesting [19] to note that, in the case of finitely generated groups, an FA-presentable group is necessarily automatic in the Epstein sense (although the converse is false). This implication does not hold for general groups, as any automatic group is finitely generated (in the group sense) by definition, whereas there are examples of FA-presentable groups that are not finitely generated (such as the Prüfer group $R_p/\mathbb{Z}$, where p is a prime and R_p is the additive group of all rationals of the form z/p^m with $z \in \mathbb{Z}$ and $m \in \mathbb{N}$; see [12] for example).

Further examples of FA-presentable groups may be found in [17] (also see [16]). For instance, there is a rank 2 torsion free abelian FA-presentable group that is indecomposable. However, we still seem to be a long way from being able to characterize the class of FA-presentable groups. It is interesting to speculate as to whether the class of FA-presentable (abelian) groups might be well behaved in the sense given above (i.e. in having a decidable isomorphism problem), wild (in the isomorphism problem being Σ_1^1-complete) or something in between.

As far as other classes of structures are concerned, it is known [12] that no infinite integral domain is FA-presentable, but the question as to which general rings are FA-presentable is also still open. The results presented in this paper place some strong restrictions as to which groups and rings could be FA-presentable.

The main result of [19] states that a finitely generated group is FA-presentable if and only if it is virtually abelian (i.e. has an abelian subgroup of finite index). In Theorem 10 below we strengthen this result considerably: we show that, given an arbitrary FA-presentable group G, every finitely generated subgroup H of G is virtually abelian; moreover, we show that there is a uniform bound on the rank of a finitely generated free abelian subgroup of G, which further restricts the possible structure of H. However, note that, since a direct power of a countable collection of isomorphic copies of a fixed finite group is FA-presentable, it is clear that the characterization of FA-presentable as being equivalent to virtually abelian does not generalize from the finitely generated case to the class of all groups.

We comment that the main result of [19] depends on two deep theorems; Gromov's classification [6] of the finitely generated groups with polynomial growth and the classification of certain types of group having a decidable first-order theory. With regards to the latter, Eršov showed [5] that a nilpotent group has decidable first-order theory if and only if it is virtually abelian. This was generalized by Romanovskii [22] to virtually polycyclic groups and then by Noskov [18], who showed that a virtually solvable group has decidable first-order theory if and only if it is virtually abelian. The proof in [19] used the result for virtually nilpotent groups which is a special case of Romanovskiĭ's theorem. In our generalization of [19] we remove the need to appeal to Romanovskiĭ's result. We prove that the Heisenberg group $\mathrm{UT}_3^3(\mathbb{Z})$ does not embed in any FA-presentable group (see Lemma 8 below) and, given the fact that any virtually nilpotent group that does not contain $\mathrm{UT}_3^3(\mathbb{Z})$ is virtually abelian, the generalization of [19] can be established.

Given all this, we go on to consider rings. The main result is that any FA-presentable ring R with identity is locally finite (i.e. any finitely generated subring is finite); see Theorem 16. In order to do this, we apply Theorem 10 to the FA-presentable matrix group $\mathrm{GL}(3, R)$. Local finiteness is a considerable restriction and allows us to prove (for example) that an FA-presentable ring with identity and no zero divisors is necessarily finite; see Corollary 17.

2. FA-presentable structures and preliminary results

To explain what is meant by an FA-presentable structure, we use convolutions as in [11]. If I is an alphabet, we first choose a *padding symbol* $\square$ such that $\square \notin I$ and then define the *convolution* of $(x_1, x_2, \ldots, x_n) \in (I^*)^n$, where $x_i = x_{i,1}x_{i,2}\ldots x_{i,p_i}$ $(x_{i,j} \in I)$, to be

$$\operatorname{conv}(x_1, \ldots, x_n) = (\bar{x}_{1,1}, \bar{x}_{2,1}, \ldots, \bar{x}_{n,1}) \ldots (\bar{x}_{1,p}, \bar{x}_{2,p}, \ldots, \bar{x}_{n,p}),$$

where $p = \max\{p_i\colon 1 \leqslant i \leqslant n\}$ and

$$\bar{x}_{i,j} = \begin{cases} x_{i,j}, & 1 \leqslant j \leqslant p_i, \\ \square, & p_i < j \leqslant p. \end{cases}$$

These elements $\operatorname{conv}(x_1, \ldots, x_n)$ are words over the alphabet

$$I_\square^n = \bigl(\bigl(I \cup \{\square\}\bigr) \times \bigl(I \cup \{\square\}\bigr) \times \cdots \times \bigl(I \cup \{\square\}\bigr)\bigr) \setminus \bigl\{(\square, \square, \ldots, \square)\bigr\}.$$

Definition 1. A structure $\mathcal{A} = (A, R_1, \ldots, R_n)$ is said to be FA-presentable (over an alphabet I) if

(1) there is a language L (over I) and a surjective map $c : L \to A$;
(2) L is accepted by a finite automaton over I;
(3) the language $L_= = \{\operatorname{conv}(x, y)\colon c(x) = c(y),\ x, y \in L\}$ is accepted by a finite automaton over $I_\square^2$;
(4) for each relation R_i in $\mathcal{A}$, the language

$$L_{R_i} = \bigl\{\operatorname{conv}(x_1, \ldots, x_{r_i})\colon \bigl(c(x_1), \ldots, c(x_{r_i})\bigr) \in R_i\bigr\}$$

is accepted by a finite automaton over $I_\square^{r_i}$.

The tuple $(I, L, c, L_=, (L_{R_i})_{1\leqslant i\leqslant n})$ is called an *automatic presentation* for $\mathcal{A}$.

When referring to a situation such as the one described in Definition 1, we often let $|a|$ denote the length of an encoding of an element $a \in A$, i.e. the length of a string α such that $c(\alpha) = a$. In fact it is well known (see [11]) that we can always choose the mapping $c : L \to A$ to be injective, in which case $|a|$ is unambiguously defined for all $a \in A$. We will normally assume that c is injective in what follows.

We need the following facts from [11]:

Proposition 2. *If $\mathcal{A} = (A, R_1, \ldots, R_n)$ is an FA-presentable structure, $\varphi(x)$ is a first-order sentence over $R_1, \ldots, R_n$ with free variable x and constants in A, B is the set*

$$\bigl\{x \in A\colon \varphi(x) \text{ is true}\bigr\}$$

and S_i is the restriction of the relation R_i to elements of B for $1 \leqslant i \leqslant n$, then $\mathcal{B} = (B, S_1, \ldots, S_n)$ is an FA-presentable structure.

Proposition 3. *If $\mathcal{A}_1$ and $\mathcal{A}_2$ are two FA-presentable structures with the same signature, then $\mathcal{A}_1 \times \mathcal{A}_2$ is also FA-presentable.*

Proposition 4. *If $\mathcal{A} = (A, R_1, \ldots, R_n)$ is an FA-presentable structure and if $\sim$ is a congruence on $\mathcal{A}$ which is recognized by a finite automaton, then the quotient structure $\mathcal{A}/{\sim}$ is FA-presentable.*

We now prove a bound on the rank of a free commutative monoid that can be embedded into an FA-presentable structure, in terms of the number of states of the relevant automaton. Note that, in this paper, all logarithms will be taken with the base 2; in particular, $\lceil \log n \rceil$ is the least $i \in \mathbb{N}$ such that $2^i \geqslant n$.

Theorem 5. *Suppose that Σ is an alphabet and that the relation $R \subseteq (\Sigma^*)^3$ is recognized by a k-state NFA. Let $M \subseteq \Sigma^*$ be a set such that $R \cap (M \times M \times \Sigma^*)$ is the graph of a binary operation f on M where $(M, f) \cong (\mathbb{N}, +)^r$; then $r \leqslant (k+1)\log|\Sigma|$.*

In Theorem 5 note that, for any $x, y \in M$, there is a unique $z \in \Sigma^*$ such that $(x, y, z) \in R$, and this element z is also in M. There is no restriction on R outside M, or on what M is. Since f is associative, we may handle products of finitely many elements of M in the usual way; thus we write xy for $f(x, y)$.

Before we prove Theorem 5, we note the following lemma, a weaker form of which was proved in [12]. The result actually is true for any monoid (M, f), not only $(\mathbb{N}, +)^r$:

Lemma 6. *For each $s_1, \dots, s_m \in M$, we have*

$$\left| \prod_{i=1}^{m} s_i \right| \leqslant \max\bigl\{|s_i|:\ 1 \leqslant i \leqslant m\bigr\} + k\lceil \log m \rceil.$$

Proof. By the pumping lemma for regular languages we have that, for each $x, y \in M$,

$$|xy| \leqslant k + \max\bigl(|x|, |y|\bigr), \tag{1}$$

otherwise there would be infinitely many elements z such that $(x, y, z) \in R$.

We use induction on m. For $m = 1$, the inequality becomes $|s_1| \leqslant |s_1|$.

If $m > 1$ let $m = u + v$ where $u = \lfloor m/2 \rfloor$. Apply (1) to $x = \prod_{i=1}^{u} s_i$ and $y = \prod_{i=u+1}^{m} s_i$; then, by induction,

$$\begin{aligned}
\left| \prod_{i=1}^{m} s_i \right| &\leqslant k + \max\bigl(|x|, |y|\bigr) \\
&= k + \max\Bigl\{ \max_{1 \leqslant i \leqslant u} |s_i| + k\lceil \log u \rceil,\ \max_{u+1 \leqslant i \leqslant m} |s_i| + k\lceil \log v \rceil \Bigr\} \\
&\leqslant \max_i |s_i| + k\lceil \log m \rceil,
\end{aligned}$$

since $1 + \max(\lceil \log u \rceil, \lceil \log v \rceil) \leqslant \lceil \log m \rceil$. □

We now prove Theorem 5.

Proof. Since $M \cong (\mathbb{N}, +)^r$, we may choose elements $a_0, a_1, \dots, a_{r-1}$ which generate M as a monoid. For each $n \geqslant \max\{|a_i|:\ 0 \leqslant i \leqslant r-1\}$ let

$$F_n = \Bigl\{ \prod_{0 \leqslant i < r} a_i^{\beta_i}:\ 0 \leqslant \beta_i < 2^n \Bigr\}.$$

By Lemma 6, each term $a_i^{\beta_i}$ has length at most

$$n + k\lceil \log \beta_i \rceil \leqslant n(1+k),$$

and each product has length at most $n(1+k) + k\lceil \log r \rceil$. Since all the products are distinct, we have

$$2^{nr} \leqslant |F_n| \leqslant |\Sigma|^{(1+k)n+k\lceil \log r \rceil+1}.$$

Thus $r \leqslant [(1+k) + \lceil \log r \rceil k/n + 1/n] \log |\Sigma|$. Since n can be chosen arbitrarily large, this shows that $r \leqslant (k+1) \log |\Sigma|$ as required. □

The natural DFA-representation of $(\mathbb{N}, +)^r$ has $k = 2^r$ states on a binary alphabet, in which case our bound for r is a generous $1 + 2^r$. It would be interesting to see if the bound obtained above is tighter in some other cases.

3. Groups

We prove our main result restricting the possible structure of an FA-presentable group G and show that every finitely generated subgroup H of such a group G must be virtually abelian. First, as in [19], we argue that H has polynomial growth, and hence, by the result of Gromov [6], is virtually nilpotent. The new idea is to now use what one could call the "Heisenberg alternative": either H is virtually abelian, or it embeds the Heisenberg group $\mathrm{UT}_3^3(\mathbb{Z})$. This simple fact is verified in the proof of Theorem 10. The main technical ingredient, Lemma 8, is to show that no FA-presentable group G embeds $\mathrm{UT}_3^3(\mathbb{Z})$; thus H is virtually abelian.

We first introduce some notation and recall some basic facts.

If G is a group and $g \in G$, then $C_G(g)$ (or just $C(g)$ if the context is clear) denotes the *centralizer* of g in G, i.e. the set $\{x \in G\colon xg = gx\}$. For any group G we let $Z(G)$ denote the *center* of G, i.e. the subgroup $C_G(G)$ of G.

When we are considering groups, we write x^y for $y^{-1}xy$. The *commutator* $[x, y]$ is defined to be the element $x^{-1}y^{-1}xy$. We will make frequent use of the commutator identity

$$[xy, z] = [x, z]^y [y, z].$$

We let $[x, y, z]$ denote $[[x, y], z]$ and, in general, let $[x_1, x_2, \ldots, x_r]$ denote

$$\big[[x_1, x_2, \ldots, x_{r-1}], x_r\big].$$

We say that a commutator of the form $[x_1, x_2, \ldots, x_r]$ has *weight* r. If G is a group and H and K are subgroups of G, then $[H, K]$ denotes the subgroup of G generated by all elements of the form $[h, k]$ with $h \in H$ and $k \in K$; in addition, we let (as usual) $\gamma_0(G) = G$ and $\gamma_{i+1}(G) = [\gamma_i(G), G]$.

The following fact will be useful in what follows:

Proposition 7. *Suppose that G is a group and that a, b, q are elements of G such that $q = [a, b]$ and*

$$[a, q] = [b, q] = 1.$$

Let $m, n \in \mathbb{N}^+$. *Then*:

(i) *If* v *is an element of* G *such that* $[a, v] = q^n$ *then* $[a^i, v] = q^{in}$.
(ii) $[a^m, b] = q^m$ *and* $[a^{-m}, b] = q^{-m}$, $[a, b^n] = q^n$ *and* $[a^m, b^n] = q^{mn}$.

Proof. (i) For $i = 1$, this is vacuous. If $[a^i, v] = q^{ni}$ inductively, then, because $a \in C(q)$, we have that

$$\left[a^{i+1}, v\right] = \left[a^i, v\right]^a [a, v] = q^{ni} q^n = q^{(n+1)i}.$$

(ii) The first equation follows from (i), where $v = b$ and $n = 1$; the next holds as $[a, q] = 1$ and so

$$\left[a^m, b\right]\left[a^{-m}, b\right] = \left[a^m, b\right]^{a^{-m}} \left[a^{-m}, b\right] = 1.$$

The third equation is proved similarly to the first; the fourth is again a special case of (i) with $v = b^n$. □

Mal'cev [15] introduced an existential formula $\mu(x, y, z; a, b)$ to give an interpretation of a ring R in the group $\mathrm{UT}_3^3(R)$ of 3×3 upper-triangular matrices

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

with entries in R. We use a modification of that formula to prove the following criterion:

Lemma 8. *Suppose that* G *is a group and that* a *and* b *are elements of* G *such that* $q = [a, b]$ *has infinite order and* $[a, q] = [b, q] = 1$; *then* G *is not FA-presentable.*

Note that the Heisenberg group $\mathrm{UT}_3^3(\mathbb{Z})$ is nilpotent of class 2 and has presentation

$$\left\langle a, b, q\colon\ [a, b] = q,\ [a, q] = [b, q] = 1\right\rangle,$$

and that the element q has finite order in any non-trivial quotient (as each normal subgroup intersects the center $\langle q\rangle$ non-trivially). Thus the hypothesis of Lemma 8 is equivalent to saying that $\mathrm{UT}_3^3(\mathbb{Z})$ embeds in G.

In order to prove Lemma 8 we first establish the following result:

Lemma 9. *Suppose that* G *is a group, and that* $a, b, q \in G$ *are as in Lemma* 8. *There is a first-order formula* $\eta(x, y, z; a, b)$ *with the property that, for any* $m, n \in \mathbb{N}^+$,

$$\forall z \in G \quad \left[\eta\left(q^m, q^n, z;\ a, b\right) \Leftrightarrow z = q^{mn}\right]. \tag{2}$$

Let

$$\eta(x, y, z; a, b) \equiv \exists u, v \in C(q)\ \left\{x \in C(a) \wedge y \in C(b) \wedge C(b) \subseteq C(v) \right.$$
$$\left. \wedge\ x = [u, b] \wedge y = [a, v] \wedge z = [u, v]\right\}.$$

Note that η can be viewed as a first-order formula; for example, $u \in C(q)$ is equivalent to $uq = qu$ and $C(b) \subseteq C(v)$ is equivalent to

$$\forall w \quad (wb = bw \Rightarrow wv = vw).$$

The difference between the original Mal'cev coding and our setting is that the witnesses u and v are allowed to be taken from anywhere in the group, not just the subgroup generated by a, b and q; however, the extra condition that $C(b) \subseteq C(v)$ ensures that this does not cause any damage.

Proof. The direction "$\Leftarrow$" in (2) follows from Proposition 7, where $u = a^m$ and $v = b^n$.

For the direction "$\Rightarrow$," suppose that $\eta(q^m, q^n, z; a, b)$ holds via the witnesses u and v; we have to show that $z = [u, v] = q^{mn}$. Note that $[a^m, v] = q^{mn}$ by Proposition 7(i) and the fact that $[a, v] = q^n$; so it suffices to show that $[u, v] = [a^m, v]$.

Since $[u, b]q^{-m} = 1$, we have $[u, b][a^{-m}, b] = 1$. Because $[u, b] \in C(a)$, we conclude that

$$[u, b]^{a^{-m}}\big[a^{-m}, b\big] = \big[ua^{-m}, b\big] = 1.$$

Let $s = ua^{-m}$; we then have $s \in C(b) \subseteq C(v)$, so that $[s, v] = 1$ and

$$[u, v] = \big[sa^m, v\big] = [s, v]^{a^m}\big[a^m, v\big] = \big[a^m, v\big].$$

Thus $[u, v] = q^{mn}$; this proves Lemma 9. $\square$

We are now in the position to prove Lemma 8. In order to apply Theorem 5, we assume, for a contradiction, that G is FA-presentable. Let R be the FA-recognizable relation given by

$$R(x, y, z) \quad \Leftrightarrow \quad \eta(x, y, z; a, b).$$

Let $p_1 < p_2 < \cdots$ denote the sequence of all prime numbers. Given $r \in \mathbb{N}$, let

$$M = \left\{q^m \colon m = \prod_{1 \leqslant i \leqslant r} p_i^{n_i} \text{ for some } n_i \geqslant 0\right\}.$$

Then, by (2), $R \cap (M \times M \times \Sigma^*)$ is the graph of a binary operation f on M where $(M, f) \cong (\mathbb{N}, +)^r$. Since r can be chosen arbitrarily large, this contradicts Theorem 5.

We are now ready to prove our main result on groups.

Theorem 10. *Let G be an infinite group with an FA-presentation over an alphabet Σ and let k the number of states of an NFA recognizing the graph of the group operation.*

(i) *Every finitely generated subgroup H of G is virtually abelian.*
(ii) *In fact any such subgroup H is a finite extension of a free abelian group $\mathbb{Z}^r$ of rank r, where*

$$r \leqslant (k + 1) \log |\Sigma|.$$

Proof. (i) Let H be a finitely generated subgroup of G; we may assume that H is infinite. H has polynomial growth as in [19], and therefore H has a nilpotent subgroup H^* of finite index by [6].

H^* has a torsion-free subgroup K of finite index (see [10, 17.2.2] for example). We are done if we can show that K is abelian.

Suppose that K is non-abelian and let $c \geqslant 2$ be its nilpotency class. So $\gamma_c(K)$ is a non-trivial subgroup of the center $Z(K)$ of K. By [10, 17.2.1], $\gamma_{c-1}(K)$ is generated by the commutators $[x_1, x_2, \dots, x_c]$ of weight c. We can choose an element $q \neq 1$ of this form, so that $q = [a, b]$ where $a = [x_1, \dots, x_{c-1}]$ and $b = x_c$. By Lemma 8, G is not FA-presentable, a contradiction.

(ii) By (i) there is an abelian subgroup A of finite index in H. We may assume that A is a free abelian group $\mathbb{Z}^r$ of rank r. Let R be the graph of the group operation, and let $M = \mathbb{N}^r$ viewed as a subset of A. Then, by Theorem 5, the required bound on r follows. □

4. Remarks and examples

The first part of Theorem 10 would be an immediate consequence of the main result in [19] if, for each FA-presentation of a group G and for each finitely generated subgroup S of G, the set of representations of elements of S was necessarily regular. However, this is not the case: a counterexample can be derived from recent work of Akiyama, Frougny and Sakarovitch [1]. For each prime q, there is an FA-presentation of the abelian group

$$R_q = \{z/q^i \colon z \in \mathbb{Z},\ i \in \mathbb{Z}\}$$

where the representations of integers do not form a regular subset. In [1] the authors consider p/q-representations of positive integers. For primes $p > q$, a p/q-representation is an expression of the form

$$\sum_{i=0}^{n} \frac{a_i}{q} \left(\frac{p}{q}\right)^i$$

where $0 \leqslant a_i < p$. Such a representation can be viewed as a string of length $n + 1$ over the alphabet $\{0, 1, \dots, p - 1\}$. They show that each positive integer has a unique p/q representation, and that the set of p/q representations of positive integers is not regular.

The non-negative rationals with a p/q expansion form a cancellative semigroup, and, by adding representations via a carry algorithm, we obtain an FA-presentation of that semigroup. Hence we obtain an FA-presentation of the abelian group G via the usual difference group construction. The group G can be viewed in a natural way as a subgroup of R_q. It is easy to see that, in fact, $G = R_q$: for each $i > 0$ there are $k, \ell \in \mathbb{Z}$ such that $kp^i + \ell q^i = 1$, and therefore $k(p/q)^i + \ell = 1/q^i$. In this presentation of R_q, the set of representations of the integers is not regular.

In the same vein, in [17] a presentation of the abelian group $\mathbb{Z} \times \mathbb{Z}$ is given where a non-trivial cyclic subgroup is not regular.

In the direction of a complete structure theorem, one could conjecture that any FA-presentable group has a finitely generated abelian subgroup that is normal such that the quotient by this normal subgroup is locally finite. The following fact lends some evidence to the conjecture:

Proposition 11. *Let G be FA-presentable, and let R be a free abelian subgroup of G of maximal rank r. Then R has finite index in any finitely generated subgroup H such that $R \leqslant H \leqslant G$.*

Proof. Suppose R has infinite index in H. By (i) of Theorem 10, we can choose an abelian torsion-free subgroup A of H of finite index in H. Then $R \cap A$ has finite index in R, and so $R \cap A$ has rank r as well. Since R has infinite index in H, $R \cap A$ has infinite index in A, and so A has rank larger than r, a contradiction. □

Example 12. In the situation described in Proposition 11, R need not be normal in H; in fact, there is an FA-presented group H such that R has infinite index in the normal closure of R in H.

Proof. The group H has the set of generators

$$\{x\} \cup \{y_i\colon i \in \mathbb{N}^+\} \cup \{z_i\colon i \in \mathbb{N}^+\}$$

and the following set of relations:

$$\begin{aligned} &[x, y_i] = 1 \quad \text{for all } i; && y_i^2 = z_i^2 = 1 \quad \text{for all } i; \\ &z_i^{-1} x z_i = x y_i \quad \text{for all } i; && [y_i, z_j] = 1 \quad \text{for all } i, j; \\ &[y_i, y_j] = 1 \quad \text{for all } i, j; && [z_i, z_j] = 1 \quad \text{for all } i, j. \end{aligned}$$

We see that $N = \langle \{x\} \cup \{y_i\colon i \in \mathbb{N}^+\}\rangle$ is a normal subgroup of H and that N is the normal closure of $R = \langle x\rangle$ in H. R is a free abelian group of rank 1 and R has infinite index in N.

However, note that the subgroup $\langle x^2\rangle$ is normal and the quotient is locally finite, so that H does not refute the conjecture above.

Next we show that the group H is FA-presentable. For any element g of H there exist $\alpha \in \mathbb{Z}$, $k \geqslant 0$ and

$$p(1), p(2), \ldots, p(k), q(1), q(2), \ldots, q(k) \in \{0, 1\}$$

such that

$$g = x^\alpha y_1^{p(1)} y_2^{p(2)} \ldots y_k^{p(k)} z_1^{q(1)} z_2^{q(2)} \ldots z_k^{q(k)}, \tag{3}$$

and, provided we have that at least one of $p(k)$ and $q(k)$ is non-zero, there is a unique expression of the form (3) for any given element g.

In showing that H is FA-presentable we use three "tracks" on our tape. We imagine that each cell of the tape is divided into three parts; a top part, a middle part and a bottom part. Reading the symbols from the top part of each cell (i.e. the top part of the first cell, followed by the top part of the second cell, and so on) gives us the first track; we have a similar situation for the other two tracks. Formally, we are using triples as our input symbols here.

We represent g on a tape with three tracks as follows. The first track represents α; we start with $+$ if $\alpha \geqslant 0$ and $-$ otherwise, and then represent $|\alpha|$ in reverse binary notation. We then have two tracks, one representing $p(1), p(2), \ldots, p(k)$ and the other $q(1), q(2), \ldots, q(k)$; on each of these two tracks we have a word in $\{0, 1\}^*$ of length k.

We note that, if

$$\begin{aligned} g &= x^m y_1^{p(1)} y_2^{p(2)} \ldots y_k^{p(k_1)} z_1^{q(1)} z_2^{q(2)} \ldots z_k^{q(k_1)}, \\ h &= x^n y_1^{s(1)} y_2^{s(2)} \ldots y_k^{s(k_2)} z_1^{t(1)} z_2^{t(2)} \ldots z_k^{t(k_2)}, \end{aligned}$$

and if $\ell = \max\{k_1, k_2\}$, then

$$gh = x^{m+n} y_1^{p(1)+q(1)n+s(1)} \dots y_\ell^{p(\ell)+q(\ell)n+s(\ell)} z_1^{q(1)+t(1)} \dots z_\ell^{q(\ell)+t(\ell)}$$

with the obvious conventions. The set of encodings of triples (g, h, gh) is readily seen to be recognizable by a finite automaton; apart from ordinary and binary addition, we only need to calculate the $q(i)n$ modulo 2, and the parity of n follows from the first digit after the $+/-$ symbol on the first track of the second tape. Thus H is FA-presentable. □

One might think that the situation in FA-presentable groups somehow mirrored the situation in FA-presentable Boolean algebras as described in [12] with, perhaps, commutators corresponding in some way to joins. That this is not the case can be seen by considering the following example of a virtually abelian group H which is perfect (i.e. which satisfies $[H, H] = H$).

Let A be a free abelian group of rank 5 generated by $\{x_1, x_2, x_3, x_4, x_5\}$ and let F be the finite group A_5 of order 60 (the alternating group on five symbols). We can form a semidirect product $A \rtimes F$ in a natural way; if σ is an element of A_5, let $\sigma^{-1} x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4} x_5^{i_5} \sigma$ be

$$x_{\sigma(1)}^{i_1} x_{\sigma(2)}^{i_2} x_{\sigma(3)}^{i_3} x_{\sigma(4)}^{i_4} x_{\sigma(5)}^{i_5}.$$

The cyclic subgroup N of $A \rtimes F$ generated by $n = x_1x_2x_3x_4x_5$ is normal in $A \rtimes F$; let G be the factor group $(A \rtimes F)/N = (A/N) \rtimes F$. Let B be the subgroup of A consisting of all the elements of the form $x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4} x_5^{i_5}$ where the exponent sum $i_1 + i_2 + i_3 + i_4 + i_5$ is divisible by five, and then let H be the subgroup $(B \rtimes F)/N = (B/N) \rtimes F$ of G.

Each element b of B is of the form $x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4} x_5^{i_5}$ with $i_1 + i_2 + i_3 + i_4 + i_5 = 5k$ for some k. Replacing b by bn^{-k} gives an element with exponent sum zero; so we may assume (by choosing our coset representative appropriately) that each element of B/N has exponent sum zero when expressed in this form.

If $b = x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4} x_5^{i_5} \in B/N$ with $i_1 + i_2 + i_3 + i_4 + i_5 = 0$, then we may write b in the form

$$x_1^{i_1} x_2^{-i_1} . x_2^{i_1+i_2} x_3^{-(i_1+i_2)} . x_3^{i_1+i_2+i_3} x_4^{-(i_1+i_2+i_3)} . x_4^{i_1+i_2+i_3+i_4} x_5^{-(i_1+i_2+i_3+i_4)}. \tag{4}$$

Now, if $\{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}$ and we consider the elements $(ij)(kl)$ of F and $x_k x_m^{-1}$ of B/N, then we see that $[(ij)(kl), x_k x_m^{-1}] = x_k x_l^{-1}$. Looking at (4), we see that any element of B/N lies in $[H, H]$. However, $F = [F, F] \leqslant [H, H]$; so $H = (B/N) \rtimes F \leqslant [H, H]$, and so H is perfect. In addition, H is finitely generated and virtually abelian, and hence H is FA-presentable by [19].

5. FA-presentable rings

We prove that each FA-presentable ring with identity is locally finite. As mentioned in the introduction, we are using the term "locally finite" with respect to rings to mean that every finitely generated subring is finite. In [8], the authors take an alternative definition; they call a ring "locally finite" if every finite subset generates a finite semigroup multiplicatively. However, as they point out, their definition is equivalent to the one we are using here. The point is that, if every finite subset generates a finite semigroup multiplicatively, then the ring must have finite characteristic k (else $2 = 1 + 1$ generates an infinite subsemigroup). Now any finite subset A

generates a finite semigroup $\{a_1, a_2, \ldots, a_n\}$ multiplicatively; every element of the subring S generated by A is then of the form $x_1 a_1 + \cdots + x_n a_n$ for some $0 \leqslant x_i < k$, and so S is finite. Our definition is the same as that given explicitly elsewhere; see [24] for example.

For any ring S with identity, we let $\mathrm{GL}(n, S)$ denote the group of all $n \times n$ invertible matrices with entries in S. The following result holds since FA-presentable structures are closed under interpretations:

Proposition 13. *If S is an FA-presentable ring with identity then* $\mathrm{GL}(n, S)$ *is FA-presentable for any $n \geqslant 2$.*

Let S be a ring with identity. For $1 \leqslant i, j \leqslant n$ with $i \neq j$ and $r \in S$, let $\epsilon_{i,j}(r)$ denote the matrix with r in position (i, j) and 0 elsewhere. We let I_n denote the identity matrix and $e_{i,j}(r)$ denote the matrix $I_n + \epsilon_{i,j}(r)$; the matrices $e_{i,j}(r)$ are called *transvections*.

The following basic facts about transvections are easily verified (or see [10] for example):

Proposition 14. *Let S be a ring with identity*; *then, for $r, s \in S$ and distinct elements $i, j, k \in \{1, 2, 3\}$, we have*:

(1) $e_{i,j}(r)^{-1} = e_{i,j}(-r)$.
(2) $e_{i,j}(r)e_{i,j}(s) = e_{i,j}(r + s)$.
(3) $[e_{i,j}(r), e_{j,k}(s)] = e_{i,k}(rs)$.

We let $\mathrm{E}(n, S)$ denote the subgroup of $\mathrm{GL}(n, S)$ generated by the set of transvections in $\mathrm{GL}(n, S)$. This is usually a proper subgroup of $\mathrm{SL}(n, S)$; equality only holds in some special cases, for instance if S is Euclidean. Proposition 14 implies the following:

Proposition 15. *If S is a finitely generated ring with identity then* $\mathrm{E}(n, S)$ *is a finitely generated group.*

We now prove:

Theorem 16. *If R is an FA-presentable ring with identity then R is locally finite.*

Proof. Let R be an FA-presentable ring with identity and let S be a finitely generated subring of R; we want to show that S is finite.

The group $G = \mathrm{GL}(3, R)$ is FA-presentable by Proposition 13; as S is a subring of R we have that $\mathrm{GL}(3, S)$ is a subgroup of G. So $E = \mathrm{E}(3, S)$ is a subgroup of G; moreover, E is finitely generated by Proposition 15, and so is virtually abelian by Theorem 10. Let A be a normal abelian subgroup of E such that A has finite index in E.

For each $i \neq j$ in $\{1, 2, \ldots, n\}$, let $S_{i,j}$ denote the set

$$\bigl\{s \in S\colon\ e_{i,j}(s) \in A\bigr\}.$$

Using Proposition 14, we see that $(S_{i,j}, +)$ is a subgroup of $(S, +)$. We have a natural embedding $\varphi : (S, +) \to (E, \times)$ defined by $s \mapsto e_{i,j}(s)$. We see that $(\mathrm{Im}\,\varphi \cap A)\varphi^{-1} = S_{i,j}$; since A has finite index in E, it follows that $S_{i,j}$ has finite index in S.

Let $I = \bigcap\{S_{i,j}\colon\ i \neq j\}$. Since each $(S_{i,j}, +)$ is a subgroup of finite index in $(S, +)$, we have that $(I, +)$ is also a subgroup of finite index in $(S, +)$. Moreover, if $s \in I$ and $r \in S$ and if $i \neq k$ in $\{1, 2, 3\}$, then $e_{i,j}(s), e_{j,k}(s) \in A$ (where $\{i, j, k\} = \{1, 2, 3\}$) and we have

$$e_{i,k}(sr) = \big[e_{i,j}(s), e_{j,k}(r)\big] \in A, \qquad e_{i,k}(rs) = \big[e_{i,j}(r), e_{j,k}(s)\big] \in A$$

(as A is a normal subgroup of E); so $sr, rs \in S_{i,k}$. Since this holds for all i and k, we see that $sr, rs \in I$, and so I is an ideal of S.

Now, if $s_1, s_2 \in I$, then $e_{1,2}(s_1), e_{2,3}(s_2) \in A$, and so

$$e_{1,3}(s_1 s_2) = \big[e_{1,2}(s_1), e_{2,3}(s_2)\big] = I_3.$$

So $s_1 s_2 = 0$ for all $s_1, s_2 \in I$.

Since $(I, +)$ has finite index in $(S, +)$, we have a finite set $\{t_1, t_2, \ldots, t_p\}$ of coset representatives for I in S; each S is then of the form $a + t_i$ for some $a \in I$ and some i with $1 \leqslant i \leqslant p$. Since S is finitely generated, there exists a finite subset $\{a_1, a_2, \ldots, a_m\}$ of I such that

$$X = \{a_1, a_2, \ldots, a_m, t_1, t_2, \ldots, t_p\}$$

generates S (as a ring).

Every element of S is now a sum of products of the form $x_1 x_2 \ldots x_k$ with $x_i \in X$ for all i. If (at least) two of the x_i lie in I, then $x_1 x_2 \ldots x_k = 0$ (as I is an ideal and $s_1 s_2 = 0$ for all $s_1, s_2 \in I$). So we need only consider sums of products of the form $x_1 \ldots x_r a_j x_{r+1} \ldots x_k$ and $x_1 \ldots x_r x_{r+1} \ldots x_k$ with $x_i \in \{t_1, t_2, \ldots, t_p\}$ for all i.

We will show that there are only finitely many (sums of) products of the form $x_1 \ldots x_k$ with $x_i \in \{t_1, t_2, \ldots, t_p\}$ for all i. As we commented at the start of this section, if every finite subset of our ring generates a finite semigroup multiplicatively, then every finite subset generates a finite subring. So, if there are only finitely many products of the form

$$x_1 \ldots x_k \quad \text{with } x_i \in \{t_1, t_2, \ldots, t_p\},$$

then there are also only finitely many products of the form $x_1 \ldots x_r a_j x_{r+1} \ldots x_k$ with $x_i \in \{t_1, t_2, \ldots, t_p\}$, and this is sufficient to establish the result.

Since $(I, +)$ is a subgroup of finite index in $(S, +)$, the quotient ring S/I is finite; let $c > 0$ be the characteristic of S/I. If we consider sums of products of the form $x_1 \ldots x_k$ as above, then there exists N such that we get no new elements in S/I by considering such sums with terms where $k \geqslant N$ (i.e. any such sum is equal in S/I to another such sum with $k < N$ for all terms). For any sequence $\rho = (i_1, i_2, \ldots, i_k)$ of non-zero natural numbers, let t_ρ denote $t_{i_1} t_{i_2} \ldots t_{i_k}$; then, for each such sequence ρ of length N, we have

$$t_\rho = b_\rho + t_\sigma \tag{5}$$

for some $b_\rho \in I$ and some sequence σ of length less than N.

We now show that, if τ is any sequence of length at least N, then

$$t_\tau = \sum_{|\rho| < N} \alpha_\rho t_\rho$$

where each α_ρ is either in $\mathbb{Z}/c\mathbb{Z}$ or $b_\rho\mathbb{Z}/c\mathbb{Z}$ for some sequence ρ of length less than N. Note that, if ρ is the empty sequence, we are taking t_ρ to be 1.

We can argue inductively here. If τ has length N, then the result is true by the above. Now assume that the result is true for sequences τ of length n (for some fixed $n \geqslant N$), and let τ be a sequence of length $n+1$. Let t_τ be $t_{i_1}t_{i_2}\dots t_{i_{n+1}}$. By hypothesis, we may write $t_{i_1}t_{i_2}\dots t_{i_n}$ in the form $\sum_{|\rho|<N}\alpha_\rho t_\rho$ as above; so $t_\tau = \sum_{|\rho|<N}\alpha_\rho t_\rho t_{i_{n+1}}$. Terms $\alpha_\rho t_\rho t_{i_{n+1}}$ with ρ of length less that $N-1$ are already of the required form. If ρ has length $N-1$, then we may use (5) to write $t_\zeta = t_\rho t_{i_{n+1}}$ in the form $b_\zeta + t_\sigma$ where ζ has length N, $b_\zeta \in I$ and σ has length less than N. Given that the product of any two elements of I is zero, $\alpha_\rho(b_\zeta + t_\sigma)$ is of the required form.

So there are only finitely many sums of products of the form $x_1\dots x_k$ (with x_i in $\{t_1, t_2, \dots, t_p\}$ for all i) as required. □

As we mentioned above, an FA-presentable integral domain is necessarily finite. We generalize this result to arbitrary rings without zero divisors (i.e. we drop the assumption of commutativity). To be more precise, we prove the following:

Corollary 17. *An FA-presentable ring with identity and no zero divisors is finite.*

Proof. Let R be an FA-presentable ring with identity and no zero divisors and let $x, y \in R$. Since R is locally finite by Theorem 16, the subring S generated by x and y is finite.

The non-zero elements of S form a finite cancellative monoid under multiplication, and hence a group; as a result, S is a finite division ring. Given Wedderburn's theorem that any finite division ring is a field, we have that $xy = yx$ and that the non-zero elements of S are all invertible. Since this holds for all x and y, R is an FA-presentable field, and hence R is finite. □

Theorem 16 does enable us to say a great deal about FA-presentable rings, although, in contrast with Corollary 17, such a ring need not be finite as the following example shows:

Example 18. Let F be any finite ring and choose an automatic presentation for F over an alphabet I in which the encoding of every element of F has the same length; let K denote the set of encodings. Let R be the direct sum of countably many copies of F, say $R = \bigoplus_{i\in\mathbb{N}^+} F_i$, so that every element r of R is of the form $f_1 + f_2 + \dots + f_n$ with $n \geqslant 0$ and $f_i \in F_i$ for each i. If α_i is the encoding of f_i in I^*, then we encode r as $\diamond\alpha_1\diamond\alpha_2\diamond\dots\diamond\alpha_n\diamond$ where $\diamond$ is a new symbol (i.e. $\diamond$ is not an element of I). If we let

$$L = \{\diamond\}K\{\diamond\} \cup \{\diamond\}K\{\diamond\}K\{\diamond\} \cup \cdots,$$

then we see that L is a set of encodings for R in $(I \cup \{\diamond\})^*$ and that we get an automatic presentation for R.

Any FA-presentable ring R with identity is locally finite; in particular, 1 generates a finite subring, and so R has characteristic $c > 0$. In the next result we see that, without loss of generality, we may assume that c is a prime power:

Proposition 19. *If R is an FA-presentable ring, then there exist $k \geqslant 1$ and primes $p_1, p_2, \dots, p_k$ such that $R = R_1 \oplus R_2 \oplus \dots \oplus R_k$ where R_i is an FA-presentable ring of characteristic $p_i^{n_i}$ (for some $n_i \geqslant 1$).*

Proof. Suppose that $c = mn$ with m and n coprime, and choose integers x and y such that $mx + ny = 1$. If $r \in R$, then $r = (mx + ny)r = xs + yt$, where $s = mr$, $t = nr$ and $ns = mt = 0$. If

$$R_1 = \{u \in R\colon\ mu = 0\} \quad \text{and} \quad R_2 = \{u \in R\colon\ nu = 0\},$$

then we have $R = R_1 + R_2$. However, if $u \in R_1 \cap R_2$, then $mu = nu = 0$ and so

$$u = (mx + ny)u = 0;$$

so $R = R_1 \oplus R_2$.

Note that R_1 and R_2 are first-order definable subrings of R and hence are FA-presentable; continuing in this way gives a direct sum decomposition of R in the required form. □

By Proposition 19, if we wish to classify FA-presentable rings, then we can assume that the characteristic is of the form p^a for some prime p and some $a \geqslant 1$ (as the FA-presentable rings are precisely the finite direct sums of such rings).

Given the fact that, if R_1 and R_2 are FA-presentable rings, then $R_1 \oplus R_2$ is FA-presentable, we might consider FA-presentable rings which are not direct sums of two FA-presentable rings. The following result imposes some strong restrictions on the structure of such a ring:

Proposition 20. *If R is a FA-presentable ring with identity which is not the direct sum of two non-trivial FA-presentable rings, then R is the disjoint union of the subsets*

$$N = \{a \in R\colon\ \exists k \in \mathbb{N}^+ \text{ such that } a^k = 0\};$$
$$G = \{a \in R\colon\ \exists k \in \mathbb{N}^+ \text{ such that } a^k = 1\}.$$

G is the set of units and $N = R - G$.

Proof. Suppose that R is such a ring and that e is an idempotent in R with $e \notin \{0, 1\}$. We see that $r = (1 - e)r + er$ for any $r \in R$ and that $1 - e$ is also an idempotent; so $R = (1 - e)R + eR$. However, if $x \in (1 - e)R \cap eR$, say $x = (1 - e)r = es$, then

$$x = (1 - e)x = (1 - e)es = es - e^2 s = 0.$$

So $R = (1 - e)R \oplus eR$. Moreover, since $1 - e \in (1 - e)R$ and $e \in eR$, neither $(1 - e)R$ nor eR is trivial.

Whilst it is not known in general that $R_1 \oplus R_2$ being FA-presentable would imply that R_1 and R_2 are both FA-presentable, we would have this here, as uR is first-order definable for any u in R, being the set of elements x satisfying the first-order sentence $\exists v\ (x = uv)$; so R is the direct sum of two non-trivial FA-presentable rings and we have a contradiction.

So we may assume that the only idempotents in R are 0 and 1. If r is any element of R, then, as R is locally finite, there exist $n, m \in \mathbb{N}^+$ such that $r^{n+m} = r^n$. If we choose k such that $km > n$, then $(a^{km})^2 = a^{2km} = a^{km}$; so a^{km} is an idempotent, and hence $a^{km} = 0$ or 1. So $R = N \cup G$.

Clearly elements of G are invertible (if $a^k = 1$ then a has inverse a^{k-1}) and elements of N are not (if $a^k = 0$ and $ab = 1$ then $0 = a^k b^k = 1$), and so $G = \{a \in R\colon\ \exists b \in R \text{ such that } ab = 1\}$ is first-order definable and $N = R - G$. □

We see that N is the set of nilpotent elements of R. If R is commutative then N is an ideal of R. In fact it is the Jacobson radical $J(R)$ of R; in particular it is a ring and is itself FA-presentable. In this situation N is the unique maximal ideal of R and so R is a local ring. The quotient R/N is an FA-presentable field, and hence finite; so G is finite. Since $1+x$ is a unit for any $x \in N$, $1+N \subseteq G$, and so N is finite as well. We record this fact as a corollary:

Corollary 21. *If R is an FA-presentable commutative ring with identity which is not the direct sum of two non-trivial FA-presentable rings, then R is finite.*

Theorem 16 does not extend to rings without an identity element as the following example shows:

Example 22. Consider any abelian group A with operation $+$ and identity 0, and make A into a ring by defining the product $x \bullet y$ to be 0 for any $x, y \in A$. Clearly $(A, +, 0, ^{-1})$ is FA-presentable as a group if and only if $(A, +, -, \bullet, 0, 1)$ is FA-presentable as a ring. We then simply choose $(A, +)$ to be any abelian FA-presentable group that is not locally finite, and the corresponding ring is not locally finite either.

Given Example 22, a complete characterization of all FA-presentable rings would include a characterization of all FA-presentable abelian groups which seems, at the moment, to be a rather difficult problem. In particular, it is not known whether the group $(\mathbb{Q}, +)$ is FA-presentable; see [13] for example. Even characterizations as to which groups in certain natural subclasses of the class of abelian groups (such as the torsion-free abelian groups) are FA-presentable would be of significant interest.

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References

[1] S. Akiyama, C. Frougny, J. Sakarovitch, Powers of rationals modulo 1 and rational base systems, Israel J. Math., in press.

[2] C.M. Campbell, E.F. Robertson, N. Ruškuc, R.M. Thomas, Automatic semigroups, Theoret. Comput. Sci. 365 (2001) 365–391.

[3] C. Delhommé, V. Goranko, T. Knapik, Automatic linear orderings, preprint, 2003.

[4] D.B.A. Epstein, J.W. Cannon, D.F. Holt, S. Levy, M.S. Paterson, W. Thurston, Word Processing in Groups, Jones and Barlett, 1992.

[5] Ju.L. Eršov, Elementary group theories, Dokl. Akad. Nauk SSSR 203 (1972) 1240–1243 (in Russian); Soviet Math. Dokl. 13 (1972) 528–532.

[6] M. Gromov, Groups of polynomial growth and expanding maps, Publ. Math. Inst. Hautes Études Sci. 53 (1981) 53–78.
[7] D.F. Holt, The Warwick automatic groups software, in: G. Baumslag, D.B.A. Epstein, R.H. Gilman, H. Short, C. Sims (Eds.), Geometric and Computational Perspectives on Infinite Groups, in: DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 25, American Mathematical Society, 1996, pp. 69–82.
[8] C. Huh, N.K. Kim, Y. Lee, Examples of strongly π-regular rings, J. Pure Appl. Algebra 189 (2004) 195–210.
[9] J.F.P. Hudson, Regular rewrite systems and automatic structures, in: J. Almeida, G.M.S. Gomes, P.V. Silva (Eds.), Semigroups, Automata and Languages, World Scientific, 1998, pp. 145–152.
[10] M. Kargapolov, J. Merzljakov, Fundamentals of the Theory of Groups, Springer-Verlag, 1979.
[11] B. Khoussainov, A. Nerode, Automatic presentations of structures, in: D. Leivant (Ed.), Logic and Computational Complexity, in: Lecture Notes in Comput. Sci., vol. 960, Springer-Verlag, 1995, pp. 367–392.
[12] B. Khoussainov, A. Nies, S. Rubin, F. Stephan, Automatic structures: Richness and limitations, in: Proceedings of the 19th IEEE Symposium on Logic in Computer Science, IEEE Computer Society, 2004, pp. 110–119.
[13] B. Khoussainov, S. Rubin, Automatic structures: Overview and future directions, J. Autom. Lang. Comb. 8 (2003) 287–301.
[14] B. Khoussainov, S. Rubin, F. Stephan, Automatic linear orders and trees, ACM Trans. Comput. Log. 6 (2005) 675–700.
[15] A. Mal'cev, On a correspondence between rings and groups, Amer. Math. Soc. Transl. 45 (1965) 221–231.
[16] A. Nies, Describing groups, Bull. Symb. Logic, in press.
[17] A. Nies, P. Semukhin, Finite automata presentable abelian groups, in: Proceedings of LFCS 2007, in press.
[18] G.A. Noskov, The elementary theory of a finitely generated almost solvable group, Izv. Akad. Nauk SSSR Ser. Mat. 47 (1983) 498–517 (in Russian); Math. USSR Izv. 22 (1984) 465–482.
[19] G.P. Oliver, R.M. Thomas, Automatic presentations for finitely generated groups, in: V. Diekert, B. Durand (Eds.), 22nd Annual Symposium on Theoretical Aspects of Computer Science, STACS '05, in: Lecture Notes in Comput. Sci., vol. 3404, Springer-Verlag, 2005, pp. 693–704.
[20] F. Otto, A. Sattler-Klein, K. Madlener, Automatic monoids versus monoids with finite convergent presentations, in: T. Nipkow (Ed.), Rewriting Techniques and Applications—Proceedings RTA '98, in: Lecture Notes in Comput. Sci., vol. 1379, Springer-Verlag, 1998, pp. 32–46.
[21] L. Pelecq, Isomorphismes et automorphismes des graphes context-free, équationnels et automatiques, PhD Thesis, Bordeaux 1 University, 1997.
[22] N.S. Romanovskiĭ, On the elementary theory of an almost polycyclic group, Math. Sb. 111 (1980) 135–143 (in Russian); Math. USSR Sb. 39 (1981).
[23] G. Sénizergues, Definability in weak monadic second-order logic of some infinite graphs, in: K. Compton, J.-E. Pin, W. Thomas (Eds.), Automata Theory: Infinite Computations, in: Dagstuhl Seminar, vol. 9202, Wadern, Germany, 1992, p. 16.
[24] A. Tripe, M. Ursul, Countable compact rings with periodic groups of units, Acta Math. Hungar. 90 (2001) 65–73.