THE JOURNAL OF SYMBOLIC LOGIC
Volume 65, Number 1, March 2000

STRUCTURAL PROPERTIES AND Σ_2^0 ENUMERATION DEGREES

ANDRÉ NIES AND ANDREA SORBI

Abstract. We prove that each Σ_2^0 set which is hypersimple relative to $\emptyset'$ is noncuppable in the structure of the Σ_2^0 enumeration degrees. This gives a connection between properties of Σ_2^0 sets under inclusion and and the Σ_2^0 enumeration degrees. We also prove that some low non-computably enumerable enumeration degree contains no set which is simple relative to $\emptyset'$.

§1. Introduction. There is a wide range of theorems in computability theory asserting that, in a certain degree structure $\mathscr{R}_r$ of computably enumerable (c.e.) sets under a reducibility $\leq_r$, a simplicity property of a computably enumerable set A implies the incompleteness of the r-degree of A. (Here a simplicity property requires that in some sense the complement of A is sparse.) An example of such a result is that a simple set cannot be btt-complete ([Pos44]). While a simple set may be tt-complete, the stronger notion of hypersimplicity of A even implies wtt-incompleteness. Downey and Jockusch [DJ87] showed that the wtt-degree of a hypersimple set H is in fact wtt-*noncuppable*, namely $K \leq_{\text{wtt}} H \oplus B$ implies $K \leq_{\text{wtt}} B$ for any computably enumerable B.

An interesting question is whether results of this kind can be obtained for $\mathscr{D}_e(\leq \mathbf{0}_e')$, the structure of enumeration degrees of Σ_2^0-sets. Since the domain consists of the sets that are computably enumerable in $\emptyset'$, one also has to relativize the simplicity properties to $\emptyset'$. For instance:

DEFINITION 1.1. A Σ_2^0 set H is *$\emptyset'$-hypersimple* if H is coinfinite and there is no function $f \leq_T \emptyset'$ bounding $p_{\overline{H}}$, where $p_{\overline{H}}$ is the function that lists the complement of H in order of magnitude.

The existence of $\emptyset'$-hypersimple sets (and of $\emptyset'$-simple sets, defined in the next section) follows by a straightforward relativization to $\emptyset'$ of Post's constructions of hypersimple (simple, respectively) sets, see [Pos44]. The reader is referred to [Coo90] for an extensive survey and bibliography on enumeration reducibility and its degree structure.

Received May 27, 1998.

Partially supported by NSF grant DMS–9500983.

Partially supported by the HC&M research network *Complexity, Logic and Recursion Theory* (*COLORET*), contract no. ERBCHRXCT930415, and by MURST 60%.

The paper was completed while the second author was visiting at the Department of Mathematics of the University of Wisconsin at Madison.

0022-4812/00/6501-0015/$1.80

§2. $\emptyset'$-hypersimple sets are noncuppable. Boldface small Latin letters $\mathbf{a}, \mathbf{b}, \mathbf{c}$ denote e-degrees. Recall that $\mathbf{0}'_e$ is the e-degree of $\overline{K}$, the complement of the halting set K.

Cupping and noncupping properties of $\mathscr{D}_e(\leq \mathbf{0}'_e)$ have been studied in details in [CSY97].

THEOREM 2.1 ([CSY97]). *There exists a noncupping element in $\mathscr{D}_e(\leq \mathbf{0}'_e)$, i.e., an e-degree $\mathbf{a} \leq \mathbf{0}'_e$ such that $\mathbf{a}$ is nonzero and*

$$(\forall \mathbf{b} \leq \mathbf{0}'_e)[\mathbf{0}'_e \leq \mathbf{a} \cup \mathbf{b} \Rightarrow \mathbf{0}'_e \leq \mathbf{b}].$$

However,

THEOREM 2.2 ([CSY97]). *Every nonzero Δ^0_2 e-degree $\mathbf{a}$ is cupping in $\mathscr{D}_e(\leq \mathbf{0}'_e)$, in fact there exists a total (hence Δ^0_2) e-degree $\mathbf{b} < \mathbf{0}'_e$ such that $\mathbf{0}'_e \leq \mathbf{a} \cup \mathbf{b}$.*

We show now that every $\emptyset'$-hypersimple e-degree is noncupping. In fact:

THEOREM 2.3. *Suppose C is a Δ^0_2 set which is not computably enumerable, H is $\emptyset'$-hypersimple and B is Σ^0_2. Then $C \leq_e B \oplus H$ implies $C \leq_e B$.*

COROLLARY 2.4. *If H is $\emptyset'$-hypersimple, then the enumeration degree of H is noncupping in $\mathscr{D}_e(\leq \mathbf{0}'_e)$.*

PROOF. Let $C = \overline{K}$. ⊣

COROLLARY 2.5. *If C is a Δ^0_2 non-computably enumerable set and H is $\emptyset'$-hypersimple, then $C \not\leq_e H$.*

PROOF OF THE THEOREM. Assume that $C = \Phi^{B \oplus H}$, for some e-operator Φ (with finite approximations $\{\Phi_s\}_{s\in\omega}$), but $C \not\leq_e B$. We will determine a Δ^0_2 function g such that, for $n \neq m$,

$$D_{g(n)} \not\subseteq H \quad \text{and} \quad D_{g(n)} \cap D_{g(m)} \subseteq H.$$

Then the Δ^0_2 function

$$f(n) = \max\left(\bigcup_{m \leq n} D_{g(m)}\right)$$

bounds $p_{\overline{H}}$.

Fix Σ^0_2 approximations $\{B_s\}_{s\in\omega}$, $\{H_s\}_{s\in\omega}$ for B, H, respectively, such that $\{H_s\}$ has infinitely many true stages, i.e., $(\exists^\infty s)[\, H_s \subseteq H]$ (see [Joc68]). For each set $R \subseteq \omega$, let $\gamma^R(x, s, t)$ be the predicate

$$x \notin C \text{ and } s \in R \text{ and } t > s \text{ and } (\exists \langle x, D\rangle \in \Phi_s)$$
$$[D \subseteq B_s \oplus H_s \text{ and } (\forall t' \geq t)[D_B \subseteq B_{t'}]],$$

where $D = D_B \oplus D_H$.

We claim that for each infinite recursive set R, there exist an x such that for some $s \in R$ and some $t > s$, $\gamma^R(x, s, t)$ holds.

For if such s, t only exist for no $x \notin C$, then we can define an enumeration reduction procedure Γ such that $C = \Gamma^B$ as follows. If an axiom $\langle x, D\rangle$ lies in Φ_s, with $s \in R$ and $D \subseteq B_s \oplus H_s$, then put $\langle x, D_B\rangle$ into Γ_s. If $x \in C$, then $x \in \Phi^{B\oplus H}$, so there is some $\langle x, D_B\rangle \in \Gamma$ such that $D_B \subseteq B$. But for each $x \notin C$, if $\langle x, D\rangle \in \Gamma_s$, then $D \not\subseteq B$ because $D \not\subseteq B_t$ for infinitely many $t \geq s$.

Clearly, if $\gamma^R(x, s, t)$ holds, then $H_s \not\subseteq H$. Moreover, if R is recursive, then $\gamma^R(x, s, t)$ is a Δ_2^0 property of x, s and t, since C is Δ_2^0.

Now we define by induction a function g having the desired properties.

Step 0. To define $D_{g(0)}$, think of R being ω. Using $\emptyset'$ as an oracle, find x, s, t satisfying $\gamma^R(x, s, t)$ and let $D_{g(0)} = H_s$.

Step $n > 0$. Now suppose $g(m)$ has been defined for all $m < n$. Our goal is to find a finite set $D = D_{g(n)}$ such that $\overline{H} \cap D \cap D_{g(m)} = \emptyset$ for all $m < n$, and $D \cap \overline{H} \neq \emptyset$. Let $E = \bigcup_{m<n} D_{g(m)}$. We start a searching procedure by stages. At stage u of the search for D, we determine

$$\alpha_u = \{x \in E : (\exists u' \geq u)[x \notin H_{u'}]\},$$

using $\emptyset'$ as an oracle. Thus α_u is the set of elements of E which have not yet "proved" to be in H, and α_u converges to $E - H$. Next we look for $x < u$, $s < t < u$ such that $\gamma^R(x, s, t)$ is satisfied for $R = \{s : H_s \cap \alpha_u = \emptyset\}$. If x, s, t fail to exist then proceed to stage $u + 1$. If they do exist, then let $D_{g(n)} = H_s$.

First of all, this search will terminate. For all sufficiently large u, $\alpha_u = E - H$. Moreover, $R = \{s : H_s \cap \alpha_u = \emptyset\}$ is infinite. So, as argued above, x, s, t must exist, and $D_{g(n)} \not\subseteq H$. Moreover $D_{g(n)} \cap E \subseteq H$, because $E - H \subseteq \alpha_u$ and $\alpha_u \cap D_{g(n)} = \emptyset$. ⊣

REMARK. Notice that in the construction above, H_s could be found "too early", namely while we are still working with an α_s which is not disjoint from H. But still $\alpha_s \cap H_s = \emptyset$ and hence $A \cap H_s \subseteq H$.

We prove next that Theorem 2.3 cannot be improved to $\emptyset'$-simple sets, by relativizing the usual construction of a tt-complete simple set. Define a Σ_2^0 set A to be $\emptyset'$-simple if A is coinfinite, and $A \cap V \neq \emptyset$, for every Σ_2^0-set V.

THEOREM 2.6. *There is a $\emptyset'$-simple set A such that $\overline{K} \leq_e A$.*

PROOF. Let $h(n) = n(n+1)/2$ and $E_n = [h(n), h(n+1))$ so that $|E_n| = n + 1$. Define a partial $\emptyset'$-recursive function φ such that $\varphi(e)$ is the first element $\geq h(e+1)$ which appears in $W_e^{\emptyset'}$, and undefined if there is no such element. Now let

$$A = \bigcup\{E_n : n \in \overline{K}\} \cup \operatorname{range}(\varphi).$$

Clearly $|E_n \cap \operatorname{range}(\varphi)| \leq n$. Thus, $n \in \overline{K} \Rightarrow E_n \subseteq A$ and therefore $\overline{K} \leq_e A$. Moreover, if $W_e^{\emptyset'}$ is infinite, then $\varphi(e)$ is defined and therefore $W_e^{\emptyset'} \cap A \neq \emptyset$. ⊣

§3. $\emptyset'$-simple free enumeration degrees. It is clear that there exist nonzero Σ_2^0 enumeration degrees containing no $\emptyset'$-hypersimple set (by Corollary 2.5, consider any enumeration degree with some nonzero Δ_2^0 predecessor): in fact there even exist properly Σ_2^0 enumeration degrees (i.e., Σ_2^0 enumeration degrees containing no Δ_2^0 sets) which are $\emptyset'$-hypersimple free: this follows from the fact (see [BCS]) that for every enumeration degree $\mathbf{a} < \mathbf{0}'_e$ there exists a properly Σ_2^0 enumeration degree $\mathbf{b}$ such that $\mathbf{a} \leq \mathbf{b} < \mathbf{0}'_e$.

We show in this section that there exist nonzero $\emptyset'$-simple free enumeration degrees below $\mathbf{0}'_e$.

THEOREM 3.1. *There exists a nonzero low enumeration degree containing no $\emptyset'$-simple set.*

PROOF OF THE THEOREM. We build a Σ^0_2 set C such that C is low and not computably enumerable, and for every Σ^0_2 set B, if $B \equiv_e C$ then there exists some infinite Σ^0_2 set E such that $E \subseteq \overline{B}$.

The requirements for the construction are:

$$\begin{array}{ll} L_\Phi: & \lim_s \Phi^{C_s}_{e,s}(e) \text{ exists (if } \Phi = \Phi_e) \\ N_W: & C \neq W \\ P_{\Phi,\Psi,B}: & B = \Phi^C \text{ and } C = \Psi^B \Rightarrow E_{\Phi,\Psi,B} \text{ infinite and } E_{\Phi,\Psi,B} \subseteq \overline{B} \end{array}$$

for all enumeration operators Φ, Ψ, and every computably enumerable set W, where $E_{\Phi,\Psi,B}$ is some Σ^0_2 set to be constructed. Notice that if L_Φ is satisfied for every Φ, then C is low, by [MC85].

The strategy for L_Φ consists in fixing some finite set $\gamma \subseteq C$ such that $e \in \Phi^\gamma$ (where $\Phi = \Phi_e$). The strategy for N_W is a standard diagonalization: choose a witness n, keep $n \in C$ until a stage s at which n is enumerated into W; then let $n \notin C$. We attack a P-requirement $P_{\Phi,\Psi,B}$ via infinitely many sub-requirements $\{P_{\Phi,\Psi,B,j}\}_{j\in\omega}$. Given j, the strategy for $P_{\Phi,\Psi,B,j}$ will be as follows:

(a) choose a witness n_j, and let $n_j \in C$;
(b) wait for a finite set θ such that $n_j \in \Psi^\theta$ and $\theta \cap \overline{B} \cap [0, j) = \emptyset$;
(c) extract n_j from C, and enumerate $\theta \cap \overline{B}$ into $E_{\Phi,\Psi,B}$.

First notice that if $B = \Phi^C$ then B is Δ^0_2 (since C is low, and every predecessor of a low enumeration degree consists entirely of Δ^0_2 sets, see [MC85]): thus $E_{\Phi,\Psi,B} \in \Sigma^0_2$. Moreover $E_{\Phi,\Psi,B} \subseteq \overline{B}$. Finally, if, for every j, there exists a finite set θ such that $\theta \cap \overline{B} \subseteq E_{\Phi,\Psi,B}$ and $\theta \cap \overline{B} \cap [j, +\infty) \neq \emptyset$, then $E_{\Phi,\Psi,B}$ is infinite.

We achieve $\theta \cap \overline{B} \cap [j, +\infty) \neq \emptyset$, by looking for a finite set θ with $n_j \in \Psi^\theta$ and $\theta \cap \overline{B} \cap [0, j) = \emptyset$. Then either we never find such a θ, getting in this case $n_j \in C - \Psi^B$; or $n_j \notin C$, and therefore $\theta \not\subseteq B$ (assuming $C = \Psi^B$), and $\theta \cap \overline{B} \cap [j, +\infty) \neq \emptyset$.

We assume throughout some fixed priority ordering of the requirements and sub-requirements, in which $P_{\Phi,\Psi,B,j}$ has higher priority than $P_{\Phi,\Psi,B,j'}$ if $j < j'$.

The tree of outcomes. The *tree of outcomes* $T \subseteq [2 \cup (\omega \times 2)]^{<\omega}$ is defined inductively as follows: $\emptyset \in T$ and $\emptyset$ is an *L-node*; if σ is an L-node then $\sigma^\frown i \in T$ and $\sigma^\frown i$ is an *N-node*, for $i \in \{0, 1\}$; if σ is an N-node then $\sigma^\frown i \in T$ and $\sigma^\frown i$ is a *P-node*, for $i \in \{0, 1\}$; finally, if σ is a P-node then $\sigma^\frown(h, i) \in T$ and $\sigma^\frown(h, i)$ is an *L-node*, for $h \in \omega$ and $i \in \{0, 1\}$.

Given $\sigma, \tau \in T$, define $\sigma \preceq \tau$ if

$$\sigma \subseteq \tau \text{ or } [\sigma(i(\sigma, \tau)) < \tau(i(\sigma, \tau))]$$

where $i(\sigma, \tau) = \min\{i : \sigma(i) \neq \tau(i)\}$ if $\sigma \not\subseteq \tau$ and $\tau \not\subseteq \sigma$: for this we define $(h, i) < (h', i')$ if $h > h'$ or $[h = h'$ and $i < i']$, for every $h, h' \in \omega$ and $i, i' \in \{0, 1\}$. Finally, let $\sigma \prec_L \tau$ if $\sigma \not\subseteq \tau$ and $\sigma \preceq \tau$.

Let $T^P = \{\sigma^\frown h : \sigma \in T \text{ and } h \in \omega \text{ and } \sigma\ P\text{-node}\}$ and let $\{\nu_\sigma\}_{\sigma \in \widehat{T}}$ be a computable partition of ω into infinite computable sets, where

$$\widehat{T} = T^P \cup \{\sigma : \sigma \in T \text{ and } \sigma\ N\text{-node}\}.$$

We extend $\preceq$ to $\widehat{T}$ in the obvious way. We assume throughout a standard requirement assignment function R, assigning to each $\sigma \in T$ a requirement $R(\sigma)$, where $R(\sigma)$ is an L- (N-, P-) requirement or sub-requirement according as σ is an L- (N-, P-) node; moreover $R(\sigma)$ has higher priority than $R(\tau)$ if $\sigma \subset \tau$.

The construction is by stages. At step s we define a string δ_s, with $|\delta_s| \leq s$ together with the values of the parameters $\gamma(\sigma, s)$, $\varepsilon(\sigma, s)$, $n(\sigma, s)$, $\theta(\sigma, s)$, $h(\sigma, s)$, $L(\sigma, s)$, $<_{\sigma,s}$, for $\sigma \in T \cup T^P$. At each stage s each parameter retains the same value as at the preceding stage, unless otherwise specified. For every $\sigma \in T \cup T^P$, $\gamma(\sigma, s)$ is a parameter for some finite set which the construction wants to fix in C; $\varepsilon(\sigma, s)$ is a parameter for some finite set which the construction wants to keep out of C; $n(\sigma, s)$ denotes the current witness to the requirement $R(\sigma)$. If at stage s, we take action at σ (i.e., $\sigma \subseteq \delta_s$), where σ is a P-node, then we give outcome (h, i) at σ if h is the canonical index of the (current assessment of the) finite set $\overline{B} \cap [0, j)$; and $i = 0$ if there exists some finite set $\theta(\sigma^\frown h, s)$ such that $n(\sigma^\frown h, s) \in \Phi_s^{\theta(\sigma^\frown h, s)}$ and $\theta(\sigma^\frown h, s) \cap D_h = \emptyset$; otherwise $i = 1$. We let $h = h(\sigma, s)$ to be the $<_{\sigma,s}$-least element of a finite set $L(\sigma, s)$, where, for $h, h' \in L(\sigma, s)$, we let $h <_{\sigma,s} h'$ if there is "more evidence" at s of being $D_h = \overline{B} \cap [0, j)$ rather than $D_{h'} = \overline{B} \cap [0, j)$.

Step 0. Let $\delta_0 = \emptyset$; for every $\sigma \in T \cup T^P$, let $n(\sigma, 0) = h(\sigma, 0) =\uparrow$, and define

$$\gamma(\sigma, 0) = \varepsilon(\sigma, 0) = \theta(\sigma, 0) = L(\sigma, 0) = <_{\sigma,0} = \emptyset.$$

Step $s + 1$. Suppose we have defined $\sigma = \delta_{s+1} \upharpoonright n$, with $n < s + 1$. In order to define $\sigma^+ = \delta_{s+1} \upharpoonright n + 1$, and the relative parameters, we distinguish the following three cases.

Case 1. σ is an L-node, say $R(\sigma) = L_\Phi$, and assume $\Phi = \Phi_e$:

1. if there exists a finite set γ such that $e \in \Phi_s^\gamma$ and

$$\gamma \cap \bigcup_{\tau \preceq \sigma} \varepsilon(\tau, s + 1) = \emptyset,$$

then let $\sigma^+ = \sigma^\frown 0$ and $\gamma(\sigma^+, s + 1) = \gamma$ for the least such γ;

2. otherwise, let $\sigma^+ = \sigma^\frown 1$.

Case 2. σ is an N-node, say $R(\sigma) = N_W$: let $n_\sigma = n(\sigma, s + 1)$ be the least number n such that $n \in v_\sigma$ and $n \notin \bigcup_{\tau \preceq \sigma} \gamma(\tau, s + 1)$.

1. if $n_\sigma \notin W^s$, then let $\sigma^+ = \sigma^\frown 1$ and $\gamma(\sigma^+, s + 1) = \{n_\sigma\}$;
2. otherwise, let $\sigma^+ = \sigma^\frown 0$ and $\varepsilon(\sigma^+, s + 1) = \{n_\sigma\}$.

Case 3. σ is a P-node; assume $R(\sigma) = P_{\Phi,\Psi,B,j}$: if $L(\sigma, s + 1) = \emptyset$, then let $h(\sigma, s + 1) = 0$. Otherwise, define $h(\sigma, s + 1)$ to be the $<_{\sigma,s}$-least element of $L(\sigma, s)$.

Defining $L(\sigma, s + 1)$. If D_h is a finite set with $\max D_h < j$, and $h = h(\sigma, s + 1)$, or $h \notin L(\sigma, s)$ and h does not have a precondition at s, then we assign to h a new precondition $p(\sigma, h, s + 1)$; if h has a precondition $p(\sigma, h, s)$ (i.e., $p(\sigma, h, s) \downarrow$), which was first assigned at a stage $v \leq s$, then we say that this precondition is *satisfied at* $s + 1$ if

$$(\forall x \in D_h)(\exists v)[t \leq v < s + 1 \quad \text{and} \quad x \notin B^v].$$

Define

$$L(\sigma, s+1) = (L(\sigma, s) - \{h(\sigma, s+1)\}) \cup \{h : p(\sigma, h, s+1)\downarrow \quad \text{and} \quad p(\sigma, h, s+1) \text{ satisfied at } s+1\}.$$

We order $L(\sigma, s+1)$ as follows: for every $h, h' \in L(\sigma, s+1)$, let $h <_{\sigma,s+1} h'$ if and only if either

- $h, h' \in L(\sigma, s)$ and $h <_{\sigma,s} h'$, or
- $h \in L(\sigma, s)$ and $h' \notin L(\sigma, s)$, or
- $h, h' \notin L(\sigma, s)$ and $h' < h$.

Then define $\widetilde{\sigma} = \sigma^\frown h(\sigma, s+1)$. Finally, let $n_{\widetilde{\sigma}} = n(\widetilde{\sigma}, s+1)$ be the least number n such that $n \in v_{\widetilde{\sigma}}$ and $n \notin \bigcup_{\tau \preceq \sigma} \gamma(\tau, s+1)$:

1. if there exists some finite set D such that $D \cap D_{h(\sigma,s+1)} = \emptyset$, and $n_{\widetilde{\sigma}} \in \Psi_s^D$, then choose the least such set D and define $\theta(\widetilde{\sigma}, s+1) = D$; let $\sigma^+ = \widetilde{\sigma}^\frown 0$ and $\varepsilon(\sigma^+, s+1) = \{n_{\widetilde{\sigma}}\}$;
2. otherwise, let $\sigma^+ = \widetilde{\sigma}^\frown 1$ and $\gamma(\sigma^+, s+1) = \{n_{\widetilde{\sigma}}\}$.

Define

$$C^{s+1} = \left(C^s \cup \bigcup_{\tau \subseteq \delta_{s+1}} \gamma(\tau, s)\right) - \bigcup_{\tau \subseteq \delta_{s+1}} \varepsilon(\tau, s).$$

This concludes the construction. We now verify that the construction works.

LEMMA 3.2. *For every n, $\liminf_s \delta_s \upharpoonright n$ exists.*

PROOF OF THE LEMMA. Assume by induction that $\sigma_n = \liminf_s \delta_s \upharpoonright n$ exists. Clearly it is enough to consider the case when σ_n is a P-node ($R(\sigma_n) = P_{\Phi,\Psi,B,j}$, say), and show that there exist h and i such that $\sigma_n{}^\frown(h, i) = \liminf_s \delta_s \upharpoonright n+1$. Let h be such that $D_h = [0, j) \cap \overline{B}$: notice that, for every v, if we assign a precondition $p(\sigma_n, h, v)$ to h at v, then there is a stage $s > v$ at which this precondition is satisfied. This shows that at infinitely many stages s, $h \in L(\sigma_n, s)$, and consequently, at infinitely many stages s', we have that $h = h(\sigma_n, s')$. On the other hand it is clear that, if t is a stage such that $B^s(x) = B^t(x)$, for every $s \geq t$ and $x < j$ with $x \in B$, then $D_{h(\sigma_n,s)} \subseteq D_h$, for every $s \geq t$. Therefore, for some $i \in \{0, 1\}$ we have that $\sigma_n{}^\frown(h, i) = \liminf_s \delta_s \upharpoonright n+1$. ⊣

By the previous lemma, let f be the *true path*, i.e., $f = \bigcup_n \sigma_n$, where $\sigma_n = \liminf_s \delta_s \upharpoonright n$.

LEMMA 3.3. *For every n, $\lim_s \gamma(\sigma_n, s)$ and $\lim_s \varepsilon(\sigma_n, s)$ exist.*

PROOF OF THE LEMMA. The claim is trivially true for $n = 0$. Assume by induction that the claim is true of n, and let t_n be a stage such that, for every $s \geq t_n$, $\gamma(\sigma_n, s) = \gamma(\sigma_n, t_n)$, and $\varepsilon(\sigma_n, s) = \varepsilon(\sigma_n, t_n)$, and $\tau \not\subseteq \delta_s$, for every $\tau \prec_L \sigma_n$. For every $\tau \preceq \sigma_n$, let $\gamma(\tau) = \lim_s \gamma(\tau, s)$ and $\varepsilon(\tau) = \lim_s \varepsilon(\tau, s)$.

Case 1. If σ_n is an L-node and $R(\sigma_n) = L_\Phi$, with $\Phi = \Phi_e$, then $\varepsilon(\sigma_{n+1}) = \emptyset$; if $\sigma_{n+1} = \sigma_n{}^\frown 1$, then $\gamma(\sigma_{n+1}) = \emptyset$; otherwise there exists a finite set γ such that

$$\gamma \cap \bigcup_{\tau \preceq \sigma_n} \varepsilon(\tau) = \emptyset$$

and $e \in \Phi^{\gamma}$: in this case $\gamma(\sigma_{n+1}) = \gamma$, for the least such γ.

Case 2. If σ_n is an N-node, then we first observe that $n_{\sigma_n} = \lim_s n(\sigma_n, s)$ exists: n_{σ_n} is the least number n such that $n \in \xi_{\sigma_n}$ and $n \notin \bigcup_{\tau \preceq \sigma_n} \gamma(\tau)$. Then $\gamma(\sigma_{n+1}) = \{n_{\sigma_n}\}$ and $\varepsilon(\sigma_{n+1}) = \emptyset$ if $\sigma_{n+1} = \sigma_n{}^\frown 1$; otherwise $\gamma(\sigma_{n+1}) = \emptyset$ and $\varepsilon(\sigma_{n+1}) = \{n_{\sigma_n}\}$.

Case 3. Assume now that σ_n is a P-node, with $R(\sigma_n) = P_{\Phi,\Psi,B,j}$. If $\sigma_{n+1} = \widetilde{\sigma}{}^\frown 0$, then as in the previous case, one easily sees that $n_{\widetilde{\sigma}} = \lim_s n(\widetilde{\sigma}, s)$ exists, and $\varepsilon(\sigma_{n+1}) = \{n_{\widetilde{\sigma}}\}$ and $\gamma(\sigma_{n+1}) = \emptyset$; otherwise $\gamma(\sigma_{n+1}) = \{n_{\widetilde{\sigma}}\}$ and $\varepsilon(\sigma_{n+1}) = \emptyset$. ⊣

LEMMA 3.4. *C is low and not computably enumerable*

PROOF OF THE LEMMA. In order to show that C is low, it is enough to show that, for every e, $\lim_s \Phi_{e,s}^{C_s}(e)$ exists. Given e, let n be such that $R(\sigma_n) = L_{\Phi_e}$. If there exist infinitely many stages s such that $e \in \Phi_{e,s}^{C_s}$, then $\sigma_n{}^\frown 0 \subset f$ and $e \in \Phi_e^{\gamma(\sigma_n)}$, with $\gamma(\sigma_n) \subseteq C$. This shows that $e \in \Phi_e^C$.

It is straightforward to check that each N-requirement is satisfied, hence C is not computably enumerable ⊣

LEMMA 3.5. $\deg_e(C)$ *does not contain any* $\emptyset'$*-simple set.*

PROOF OF THE LEMMA. We show that for every B such that $B \equiv_e C$, there exists an infinite Σ_2^0 set E such that $E \subseteq \overline{B}$.

Given any Σ_2^0 set B and any pair of enumeration operators Φ, Ψ, define

$$E_{B,\Phi,\Psi} = \{x : (\exists s)(\exists \sigma)(\exists j)[R(\sigma) = P_{\Phi,\Psi,B,j} \quad \text{and} \quad x \in \theta(\sigma, s) \cap \overline{B}]\}.$$

Assume now that $B \equiv_e C$: let Φ, Ψ be enumeration operators such that $B = \Phi^C$ and $C = \Psi^B$. Since $B \leq_e C$, and C is low, we have that $B \in \Delta_2^0$. Hence $E_{B,\Phi,\Psi}$ is a Σ_2^0 set. Moreover, by definition, $E_{B,\Phi,\Psi} \subseteq \overline{B}$. It is left to show that $E_{B,\Phi,\Psi}$ is infinite. To this end, let j be given, and let $\sigma \subset f$ be such that $R(\sigma) = P_{\Phi,\Psi,B,j}$; let h be such that $\widetilde{\sigma} = \sigma{}^\frown h \subset f$. The construction ensures that there are infinitely many stages s such that $(\sigma^+ =)\widetilde{\sigma}{}^\frown 0 \subseteq \delta_s$, at which we find a finite set $\theta = \theta(\widetilde{\sigma}, s)$ such that $\theta \cap \overline{B} \cap [0, j) \neq \emptyset$ and $\theta \not\subseteq B$ (since $n_{\widetilde{\sigma}} \in \Psi_s^{\theta}$, but $n_{\widetilde{\sigma}} \notin C$). Then for each such s there exists $x \geq j$ such that $x \in \theta$ but $x \notin B$, hence $x \in E_{B,\Phi,\Psi}$. This shows that $E_{B,\Phi,\Psi}$ contains arbitrarily large numbers, i.e., $E_{B,\Phi,\Psi}$ is infinite. So $E_B = E_{B,\Phi,\Psi}$ is the desired set. ⊣

REFERENCES

[BCS] S. BEREZNYUK, R. COLES, and A. SORBI, *The distribution of properly Σ_2^0 enumeration degrees*, this JOURNAL, to appear.

[Coo90] S.B. COOPER, *Enumeration reducibility, nondeterministic computations and relative computability of partial functions*, ***Recursion theory week, Oberwolfach 1989*** (K. Ambos-Spies, G. Müller, and G.E. Sacks, editors), Lecture Notes in Mathematics, vol. 1432, Springer-Verlag, Heidelberg, 1990, pp. 57–110.

[CSY97] S.B. COOPER, A. SORBI, and X. YI, *Cupping and noncupping in the enumeration degrees of Σ_2^0 sets*, ***Annals of Pure and Applied Logic***, vol. 82 (1997), pp. 317–342.

[DJ87] R.G. DOWNEY and CARL G. JOCKUSCH, JR., **T**-*degrees, jump classes, and strong reducibilities*, ***Transactions of the American Mathematical Society***, vol. 30 (1987), pp. 103–137.

[Joc68] C.G. JOCKUSCH, JR., *Semirecursive sets and positive reducibility*, ***Transactions of the American Mathematical Society***, vol. 131 (1968), pp. 420–436.

[MC85] K. McEvoy and S. B. Cooper, *On minimal pairs of enumeration degrees*, this Journal, vol. 50 (1985), pp. 983–1001.

[Pos44] E.L. Post, *Recursively enumerable sets of positive integers and their decision problems*, ***Bulletin of the American Mathematical Society***, vol. 50 (1944), pp. 284–316.

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF CHICAGO
CHICAGO, ILLINOIS 60637-1514
E-mail: nies@math.uchicago.edu

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SIENA
53100 SIENA, ITALY
E-mail: sorbi@unisi.it