Branching in the enumeration degrees of the Σ_2^0 sets

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Abstract

We show that every incomplete Σ_2^0 enumeration degree is meet-reducible in the structure of the enumeration degrees of the Σ_2^0 sets.

1 Introduction

Informally, a set A is enumeration reducible to a set B if there is an effective procedure for enumerating A, given *any* enumeration of B. Following **?**), and **?**), this is usually formalized using the notion of enumeration operator:

A mapping $\Phi : 2^\omega \longrightarrow 2^\omega$ is an *enumeration operator* (or, simply an *e-operator*), if there exists a recursively enumerable set W such that, for each set B,

$$\Phi^B = \{x (\exists u)[\langle x, u\rangle \in W \,\&\, D_u \subseteq B]\},$$

where D_u denotes the finite set with canonical index u.

If the r.e. set W_z defines the e-operator Φ in the sense of the above definition, then we let $\Phi = \Phi_z$. Let $\{W_z z \in \omega\}$ be the standard enumeration of the r.e. sets: we get a corresponding enumeration $\{\Phi_z z \in \omega\}$ of the e-operators. If $\{W_z^s s \in \omega\}$ is a recursive enumeration of W_z (in the sense of **?**, p. 34)), then we get a corresponding *recursive enumeration* $\{\Phi_z^s s \in \omega\}$ of the e-operator Φ_z. We will refer in the following to some fixed recursive sequence $\{W_{z,s} z, s \in \omega\}$ of finite sets, such that, for every z, $\{W_z^s s \in \omega\}$ is a recursive enumeration of W_z.

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Given sets A, B, we say that A is *enumeration reducible* (or, simply, *e-reducible*) *to* B (notation: $A \leq_e B$), if there exists some e-operator Φ such that $A = \Phi^B$.

It is easily seen that $\leq_e$ is a preordering relation. Let $\equiv_e$ denote the equivalence relation generated by $\leq_e$. The $\equiv_e$-equivalence class of a set A (denoted by $\deg_e(A)$) is called the *enumeration degree* (or, simply, the *e-degree*) of A. On e-degrees the reducibility $\leq_e$ originates a partial ordering relation (denoted by $\leq$). We therefore get a degree structure $\langle \mathfrak{D}_e, \leq \rangle$, where $\mathfrak{D}_e$ is the collection of all e-degrees and $\leq$ is defined by: $[A]_e \leq [B]_e$ if and only if $A \leq_e B$. In fact $\mathfrak{D}_e$ is an upper semilattice with least element $\mathbf{0}_e$ and binary operation $\cup$: the least element $\mathbf{0}_e$ is the e-degree of the r.e. sets and $[A]_e \cup [B]_e = [A \oplus B]_e$, with $A \oplus B = \{2xx \in A\} \bigcup \{2x+1x \in B\}$. The reader may consult **?**) and **?**) for an extensive survey and bibliography on the e-degrees.

An important class of e-degrees is constituted by the Σ^0_2 e-degrees, i.e. the e-degrees of the Σ^0_2 sets. It is known, see **?**) and **?**), that the Σ^0_2 e-degrees coincide with the structure $\mathfrak{G} = \mathfrak{D}_e(\leq \mathbf{0}'_e)$, where $\mathbf{0}'_e = \deg_e(\overline{K})$, $\overline{K}$ being the complement of the halting set K.

Although, under several respects, $\mathfrak{G}$ can be viewed as the e-degree theoretic analog of the structure $\mathfrak{R}$ of the Turing degrees of the r.e. sets (as suggested for instance by Cooper: see the density theorem for $\mathfrak{G}$, **?**)), there are striking elementary differences between the two structures. For instance, **?**) shows that there exist (in fact, low) e-degrees $\mathbf{a}, \mathbf{b} \in \mathfrak{G}$ such that $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'_e$ and $\mathbf{a} \cap \mathbf{b} = \mathbf{0}_e$.

We show in this paper another elementary difference between $\mathfrak{G}$ and $\mathfrak{R}$, that refers to the notion of branching.

Let $\mathfrak{P} = \langle P, \leq \rangle$ be a partial order. We say that an element $c \in P$ is *branching* (or *meet-reducible*) if

$$(\exists a \in P)(\exists b \in P)[c < a, b \,\&\, c = a \wedge b].$$

An element $c \in P$ is called *nonbranching* if it is not branching.

?) proved the existence of incomplete nonbranching elements in $\mathfrak{R}$. Subsequently, **?**) proved that the nonbranching elements of $\mathfrak{R}$ are dense. **?**) proved the density of the branching elements of $\mathfrak{R}$.

We prove here a rather surprising result for $\mathfrak{G}$: all elements $\mathbf{a} \in \mathfrak{G}$ such that $\mathbf{a} < \mathbf{0}'_e$ are branching in $\mathfrak{G}$. In fact, we prove

For every pair of incomplete Σ^0_2 enumeration degrees $\mathbf{c}_1, \mathbf{c}_2$ there exists a pair $\mathbf{a}, \mathbf{b}$ of enumeration degrees below $\mathbf{0}'_e$ such that, for every $i \in \{1, 2\}$:

(1) $\mathbf{a} \cup \mathbf{c}_i, \mathbf{b} \cup \mathbf{c}_i \not\leq \mathbf{c}_i$, and

(2) $\mathbf{c}_i = (\mathbf{a} \cup \mathbf{c}_i) \cap (\mathbf{b} \cup \mathbf{c}_i)$.

For simplicity, we will prove the following weaker version of the above theorem, and we then will explain how to modify the proof, in order to obtain Theorem 1 above:

For every incomplete Σ^0_2 enumeration degree $\mathbf{c}$ there exist enumeration degrees $\mathbf{a}, \mathbf{b}$ below $\mathbf{0}'_e$ such that:

(1) $\mathbf{a} \cup \mathbf{c}, \mathbf{b} \cup \mathbf{c} \not\leq \mathbf{c}$, and

(2) $\mathbf{c} = (\mathbf{a} \cup \mathbf{c}) \cap (\mathbf{b} \cup \mathbf{c})$.

In the following, suppose that L is a Σ^0_2 set such that $L <_e \overline{K}$, and suppose we are given some Σ^0_2 approximation $\{L^s s \in \omega\}$ to L, i.e. a recursive sequence of finite sets such that

$$L = \{x(\exists t)(\forall s \geq t)[x \in L^s]\}.$$

For more on Σ^0_2-approximations, see ?). Finally, Let $\overline{K}^s = \{x \leq sx \notin K^s\}$ (where $\{K^s s \in \omega\}$ is a recursive approximation to the halting set K).

2 The requirements

In the following we will exhibit a construction by stages. At stage s of the construction, given any expression $\mathcal{A}$, we will often write $\mathcal{A}[s]$ to denote the evaluation of the expression at stage s: see ?, p. 315) for this notation.

If at stage s we define the current value of a set $X[s]$, we will write $x \searrow X[s]$ to mean that we enumerate x (or x gets enumerated) into $X[s]$ (i.e. $x \in X[s]$) and $x \nearrow X[s]$ to mean that we extract x (or x gets extracted) from $X[s]$ (i.e. $x \notin X[s]$). If E is a finite set, we use similar notations: $E \searrow X[s]$ (i.e. $x \searrow X[s]$, all $x \in E$) and $E \nearrow X[s]$ (i.e. $x \nearrow X[s]$, all $x \in E$).

Let $\{(\Phi_i, \Psi_i)\}_{i \in \omega}$ be an effective listing of all pairs of e-operators.

In order to prove Theorem 1, we want to construct Σ^0_2 sets A, B satisfying the following requirements, for every $i, k \in \omega$.

$$\begin{array}{ll} \mathcal{P}_i: & Z_i = \Phi_i^{A \oplus L} = \Psi_i^{B \oplus L} \Rightarrow Z_i = \Gamma_i^L \\ \mathcal{N}_k^A: & A = \Phi_k^L \Rightarrow \overline{K} = \Delta_{A,k}^L \\ \mathcal{N}_k^B: & B = \Phi_k^L \Rightarrow \overline{K} = \Delta_{B,k}^L \end{array}$$

where $\Gamma_i, \Delta_{A,k}, \Delta_{B,k}$ are e-operators to be constructed.

We say that a requirement $\mathcal{R}$ is a $\mathcal{P}$-requirement if, for some i, $\mathcal{R} = \mathcal{P}_i$; in a similar way, we talk about $\mathcal{N}$-requirements, $\mathcal{N}^A$-requirements and $\mathcal{N}^B$-requirements.

We order the requirements with the following linear ordering (called the *priority* ordering of the requirements):

$$\ldots \mathcal{P}_i < \mathcal{N}_i^A < \mathcal{N}_i^B < \mathcal{P}_{i+1} < \mathcal{N}_{i+1}^A < \mathcal{N}_{i+1}^B, \ldots$$

with $i \in \omega$.

3 The strategies

We briefly outline the strategies used to meet the requirements.

3.1 The atomic modules

3.1.1 The requirement $\mathcal{P}_i$

For simplicity, let us drop the subscript i; let $Z = \Phi^{A \oplus L} \cap \Psi^{B \oplus L}$;

If all numbers $y < x$ have been chosen, then choose x;

1. if $Z(x) = \Gamma^L(x)$, then do nothing;
2. if $x \in Z - \Gamma^L$, then choose finite sets $\alpha, \lambda^A, \beta, \lambda^B$ such that

   $$\langle x, \alpha \oplus \lambda^A \rangle \in \Phi \quad \langle x, \beta \oplus \lambda^B \rangle \in \Psi$$

   and

   $$\alpha \oplus \lambda^A \subseteq A \oplus L \quad \beta \oplus \lambda^B \subseteq B \oplus L,$$

   and enumerate the axiom $\langle x, \lambda^A \cup \lambda^B \rangle \in \Gamma$;
3. if $x \in \Gamma^L - Z$ and $x \in \Phi^{A \oplus L} - \Psi^{B \oplus L}$, then choose some finite set F such that

   (a) $F \subseteq A$ and $x \in \Phi^{F \oplus L}$: in this case restrain $F \subseteq A$;

   (b) wait for $x \nearrow \Phi^{A \oplus L}$ (thus $x \nearrow \Gamma^L$, due to some L-change in the Γ-use of x);

   (c) drop any restraint;

   if $x \in \Gamma^L - Z$ and $x \in \Psi^{B \oplus L} - \Phi^{A \oplus L}$, then choose some finite set F such that

(i) $F \subseteq B$ and $x \in \Psi^{F \oplus L}$: in this case restrain $F \subseteq B$;

(ii) wait for $x \nearrow \Psi^{B \oplus L}$ (thus $x \nearrow \Gamma^L$, due to some L-change in the Γ-use of x);

(iii) drop any restraint.

3.1.2 The requirement $\mathcal{N}_k^A$

For simplicity let us drop the indices A, k;

If all numbers $y < x$ have been chosen, and no such y is currently at 2(a), or 3(i) of the basic module below, then choose x;

1. if $\overline{K}(x) = \Delta^L(x)$, then do nothing;

2. if $x \in \overline{K} - \Delta^L$, then choose a number c_x and define $c_x \in A$:

   (a) wait for $c_x \searrow \Phi^L$;

   (b) choose an axiom $\langle x, \lambda \rangle \in \Phi$ such that $\lambda \subseteq L$ and enumerate the axiom $\langle x, \lambda \rangle \in \Delta$;

   (c) wait for $x \nearrow \Delta^L$;

   (d) return to (a);

3. if $x \in \Delta^L - \overline{K}$ then extract c_x from A;

   (i) wait for $c_x \nearrow \Phi^L$;

   (ii) return to 1.

The numbers c_x will be called *followers*.

3.1.3 The requirement $\mathcal{N}_k^B$

The module in this case is of course similar to the module for $\mathcal{N}_k^A$, but replacing A wiwth B and $\Delta_{A,k}$ with $\Delta_{B,k}$. We skip the obvious details.

3.2 Analysis of outcomes

We briefly discuss the possible outcomes of the above strategies.

3.2.1 The requirement $\mathcal{P}_i$

For simplicity, let us drop the subscript i.

If no x gets stuck at 3(b) (with $x \notin \Psi^{B\oplus L}$) or 3(ii) (with $x \notin \Phi^{A\oplus L}$), then, for every x, we get $Z(x) = \Gamma^L(x)$. In particular, for any given x, infinitely many loops through 3(c) or 3(iii) yield $x \notin Z \cup \Gamma^L$.

Otherwise, for some x, we get the finitary outcome 3(b) or the finitary outcome 3(ii), which imply $x \in \Phi^{A\oplus L} - \Psi^{B\oplus L}$ or $x \in \Psi^{B\oplus L} - \Phi^{A\oplus L}$, respectively.

3.2.2 The requirement $\mathcal{N}_k^A$

For simplicity, let us drop the subscript k.

The finitary outcome 2(a) corresponds to $c_x \in A - \Phi^L$; infinitely many loops through 2(d), in relation to some x corresponds to the case $c_x \in A - \Phi^L$. The finitary outcome 3(i) corresponds to $c_x \in \Phi^L - A$.

If no x gets stuck at 2(a) or 3(i), and does not yield infinitely many loops through 2(d), then we get $\overline{K} = \Delta^L$ (contradicting that $L <_e \overline{K}$).

3.2.3 The requirement $\mathcal{N}_k^B$

See the discussion relative to the outcomes of $\mathcal{N}_k^A$, but replacing A with B and $\Delta_{A,k}$ with $\Delta_{B,k}$.

3.3 Interactions between requirements

The extracting activity of the $\mathcal{N}$-requirements conflicts with the fixing activity of the $\mathcal{P}$-requirements. We explain below the nature of these conflicts are and how to combine the strategies for the requirements in order to solve the conflicts.

3.4 A $\mathcal{P}$-requirement below an $\mathcal{N}$-requirement. The modified $\mathcal{P}$-module

We consider only the case of an $\mathcal{N}$-requirement of the form $\mathcal{N}_k^A$: the case of a requirement of the form $\mathcal{N}_k^B$ is similar.

The problematic case is when we go through 3. of the basic module of $\mathcal{N}_k^A$ on behalf of infinitely many numbers x, ending up with an infinite set V (consisting of all the numbers c_x corresponding to those numbers x such that $x \notin \overline{K}$) being extracted from A. How does $\mathcal{P}_i$, acting after $\mathcal{N}_k^A$, account for this infinitary extracting activity?

3.4.1 Updating Γ_i

Activity on behalf of $\mathcal{P}_i$ mostly consists in enumerating Γ_i-axioms. We need to consider what happens when, for some z, at some stage, $z \in Z_i = \Phi_i^{A\oplus L} \cap \Psi_i^{B\oplus L}$ and we consequently enumerate an axiom $\langle z, \lambda\rangle \in \Gamma_i$, with $\lambda \subseteq L$, in order to have $z \in \Gamma_i^L$. Subsequent extracting activity demanded by $\mathcal{N}_k^A$ may force z to leave $\Phi_i^{A\oplus L}$ and, thus, Z_i, so that we need to extract z from Γ_i^L.

Since $L <_e \overline{K}$, it follows from the discussion about the outcomes of $\mathcal{N}_k^A$ that there must exist some c_u such that $A(c_u) \neq \Phi_k^L(c_u)$. Suppose that we are able to pin down such a number c_u through a suitable length of agreement function $\ell(k,s)$ (with $\liminf_s \ell(k,s) = c_u$) and that we are eventually working at stages s such that $\ell(k,s) \geq c_u$. Let s_0 be such that

$$(\forall s \geq s_0)[\ell(k,s) \geq c_u \,\&\, (\forall c_v \leq c_u)[c_v \in A \Rightarrow c_v \in A^s]].$$

Enumerating Γ_i-axioms at stages $s \geq s_0$ At stage $s \geq s_0$, let $\ell(k,s) = c_v$. We distinguish two cases.

1. if $c_v \notin \Phi_k^L$, then we extract all (already appointed) $c_w \geq c_v$ from A^s ;
2. if $c_v \in \Phi_k^L$, the we extract the finite set $\{c_w c_w \leq c_v \,\&\, w \in \overline{K}\}$.

We have now two possibilities related to c_u.

The case in which $c_u \notin \Phi_k^L$ corresponds to a possibly infinitary outcome for $\mathcal{N}_k^A$, since there can be infinitely many stages s such that $c_u \in \Phi_k^L[s]$:

1. while working at stages $s \geq s_0$ such that $\ell(k,s) > c_u$, we assume $c_u \in \Phi_k^L[s]$, thus we enumerate axioms $\langle z, \lambda\rangle \in \Gamma_i$, with λ containing the Φ_k^L-use of c_u (i.e. $c_u \in \Phi_k^\lambda$). Since $c_u \notin \Phi_k^L$, it follows that none of these axioms applies to get $z \in \Gamma_i^L$.
2. if we work at some stage stage $s \geq s_0$ such that $\ell(k,s) = c_u$ then in this case, since we extract all $c_v \geq c_u$ from A^s (and eventually from A, since this action is repeated infinitely often), it follows that for every axiom $\langle z, \lambda\rangle \in \Gamma_i$, which is enumerated at such a stage, and for every $c_v \geq c_u$, we have that $c_v \notin \lambda$, thus no such number c_v is used in getting $z \in Z_i$ at stage $s \geq s_0$ s, therefore we can freely extract the recursive set $\{c_v c_v \geq c_u\}$, without any danger of extracting z from Z_i and, thus, without any interference of $\mathcal{N}_k^A$ with $\mathcal{P}_i$.

The case $c_u \in \Phi_k^L$ corresponds to the finitary outcome in which we appoint only finitely many followers c_z on behalf of $\mathcal{N}_k^A$, which can easily be accounted for by $\mathcal{P}_i$.

It follows from the preceding discussion that the axioms $\langle z, \lambda\rangle \in \Gamma_i$ which are enumerated at $s \geq s_0$ will assume (the true assumption) that only the elements of the finite set $\{c_v c_v \leq c_u \,\&\, v \in \overline{K}\}$ are in A, among all numbers c_z's.

In conclusion, there are only finitely many numbers z such that (letting V be the possibly infinite set eventually extracted by $\mathcal{N}_k$) we have that $z \nearrow Z_i$, due to $V \nearrow A$. These numbers are included among those numbers y for which we enumerate axioms $\langle y, \lambda\rangle \in \Gamma_i$ before stage s_0.

3.5 The $\mathcal{N}$-requirements acting after a $\mathcal{P}$-requirement

We consider only the case of an $\mathcal{N}$-requirement of the form $\mathcal{N}_k^A$, the case of $\mathcal{N}_k^B$ being similar.

It is clear from the discussion in the previous subsection that the extracting activity of $\mathcal{N}_k^A$ interferes with the strategy of a $\mathcal{P}_i$ in that the $\mathcal{N}_k^A$-extractions may result in forcing numbers z to leave $\Phi_i^{A\oplus L}$; thus $z \nearrow Z_i$, and this implies that we need $z \nearrow \Gamma_i^L$, if we hope to maintain the equation $Z_i = \Gamma_i^L$.

First of all we notice that, for the same reasons as in Subsection 3.4, we do not have to worry for those numbers z such that we appoint axioms of the form $\langle z, \lambda\rangle \in \Gamma_i$ only while acting after $\mathcal{N}_k^A$ and at stages at which $\ell(k, s) \geq c_u$, where c_u is a number as in the discussion of Subsection 3.4, for which $A(c_u) \neq \Phi_k^A(c_u)$.

We deal with the other cases as follows: if $z \nearrow \Phi_i^{A\oplus L}$, due to the extracting activity of $\mathcal{N}_k^A$, then we restrain some finite $F \subseteq B$ such that $z \in \Psi^{F\oplus L}$. If, following this action, no L-change occurs yielding $c_z \nearrow \Psi^{F\oplus L}$ (and thus $z \nearrow \Gamma_i^L$), then we win the requirement $\mathcal{P}$, since we get $z \in \Psi^{B\oplus L} - \Phi_i^{A\oplus L}$; otherwise we get $z \nearrow \Gamma_i^L$, thus restoring the equation $Z_i(z) = \Gamma_i^L(z)$: in this latter case we drop any previous restraints.

This restraining activity does not prevent lower priority requirements from being satisfied, since it is finitary, referring, as we shall show, only to finitely many numbers z's.

A more detailed discussion of how to combine the strategies will be in reference to the tree of outcomes, described in next section.

4 The tree of outcomes

In this section we define the tree of outcomes, which is going to be a subset $T \subseteq \omega^{<\omega}$. Let $\mathbb{R}$ be the set of all requirements, and let $\mathbb{P}$ and $\mathbb{N}$ denote the $\mathcal{P}$-requirements and the $\mathcal{N}$-requirements, respectively. The set $\mathbb{N}$ is partitioned into $\mathbb{N}^A$ e $\mathbb{N}^B$, the sets of the $\mathcal{N}^A$- and the $\mathcal{N}^B$-requirements, respectively. Together with the tree of outcomes, we will define also the *requirement assignment* function, i.e. a function $\mathcal{R} : T \longrightarrow \mathbb{R} \cup (\mathbb{N} \times \mathbb{P}) \cup (\mathbb{P} \times \omega)$.

The elements of T will be called *strings* or *nodes*. We will distinguish the $\mathcal{P}$-*nodes*, the $\mathcal{N}$-*nodes* (partitioned into the $\mathcal{N}^{\mathcal{A}}$- and the $\mathcal{N}^{\mathcal{B}}$-nodes), the $(\mathcal{N},\mathcal{P})$-*nodes* (again, partitioned into $(\mathcal{N}^A,\mathcal{P})$-*nodes* and $(\mathcal{N}^B,\mathcal{P})$-*nodes*), and the Γ-*nodes*. If σ is a $\mathcal{P}$-node, then $\mathcal{R}(\sigma)$ is a $\mathcal{P}$-requirement; if σ is an $\mathcal{N}$-node, then $\mathcal{R}(\sigma)$ is an $\mathcal{N}$-requirement; if σ is an $(\mathcal{N},\mathcal{P})$-node, then $\mathcal{R}(\sigma) \in \mathbb{N} \times \mathbb{P}$, i.e. $\mathcal{R}(\sigma) = (\mathcal{N},\mathcal{P})$, where $\mathcal{N}$ is an $\mathcal{N}$-requirement and $\mathcal{P}$ is a $\mathcal{P}$-requirement; finally, if σ is a Γ-node, then $\mathcal{R}(\sigma) = (\mathcal{P}, x)$, where $\mathcal{P}$ is a $\mathcal{P}$-requirement and $x \in \omega$.

The meaning of the Γ-nodes and the $(\mathcal{N},\mathcal{P})$-nodes will be explained in Section 4.2.

T and $\mathcal{R}$ are defined by induction as follows (we assume that $\mathbb{N} \times \mathbb{P}$ and $\mathbb{P} \times \omega$ are lexicographically ordered, with respect to the priority ordering of $\mathbb{R}$ and the natural ordering of ω; when referred to requirements, the term "least" below, unless otherwise specified, refers again to the priority ordering of the requirements):

1. $\emptyset \in T$; $\emptyset$ is a $\mathcal{P}$-node; $\mathcal{R}(\sigma) = \mathcal{P}_0$;

2. if $\sigma \in T$ and σ is a $\mathcal{P}$-node, then $\sigma\widehat{\ }0 \in T$; $\sigma\widehat{\ }0$ is a Γ-node; finally,

$$\mathcal{R}(\sigma\widehat{\ }0) = \text{least}\{(\mathcal{P}, x) \in \mathbb{P} \times \omega(\forall \tau \subseteq \sigma)[\mathcal{R}(\tau) \in \mathbb{P} \times \omega \\ \Rightarrow \mathcal{R}(\tau) < (\mathcal{P}, x) \,\&\, (\exists \tau \subseteq \sigma)[\tau \in \mathbb{P} \,\&\, \mathcal{R}(\tau) = \mathcal{P}]\}.$$

3. if $\sigma \in T$ and σ is a Γ-node, then $\sigma\widehat{\ }0 \in T$ and $\sigma\widehat{\ }1 \in T$; for $i = 0, 1$,

$$\mathcal{R}(\sigma) = \text{least}\{\mathcal{N} \in \mathbb{N}(\forall \tau \subseteq \sigma)[\mathcal{R}(\tau) \in \mathbb{N} \Rightarrow \mathcal{R}(\tau) < \mathcal{N}\}];$$

4. if $\sigma \in T$ and σ is an $\mathcal{N}^{\mathcal{A}}$-node, then, for every $n \in \omega$, $\sigma\widehat{\ }n \in T$; $\sigma\widehat{\ }n$ is an $(\mathcal{N}^{\mathcal{A}},\mathcal{P})$-node; $\mathcal{R}(\sigma\widehat{\ }n) = (\mathcal{R}(\sigma), \mathcal{P}_0)$.

   Finally, let $o(\sigma\widehat{\ }n) = \sigma$;

5. if $\sigma \in T$ and σ is an $\mathcal{N}^{\mathcal{B}}$-node, then, for every $n \in \omega$, $\sigma\widehat{\ }n \in T$; $\sigma\widehat{\ }n$ is an $(\mathcal{N}^{\mathcal{B}},\mathcal{P})$-node; $\mathcal{R}(\sigma\widehat{\ }n) = (\mathcal{R}(\sigma), \mathcal{P}_0)$.

   Finally, let $o(\sigma\widehat{\ }n) = \sigma$;

6. if $\sigma \in T$ and σ is an $(\mathcal{N}^{\mathcal{A}},\mathcal{P})$-node, and, say, $\mathcal{R}(\sigma) = (\mathcal{N}^{\mathcal{A}},\mathcal{P})$ with $\mathcal{P} \leq \mathcal{N}^{\mathcal{A}}$, and

$$\{\mathcal{R} \in \mathbb{P}\mathcal{P} < \mathcal{R} \leq \mathcal{N}^{\mathcal{A}}\} \neq \emptyset$$

then let $\sigma^\frown n \in T$, for every $n \in \omega$; the nodes $\sigma^\frown n$ are $(\mathcal{N}^{\mathcal{A}}, \mathcal{P})$-nodes; we define $\mathcal{R}(\sigma^\frown n)$ to be $(\mathcal{N}^{\mathcal{A}}, \mathcal{P}')$, where $\mathcal{P}'$ is the least requirement $\mathcal{R} \in \mathbb{P}$ such that $\mathcal{P} < \mathcal{R} \leq \mathcal{N}^{\mathcal{A}}$.

Let $o(\sigma^\frown n) = o(\sigma)$, for every $n \in \omega$;

7. if $\sigma \in T$ and σ is an $(\mathcal{N}^{\mathcal{A}}, \mathcal{P})$-node, and, say, $\mathcal{R}(\sigma) = (\mathcal{N}^{\mathcal{A}}, \mathcal{P})$ with $\mathcal{P} \leq \mathcal{N}$, and
$$\{\mathcal{R} \in \mathbb{P} \mathcal{P} < \mathcal{R} \leq \mathcal{N}^{\mathcal{A}}\} = \emptyset$$
then let $\sigma^\frown n \in T$, for every $n \in \omega$; the nodes $\sigma^\frown n$ are $\mathcal{N}^{\mathcal{B}}$-nodes; we define $\mathcal{R}(\sigma^\frown n)$ to be the least $\mathcal{N}^{\mathcal{B}}$-requirement $\mathcal{N} > \mathcal{N}^{\mathcal{A}}$;

8. if $\sigma \in T$ and σ is an $(\mathcal{N}^{\mathcal{B}}, \mathcal{P})$-node, and, say, $\mathcal{R}(\sigma) = (\mathcal{N}^{\mathcal{B}}, \mathcal{P})$ with $\mathcal{P} \leq \mathcal{N}^{\mathcal{B}}$, and
$$\{\mathcal{R} \in \mathbb{P} \mathcal{P} < \mathcal{R} \leq \mathcal{N}^{\mathcal{B}}\} \neq \emptyset$$
then let $\sigma^\frown n \in T$, for every $n \in \omega$; the nodes $\sigma^\frown n$ are $(\mathcal{N}^{\mathcal{B}}, \mathcal{P})$-nodes; we define $\mathcal{R}(\sigma^\frown n)$ to be $(\mathcal{N}^{\mathcal{B}}, \mathcal{P}')$, where $\mathcal{P}'$ is the least requirement $\mathcal{R} \in \mathbb{P}$ such that $\mathcal{P} < \mathcal{R} \leq \mathcal{N}^{\mathcal{B}}$.

   Let $o(\sigma^\frown n) = o(\sigma)$, for every $n \in \omega$;

9. if $\sigma \in T$ and σ is an $(\mathcal{N}^{\mathcal{B}}, \mathcal{P})$-node, and, say, $\mathcal{R}(\sigma) = (\mathcal{N}^{\mathcal{B}}, \mathcal{P})$ with $\mathcal{P} \leq \mathcal{N}$, and
$$\{\mathcal{R} \in \mathbb{P} \mathcal{P} < \mathcal{R} \leq \mathcal{N}\} = \emptyset$$
then let $\sigma^\frown n \in T$, for every $n \in \omega$; the nodes $\sigma^\frown n$ are $\mathcal{N}^{\mathcal{B}}$-nodes; we define $\mathcal{R}(\sigma^\frown n)$ to be the least $\mathcal{P}$-requirement $\mathcal{P} \geq \mathcal{N}^{\mathcal{B}}$.

We will sometimes write $\sigma \in T^{\mathbb{N}}$, $\sigma \in T^{\mathbb{N}^A}$, $\sigma \in T^{\mathbb{N}^B}$, σ if σ is an $\mathcal{N}$-, $\mathcal{N}^A$-, $\mathcal{N}^B$-node, respectively.

We notice:

1. if σ is an $(\mathcal{N}, \mathcal{P})$-node, then $o(\sigma)$ denotes the largest $\mathcal{N}$-node $\tau \subset \sigma$;

2. if σ is an $\mathcal{N}$-node, and $\mathcal{R}(\sigma) = \mathcal{N}_k$, then σ is immediately followed by $k+1$ $(\mathcal{N}, \mathcal{P})$-nodes $\tau_0, \dots, \tau_k$, where $\mathcal{R}(\tau_i) = (\mathcal{N}_k, \mathcal{P}_i)$, with $\mathcal{P}_i$ the i-th $\mathcal{P}$-requirement in order of priority;

3. if f is any infinite path through T and $\sigma \subset f$ is a $\mathcal{P}$-node, then for every $x \in \omega$ there exists exactly one Γ-node τ such that $\sigma \subset \tau \subset f$ and $\mathcal{R}(\tau) = (\mathcal{R}(\sigma), x)$, and for no $\tau' \subset \sigma$ can we have $\mathcal{R}(\tau') = (\mathcal{R}(\sigma), y)$, for any y.

4. if $\mathcal{R}(\sigma) = \mathcal{P}_i$, we will happen sometimes to write
$$Z_\sigma = \Phi_\sigma^{A\oplus L} = \Psi_\sigma^{B\oplus L} \Rightarrow Z_\sigma = \Gamma_\sigma^L$$
instead of
$$Z_i = \Phi_i^{A\oplus L} = \Psi_i^{B\oplus L} \Rightarrow Z_i = \Gamma_\sigma^L$$
(and similarly Z_σ for Z_i, Φ_σ for Φ_i, etc.) Similarly we may write
$$\mathcal{N}_\sigma^A : A = \Phi_\sigma^L \Rightarrow \overline{K} = \Delta_{A,\sigma}^L$$
instead of
$$\mathcal{N}_k^A : A = \Phi_k^L \Rightarrow \overline{K} = \Delta_{A,\sigma}^L$$
(and similarly Φ_σ for Φ_k, etc.) if $\mathcal{R}(\sigma)$ is an $\mathcal{N}^{\mathcal{A}}$-requirement; we use similar notations for $\mathcal{N}^{\mathcal{B}}$-nodes.

Let $\{\xi_\sigma \sigma \in T\}$ be a recursive partition of ω into infinite recursive sets. The elements of ξ_σ may be chosen to be appointed as *σ-followers*.

4.1 Notation and terminology for strings

We use standard terminology and notations for strings. In particular, given any $\sigma \in T$, let $|\sigma|$ denote the length of σ.

If $\sigma \in T$, and $n \in \omega$ is such that $\sigma^\frown n \in T$, then n is an *outcome at σ*.

Given $\sigma, \tau \in T$, let $\sigma \preceq \tau$ if and only if either $\sigma \subseteq \tau$ or $y(\sigma,\tau)\downarrow$ and $\sigma(y(\sigma,\tau)) \leq \tau(y(\sigma,\tau))$, where $y(\sigma,\tau) = \mu y.[y < |\sigma|, |\tau|. \sigma(y) \neq \tau(y)]$. We say that σ is *to the left of* τ (notation: $\sigma \prec_L \tau$), if $\sigma \preceq \tau$, but $\sigma \not\subseteq \tau$. Given a string σ and a number y, the symbol $\sigma \restriction y$ denotes the initial segment of σ having length y. If $|\sigma| > 0$, then let $\sigma^- = \sigma \restriction |\sigma| - 1$.

Finally, if $\tau \subseteq \sigma$ and $\tau = \tau^{-\frown} x$, then we say that x is the *outcome at τ^- along σ*.

4.2 Analysis of tree outcomes.

We briefly describe the intended meaning of the outcomes of the tree of outcomes.

1. If σ is a $\mathcal{P}$-node (say $\mathcal{R}(\sigma) = \mathcal{P}_i$), then we observe that we have no distinct outcomes at σ. We regard σ as just the node at which we start our strategy for the corresponding $\mathcal{P}$-requirement, by routinely updating the operator Γ_σ. The eventual success of the strategy will need the cooperation of the lower priority $\mathcal{N}$-requirements. The updating strategy will be dispersed through the infinitely many Γ-nodes $\tau \supseteq \sigma$ with $\mathcal{R}(\tau) = (\mathcal{P}_i, x)$, for some x.

2. Let σ be a Γ-node (say $\mathcal{R}(\sigma) = (\mathcal{P}_i, x)$). The Γ-node σ is devoted to defining suitable axioms $\langle x, \lambda\rangle \in \Gamma_\tau$, where $\tau \subset \sigma$ is such that $\mathcal{R}(\tau) = \mathcal{P}_i$. The tree outcome 1 corresponds to the finitary outcome $x \in \Phi_i^{A\oplus L} \cap \Psi_i^{B\oplus L}$. We use the symbols

$$\alpha(\sigma\hat{\ }1, s),\ \beta(\sigma\hat{\ }1, s),\ \lambda^A(\sigma\hat{\ }1, s),\ \lambda^B(\sigma\hat{\ }1, s)$$

to denote suitably chosen finite sets $\alpha, \beta, \lambda^A, \lambda^B$ such that, at stage s,

$$x \in \Phi_i^{\alpha\oplus\lambda^A} \cap \Psi_i^{\beta\oplus\lambda^B}$$

and $\alpha \oplus \lambda^A \subseteq A \oplus L$, and $\beta \oplus \lambda^B \subseteq B \oplus L$. We enumerate an axiom $\langle x, \lambda(\sigma\hat{\ }1, s)\rangle \in \Gamma_\tau$, where $\lambda(\sigma\hat{\ }1, s) \supseteq \lambda^A \cup \lambda^B$, and $\lambda(\sigma\hat{\ }1, s)$ is large enough to contain all finite sets $\lambda(\rho, s)$ such that $\rho \subset \sigma$ and $\lambda(\rho, s) \subseteq L^s$. The sets $\lambda(\rho, s)$ have the following meaning.

If ν is a Γ-node, and $\rho = \nu\hat{\ }1 \subseteq \sigma$, then $\lambda(\rho, s) = \lambda^A(\rho, s) \cup \lambda^B(\rho, s)$.

If $\nu \subseteq \sigma$ is an $\mathcal{N}$-node, and ℓ is the outcome at ν along σ, i.e. $\rho = \nu\hat{\ }\ell \subseteq \sigma$, then we single out some ν-followers c such that $c \in \Phi_\nu^L[s]$, we will denote by $\lambda(\nu, c, s)$ a suitably chosen finite set such that $\lambda(\nu, c, s) \subseteq L^s$ and $c \in \Phi_\nu^{\lambda(\nu,c,s)}$: if $C(\rho, s)$ the set of all such ν-followers, then finally let

$$\lambda(\rho, s) = \bigcup_{c\in C(\rho,s)} \lambda(\nu, c, s).$$

Similarly, if ν is an $(\mathcal{N}, \mathcal{P})$-node (where, say, $\mathcal{P} = \mathcal{P}_i$, and $\pi \subseteq \sigma$ is the corresponding $\mathcal{P}$-node) such that $\nu \subset \sigma$, n is the outcome at ν along σ, i.e. $\rho = \nu\hat{\ }n \subseteq \sigma$, then we will denote by $\lambda(\rho, s)$ the finite set

$$\lambda(\rho, s) = \bigcup_{y\in E(\nu,s)} \lambda(\nu, y, s),$$

(dove $E(\nu, s)$ is the current guess at the (finite) set of elements leaving Z_i as a consequence of the extracting activity of $\mathcal{R}_{o(\nu)}$), but $E(\nu, s) \subseteq \Gamma_\pi^L$ at stage s: by $\lambda(\nu, y, s)$ we mean some suitably chosen finite set such that $y \in \Gamma_\pi^{\lambda(\nu,y,s)}$ and $\lambda(\nu, y, s) \subseteq L^s$. It follows that $\lambda(\rho, s) \subseteq L^s$.

Notice that any L-change at some later stage t, relative to any of these sets $\lambda(\rho, s)$ (i.e. $\lambda(\rho, s) \not\subseteq L^t$), will entail $\lambda(\sigma\hat{\ }1, s) \not\subseteq L^t$. This is a crucial point for the success of $\mathcal{R}_\sigma$: if τ is on the true path, then the construction guarantees that all axioms $\langle x, \lambda\rangle \in \Gamma_\tau$ defined while acting at a stage s at some string to the right of the true path is such that λ will contain some set $\lambda(\rho, s)$ such that $\lambda(\rho, s) \not\subseteq L$.

At $\sigma\widehat{\ }1$ we restrain $\alpha \subseteq A$ and $\beta \subseteq B$. Following this restraining action, if $\lambda(\sigma\widehat{\ }1, s) \subseteq L$, then the only $\mathcal{N}$-requirements that are entitled to force $x \nearrow Z_i$ are those of higher priority than $\mathcal{R}_\sigma$.

Notice that we drop any restraint when we move past $\sigma\widehat{\ }0$: the tree outcome 0 corresponds to the case $x \notin \Phi_i^{A\oplus L} \cap \Psi_i^{B\oplus L}$.

3. Let σ be an $\mathcal{N}$-node. Assume for simplicity that σ is an $\mathcal{N}^{\mathcal{A}}$-node, the case of an $\mathcal{N}^{\mathcal{B}}$-node being similar. We define a length of agreement function $\ell(\sigma, s)$, and we show (with σ on the true path),

$$A = \Phi_\sigma^L \Leftrightarrow \lim_s \ell(\sigma, s) = +\infty.$$

On the other hand, the construction guarantees that

$$A = \Phi_\sigma^L \Rightarrow \overline{K} \leq_e L :$$

indeed, if $A = \Phi_\sigma^L$ then $\overline{K} =^* \Delta_\sigma^L$ (i.e. equality modulo a finite set), with Δ_σ the e-operator built at σ by the construction.

¿From the preceding remarks it follows that $\liminf_s \ell(\sigma, s) = \ell$ is finite. We give outcome $\ell(\sigma, s)$ at σ at s.

The outcome $\ell(\sigma, s)$ will be of the form $\ell = \langle c, u\rangle$: we aim at getting either $c \in A - \Phi_\sigma^L$ (and in this case, for every $s \geq u$, $c \in A^s$), or $c \in \Phi_\sigma^L - A$ (and in this case, for every $s \geq u$, $c \in \Phi_\sigma^L[s]$). The numbers c (called σ-followers, see Definition 4) will be chosen from ξ_σ: the follower of z, when chosen at some stage s, will be denoted by $c(\sigma, z, s)$: since it is never changed, after being appointed, i.e. $c(\sigma, z, t) = c(\sigma, z, s)$, for all $t \geq s$, we will simply write $c(\sigma, z)$ instead of $c(\sigma, z, t)$ at all $t \geq s$.

If $\ell = \langle c, u\rangle$ is the outcome at σ at stage s, then (located at $\sigma\widehat{\ }\ell$) we extract from A^s a finite set $V(\sigma\widehat{\ }\ell, s)$, where $V(\sigma\widehat{\ }\ell, s)$ consists of

- all the numbers $c' \leq \ell$ such that c' has been appointed for some z, i.e. $c' = c_z$, and $z \notin \overline{K}^s$, if $c \in \Phi_\sigma^L[s] - A^s$;
- all the numbers $c' \leq \ell$ such that c' has been appointed for some z, i.e. $c' = c_z$, and $z \notin \overline{K}^s$ together with all $c' \geq \ell$, if $c \in A^s - \Phi_\sigma^L[s]$.

We refer the reader to Subsection 3.4 for a discussion relative to this extracting activity. Notice however that, for the actual definition of $\ell(\sigma, s)$, if $\ell = \liminf_s \ell(\sigma, s)$ then we do not have $A(\ell) \neq \Phi_\sigma^L(\ell)$, but rather $A(c) \neq \Phi_\sigma^L(c)$, if $\ell = \langle c, u\rangle$, for some u.

4. Let σ be an $(\mathcal{N}, \mathcal{P})$-node. Assume for simplicity that σ is an $(\mathcal{N}^{\mathcal{A}}, \mathcal{P})$-node, the case of an $(\mathcal{N}^{\mathcal{B}}, \mathcal{P})$-node being similar.

   Let $\mathcal{R}(\sigma) = (\mathcal{N}_k^A, \mathcal{P}_i)$, and let $\pi \subseteq \sigma$ be such that $\mathcal{R}(\sigma) = \mathcal{P}_i$. At this node we record the effects on $\mathcal{R}(\pi)$ of the extracting activity done on behalf of $\mathcal{R}(o(\sigma))$, with $\mathcal{N}_k^A$ of lower priority than $\mathcal{P}_i$ (recall that $\mathcal{R}(o(\sigma)) = \mathcal{N}_k^A$). For sismplicity, let $\nu = o(\sigma)$ and $\nu^+ = \nu^\frown \ell \subseteq \sigma$). Suppose that at stage s we need to extract $V(\nu^+, s)$ from A, as demanded by the strategy for $\mathcal{R}(\nu)$ (for simplicity, let $V = V(\nu^+, s)$). Let us use the symbol $E(\sigma, s)$ to denote the finite set of elements such that we have:

$$V \nearrow A \Rightarrow E(\sigma, s) \nearrow Z_i \,\&\, E(\sigma, s) \subseteq \Gamma_\pi^L$$

   (where, for any $x \in E(\sigma, s)$, axioms of the form $\langle x, \lambda \rangle \in \Gamma_\pi$ have been previously defined).

   We give outcome $h = h(\sigma, s)$ at σ at s, where h is the canonical index of $E(\sigma, s)$.

   (In order to define $E(\sigma, s)$, we use an auxiliary set $H(\sigma, s)$, which keeps track of those numbers x such that there has been evidence at stages $t < s$ that the extraction of $V(\nu^+, t)$ from A entails $x \nearrow \Phi_i^{A \oplus L}$. The elements of $H(\sigma, s)$ are ordered by $\prec_\sigma^s$: intuitively, $x \prec_\sigma^s x'$ if x has stayed in Γ_π^L, longer than x'. At stage $s > 0$, we let $E(\sigma, s)$ be the longest initial segment - according to $\prec_\sigma^s$ - of $H(\sigma, s-1)$ consisting of the numbers that are still in Γ_π^L at stage s. At the end of stage s we update $H(\sigma, s-1)$ by enumerating in $H(\sigma, s)$ the elements of $E(\sigma, s)$ plus the numbers x such that, at stage s,the extraction of V from A entails $x \nearrow \Phi_i^{A \oplus L}$).

   If $E(\sigma, s) \neq \emptyset$, we restrain at $\sigma^\frown h$ some finite set $F \subseteq B$ such that $E \subseteq \Psi_i^{F \oplus L}$. We use the symbol $\beta(\sigma^\frown h, s)$ ($\alpha(\sigma^\frown h, s)$ if σ is an $(\mathcal{N}^{\mathcal{B}}, \mathcal{P})$-node) to denote such a finite set F: in fact, for every $x \in E(\sigma, s)$, we suitably choose finite sets $\beta(\sigma^\frown h, x, s)$ ($\alpha(\sigma^\frown h, x, s)$ if σ is an $(\mathcal{N}^{\mathcal{B}}, \mathcal{P})$-node) and $\lambda(\sigma^\frown h, x, s)$ such that $x \in \Psi^{\beta(\sigma^\frown h, x, s) \oplus \lambda(\sigma^\frown h, x, s)}$ and $\beta(\sigma^\frown h, x, s) \oplus \lambda(\sigma^\frown h, x, s) \subseteq B \oplus L[s]$, and we restrain $\beta(\sigma^\frown h, x, s)$ ($\alpha(\sigma^\frown h, x, s)$ if σ is an $(\mathcal{N}^{\mathcal{B}}, \mathcal{P})$-node), and we let $\beta(\sigma^\frown h, s) = \bigcup_{x \in E(\sigma, s)} \beta(\sigma^\frown h, x, s)$ ($\alpha(\sigma^\frown h, s) = \bigcup_{x \in E(\sigma, s)} \alpha(\sigma^\frown h, x, s)$, if σ is an $(\mathcal{N}^{\mathcal{B}}, \mathcal{P})$-node).

   If σ is on the true path, we will show that $h = \liminf_s h(\sigma, s)$ exists. There are two possibilities:

   - If we get outcome 0 at σ infinitely often, then there is no damage caused to $\mathcal{R}(\pi)$ by the extracting activity done on behalf of $\mathcal{R}(\sigma)$, since, for all possible x such that $x \nearrow Z_i$ due to $\mathcal{R}(\sigma)$-extractions, we get $x \nearrow \Gamma^L_\pi$ due to infinitely many L-changes.

- Otherwise $D_h \neq \emptyset$. Then, *either* for some $x \in D_h$ our restraining activity at $\sigma\frown h$ gives $x \in \Psi^{B\oplus L}_\pi - \Phi^{A\oplus L}_\pi$: this yields an outright win of $\mathcal{R}(\pi)$; *or* $x \in \Phi^{A\oplus L}_\pi \cap \Psi^{B\oplus L}_\pi$, for all $x \in D_h$, showing that $\mathcal{R}(\sigma)$-extractions do not interfere with the equataion $Z_i = \Gamma^L_\pi$.

5 The construction

The construction is by stages, aiming to define suitable recursive sequences of finite sets $\{A^s s \in \omega\}$ and $\{B^s s \in \omega\}$, such that the Σ^0_2 sets

$$\begin{aligned} A &= \{x(\exists t)(\forall s \geq t)[x \in A^s]\} \\ B &= \{x(\exists t)(\forall s \geq t)[x \in B^s]\} \end{aligned}$$

satisfy the requirements of Section 2.

At stage s we define a string $\delta_s \in T$ (with $|\delta_s| = s$), together with the values of several parameters. The intuitive meaning of all parameters (except $\Pi(\sigma, z, s)$, where $\sigma \in T$ and $z \in \omega$) is explained in the previous section: $\Pi(\sigma, z, s)$ is an auxiliary parameter needed to define $\ell(\sigma, s$.

For every $\sigma \in T$ and stage s, let

$$t(\sigma, s) = \begin{cases} \max\{t < s\sigma \subseteq \delta_t\} & \text{if any} \\ s & \text{otherwise} \end{cases}$$

Throughout the following, while acting at σ at stage s, given any e-operator Φ and any Σ^0_2 set X with a Σ^0_2-approximation $\{X^t\}_{t\in\omega}$, we write $x \in \Phi^X[s]$ if

$$(\exists F)[\langle x, F\rangle \in \Phi_s \,\&\, (\forall u)[t(\sigma, s) \leq u \leq s \Rightarrow F \subseteq X^u]].$$

It is clear that if we act infinitely many times at σ, then we get that $\{\Phi^X[s]\}_{s\in\omega}$ is a Σ^0_2 approximation to Φ^B.

Similarly, we will write $x \in L[s]$, if

$$(\forall u)[t(\sigma, s) \leq u \leq s \Rightarrow x \in L^u].$$

Let $P(x, s)$ be some relation. If $P(x, s)$ holds, then let

$$t_x(s) = \text{least } \{tP(x, t) \,\&\, (\forall u)[t \leq u \leq s \Rightarrow P(x, u)]\}:$$

If x and s are as before, we say that we *consistently choose* x at stage s if x is the least number among those with minimal $t_x(s)$ (in fact x can be (the code of) a finite set or a pair of finite sets, etc.).

At step s, any parameter p retains the same value as at the preceding stage, unless otherwise specified by the construction. Any parameter p is by default undefined (i.e. $p =\uparrow$ if p ranges through the numbers, and $p = \emptyset$, if p ranges through the finite sets).

The e-operators $\Gamma_\sigma, \Delta_{A,\sigma}, \Delta_{B,\sigma}$ will be defined through recursive approximations (modulo identification of each e-operator with the corresponding r.e. set): at stage s we define $\Gamma^s_\sigma, \Delta^s_{A,\sigma}, \Delta^s_{B,\sigma}$.

5.1 Step 0

Let $\delta_0 = \emptyset$. For every $\sigma \in T$ let

$$V(\sigma,0) = H(\sigma,0) = E(\sigma,0) = \alpha(\sigma,0) = \beta(\sigma,0) = \\ \lambda(\sigma,0) = \lambda^A(\sigma,0) = \lambda^B(\sigma,0) = \prec^0_\sigma = \emptyset.$$

Let $h(\sigma,0) = 0$. For every $\sigma \in T$ and $z \in \omega$, let

$$\alpha(\sigma,z,0) = \beta(\sigma,z,0) = \lambda(\sigma,z,0) = \emptyset;$$

let $\Pi(\sigma,z,0) = \omega$; let also $c(\sigma,z,0) =\uparrow$, $\ell(\sigma,0) = 0$, $\Gamma^0_\sigma = \Delta^0_{A,\sigma} = \Delta^0_{B,\sigma} = \emptyset$.

Finally, let $A^0 = \emptyset$ and $B^0 = \emptyset$.

5.2 Step $s+1$

Assume that we have already defined $\delta_{s+1}\restriction n$ (with $\delta_{s+1}\restriction 0 = \emptyset$). If $n < s+1$ then we proceed and define $\sigma^+ = \delta_{s+1} \restriction n+1$ according to which of the following cases applies. Otherwise we go to step $s+2$.

In the following, let

$$\begin{aligned} {}^\sigma A^{s+1} &= (A^s \cup \textstyle\bigcup_{\tau\subseteq\sigma} \alpha(\tau,s+1)) - \bigcup_{\tau\subseteq\sigma,\tau^-\in T^{\mathbb{N}A}} V(\tau,s+1) \\ {}^\sigma B^{s+1} &= (B^s \cup \textstyle\bigcup_{\tau\subseteq\sigma} \beta(\tau,s+1)) - \bigcup_{\tau\subseteq\sigma,\tau^-\in T^{\mathbb{N}B}} V(\tau,s+1), \end{aligned}$$

i.e. ${}^\sigma A^{s+1}$ and ${}^\sigma B^{s+1}$ are the current values (at stage $s+1$, before acting at σ) of A^{s+1} and B^{s+1}.

5.2.1 σ is a $\mathcal{P}$-node

Let $\sigma^+ = \sigma^\frown 0$. Go and define $\delta_{s+1}\restriction n+2$, if $n \leq s+1$.

5.2.2 σ is a Γ-node

Assume that $\mathcal{R}(\sigma) = (\mathcal{P}_i, x)$, where

$$\mathcal{P}: \qquad Z_i = \Phi_i^{A\oplus L} = \Psi_i^{B\oplus L} \Rightarrow Z_i = \Gamma_\pi^L$$

with $\pi \subseteq \sigma$ such that $\mathcal{R}(\pi) = \mathcal{P}_i$. In the following, drop the subscript i and write $\Gamma = \Gamma_\pi$.

We distinguish three cases:

1. If $x \in \Phi^{\sigma A\oplus L}[s+1] \cap \Psi^{\sigma B\oplus L}[s+1] - \Gamma^L[s+1]$, then let $\sigma^+ = \sigma\widehat{\ }1$ and choose consistently (according to Definition 5) finite sets

$$\alpha(\sigma^+, s+1), \beta(\sigma^+, s+1), \lambda^A(\sigma^+, s+1), \lambda^B(\sigma^+, s+1)$$

such that

$$\begin{array}{l} \langle x, \alpha(\sigma^+, s+1) \oplus \lambda^A(\sigma^+, s+1)\rangle \in \Phi^s \\ \langle x, \beta(\sigma^+, s+1) \oplus \lambda^B(\sigma^+, s+1)\rangle \in \Psi^s \end{array}$$

and

$$\begin{array}{l} \alpha(\sigma^+, s+1) \oplus \lambda^A(\sigma^+, s+1) \subseteq ({}^\sigma A \oplus L)[s+1] \\ \beta(\sigma^+, s+1) \oplus \lambda^B(\sigma^+, s+1) \subseteq ({}^\sigma B \oplus L)[s+1]. \end{array}$$

Let $\lambda(\sigma^+, s+1) = \lambda^A(\sigma^+, s+1) \cup \lambda^B(\sigma^+, s+1)$.

Γ-updating. Finally, enumerate $\langle x, \lambda\rangle \in \Gamma^{s+1}$, where

$$\lambda = \lambda(\sigma^+, s+1) \cup$$
$$\bigcup\{\lambda(\rho, s+1) \rho \subseteq \sigma \ \& [\rho^- \in T^{\mathbb{N}} \text{ or an } (\mathcal{N}, \mathcal{P})\text{-node or a } \Gamma\text{-node}]\}.$$

2. If $x \in \Gamma^L[s+1] - (\Phi^{\sigma A\oplus L}[s+1] \cap \Psi^{\sigma B\oplus L}[s+1])$, but there exist finite subsets $\alpha, \beta, \lambda^A, \lambda^B$ such that $\lambda^A, \lambda^B \subseteq L[s+1]$ and

$$\langle x, \alpha \oplus \lambda^A\rangle \in \Phi^s \qquad \langle x, \beta \oplus \lambda^B\rangle \in \Psi^s$$

and

$$\alpha \cap \bigcup\{V(\nu, s+1) \nu \subseteq \sigma \,\&\, \nu^- \in T^{\mathbb{N}^A}\} = \emptyset$$

$$\beta \cap \bigcup\{V(\nu, s+1) \nu \subseteq \sigma \,\&\, \nu^- \in T^{\mathbb{N}^B}\} = \emptyset$$

then let $\sigma^+ = \sigma\widehat{\ }1$ and choose consistently some such finite sets

$$\alpha(\sigma^+, s+1), \beta(\sigma^+, s+1), \lambda^A(\sigma^+, s+1), \lambda^B(\sigma^+, s+1)$$

and enumerate $\alpha(\sigma^+, s+1) \subseteq A^{s+1}$ and $\beta(\sigma^+, s+1) \subseteq B^{s+1}$.

3. otherwise, let $\sigma^+ = \sigma\hat{\ }0$. Note that in this case

$$\alpha(\sigma^+, s+1) = \beta(\sigma^+, s+1) = \lambda^A(\sigma^+, s+1) = \lambda^B(\sigma^+, s+1) = \emptyset.$$

Whatever the case, define $\lambda(\sigma^+, s+1) = \lambda^A(\sigma^+, s+1) \cup \lambda^B(\sigma^+, s+1)$.

Go and define $\delta_{s+1} \restriction n+2$, if $n+2 \leq s+1$.

5.2.3 σ is an $\mathcal{N}^A$-node

Assume that $\mathcal{R}(\sigma) = \mathcal{N}$, where

$$\mathcal{N}: \qquad A = \Phi^L \Rightarrow \overline{K} = \Delta^L.$$

For simplicity, we will omit the subscripts k and σ, thus writing Φ for Φ_k, and Δ for Δ_σ.

In order to measure the agreement between A and Φ^L, we now introduce a suitable length of agreement function. Let $D(\sigma, s+1)$ the collection of σ-followers so far defined, i.e.

$$D(\sigma, s+1) = \{c (\exists z)[c = c(\sigma, z, s+1)]\};$$

assume that at stage s we have also defined the (cofinite) set $\Pi(\tau, z, s)$ of numbers, for every $\tau \in T$ and $z \in \omega$.

Let

$$\begin{aligned} \ell(\sigma, s+1) = \text{least}\{\langle c,t\rangle c \in D(\sigma, s+1)\,\& \\ [(\forall u)[t \leq u \leq s \Rightarrow \\ [[c \in A^u \,\&\, c \in {}^\sigma A^{s+1} \,\&\, (\exists v \in \Pi(\sigma, \langle c,t\rangle, s))[t \leq v \leq s+1 \,\&\, c \notin \Phi^L[v]]] \vee \\ (\forall u)[t \leq u \leq s+1 \Rightarrow \\ [c \in \Phi^L[u] \,\&\, (\exists v \in \Pi(\sigma, \langle c,t\rangle, s))[[t \leq v \leq s \,\&\, c \notin A^v \text{ or } c \notin {}^\sigma A^{s+1}]]]\}. \end{aligned}$$

If no such $\langle c,t\rangle$ exists, then let $\ell(\sigma, s+1) = s+1$.

Assume that $\ell(\sigma, s+1) = \langle c,t\rangle$. In the previous definition, let v be the least stage such that $v \in \Pi(\sigma, \langle c,t\rangle, s)$ and $t \leq v \leq s+1$ and

$$(\forall u)[t \leq u \leq s \Rightarrow [c \in A^u \,\&\, c \in {}^\sigma A^{s+1} \,\&\, c \notin \Phi^L[v]]$$

or

$$(\forall u)[t \leq u \leq s+1 \Rightarrow [c \in \Phi^L[u] \,\&\, [c \notin A^v \text{ or } c \notin {}^\sigma A^{s+1}]]]$$

(let $v = s+1$, if, for every $u \leq s$. $c \in A^u$ but $c \notin {}^\sigma A^{s+1}$).

Let $\Pi(\sigma, \langle c,t\rangle, s+1) = \{r r > v\}$.

We notice that if there exist infinitely many stages $s+1$ such that $\ell(\sigma, s+1) = \langle c, t\rangle$, then either $c \in A - \Phi^L$ or $c \in \Phi^L - A$. Indeed, it is clear that either $c \in A$ or $c \in \Phi^L$. If for instance $c \in A$, then, since $\lim_s \min \Pi(\sigma, \langle c, t\rangle, s) = +\infty$, we have that there exist infinitely many stages v such that $c \notin \Phi^L[v]$ (the exist infinitely many distinct v since $v \in \Pi(\sigma, \langle c, t\rangle, s)$ at infinitely many stages s): a similar argument works if $c \in \Phi^L$.

Let $\sigma^+ = \sigma * \ell(\sigma, s+1)$.

If $\ell(\sigma, s+1) = \langle c, t\rangle$ and $c \in \Phi^L$, then we say that ℓ is a *finitary outcome*.

We say that $s+1$ is σ-*expansionary* if

$$\ell(\sigma, s+1) > \max\{\ell(\sigma, t) t \leq s \,\&\, \sigma \subseteq \delta_t\}.$$

We distinguish the following two cases. Let $\ell(\sigma, s+1) = \ell$ and assume that $\ell = \langle c, t\rangle$:

(a) $s+1$ is σ-expansionary.

In this case, let

$$x = \text{least}\{y y \in \overline{K}^{s+1} - \Delta^L[s+1] \,\&\, c(\sigma, y, s)\uparrow\} :$$

define $c(\sigma, x, s+1)$ to be a new $c \in \xi_\sigma$. For simplicity, for $z = x$ and for all z such that $c(\sigma, z, s)\downarrow$, let $c_z = c(\sigma, z, s+1)$. Then

- let $c_x \in A^{s+1}$;
- consider all number z such that
  1. $c_z \leq \ell$;
  2. $z \in \overline{K}^{s+1} - \Delta^L[s+1]$;

  For each such c_z, if $c_z \in \Phi^L[s+1]$, then choose consistently a finite set $\lambda(\sigma, z, s+1)$ such that $\lambda(\sigma, z, s+1) \subseteq L^{s+1}$ and

  $$\langle z, \lambda(\sigma, z, s+1)\rangle \in \Phi^{s+1}.$$

  Enumerate $\langle z, \lambda(\sigma, z, s+1)\rangle \in \Delta^{s+1}$.

(b) If $s+1$ is not σ-expansionary, then we further distinguish two cases:

1. $c \in \Phi^L[s+1] - {}^{\sigma}A^{s+1}$. In this case, let

   $$V(\sigma^+, s+1) = \{c \in D(\sigma, s+1) c \leq \ell \,\&\, z \notin \overline{K}^{s+1}\}.$$

   Let $V(\sigma^+, s+1) \nearrow A^{s+1}$, by letting $y \notin A^{s+1}$, for every $y \in V(\sigma^+, s+1)$.

2. If $c \in {}^{\sigma}A^{s+1} - \Phi^L[s+1]$, then let

$$V(\sigma^+, s+1) = \{c \in D(\sigma, s+1)[c \leq \ell \,\&\, z \notin \overline{K}^{s+1}] \text{ or } c \geq \ell\} :$$

Let $V(\sigma^+, s+1) \nearrow A^{s+1}$.

In both cases, let $c_z \in A^{s+1}$, for all z such that $z \in \overline{K}^{s+1}$ and $c_z \leq \ell$.
Let

$$C(\sigma^+, s+1) = \{c(\exists z)(\exists u)[c = c(\sigma, z, s+1) \,\&\, \\ \langle c, u\rangle \leq \ell \,\&\, c \in \Phi^L[s+1]] \,\&\, (\forall t)[u \leq t \leq s \Rightarrow c \in A^t \,\&\, c \in {}^{\sigma}A^{s+1}]\}.$$

For every $c \in C(\sigma^+, s+1)$ choose consistently (see Definition 5) a finite set $\lambda(\sigma^+, c, s+1)$ such that

$$\langle c, \lambda(\sigma^+, c, s+1)\rangle \in \Phi^{s+1} \quad \& \quad \lambda(\sigma^+, c, s+1) \subseteq L[s+1].$$

Finally, let

$$\lambda(\sigma^+, s+1) = \bigcup_{c \in C(\sigma^+, s+1)} \lambda(\sigma^+, c, s+1).$$

Go and define $\delta_{s+1} \restriction n+2$, if $n+2 \leq s+1$.

5.2.4 σ is an $(\mathcal{N}^A, \mathcal{P})$-node

Assume that $\mathcal{R}(\sigma) = (\mathcal{N}, \mathcal{P})$: let $\nu = o(\sigma)$, and let $\pi \subseteq \nu$ be such that $\mathcal{R}(\pi) = \mathcal{P}$; finally assume that (omitting obvious subscripts)

$$\mathcal{N}: \qquad A = \Psi^L \Rightarrow \overline{K} = \Delta^L,$$

and

$$\mathcal{P}: \qquad Z = \Phi^{A \oplus L} = \Psi^{B \oplus L} \Rightarrow Z = \Gamma^L.$$

Let also

$$V^A_{s+1} = \bigcup_{\nu \subseteq \sigma, \nu^- \in T^{\mathbb{N}A}} V(\nu, s+1),$$

$$V^B_{s+1} = \bigcup_{\nu \subseteq \sigma, \nu^- \in T^{\mathbb{N}B}} V(\nu, s+1),$$

Updating $H(\sigma, s)$ If there exists $x \in H(\sigma, s)$ such that

$$(\exists u)[t(\sigma, s+1) \leq u \leq s+1 \,\&\, x \notin \Gamma^L[u]]$$

then let $\hat{x}$ be the least such number and let

$$E(\sigma, s+1) = \{y \in H(\sigma, s)\hat{x} \prec_\sigma^s \hat{x}\};$$

if no such x exists, then let $E(\sigma, s+1) = H(\sigma, s+1)$.

Let

$$H(\sigma, s+1) = E(\sigma, s+1) \cup \{yy \in \Gamma^L[s+1] \,\&\, y \notin \Phi^{(\omega - V^A)\oplus\omega}[s+1]\}.$$

Given any $x \in H(\sigma, s+1)$, let

$$e(\sigma, x, s+1) = \min\{t \leq s+1 (\forall u)[t \leq u \leq s+1 \Rightarrow x \in \Gamma^L[u]]\}.$$

Finally, for every $x, x' \in H(\sigma, s+1)$, let $x \prec_\sigma^{s+1} x'$ if and only if

$$e(\sigma, x, s+1) < e(\sigma, x', s+1) \text{ or } [e(\sigma, x, s+1) = e(\sigma, x', s+1) \,\&\, x < x'].$$

Let $h(\sigma, s+1)$ be the canonical index of $E(\sigma, s+1)$. Define

$$\sigma^+ = \sigma\widehat{\ }h(\sigma, s+1).$$

For every $y \in E(\sigma, s+1)$, let

$$\beta(\sigma^+, y, s+1) = \bigcup\{\beta(\pi', t)\pi \subseteq \pi' \,\&\, \pi' \text{ is a } \Gamma\text{-node} \\ \&\, \mathcal{R}(\pi') = \langle \mathcal{R}(\pi), y\rangle \,\&\, 0 < t \leq s+1 \,\&\, (\exists\lambda)[\langle y, \lambda\rangle \in \Gamma_\pi^t - \Gamma_\pi^{t-1}\} - V_{s+1}^B.$$

Let

$$\beta(\sigma^+, s+1) = \bigcup_{y \in E(\sigma, s+1)} \beta(\sigma^+, y, s+1):$$

let $\beta(\sigma^+, s+1) \searrow B^{s+1}$, by defining $y \in B^{s+1}$, for every $y \in \beta(\sigma^+, s+1)$.

Since $E(\sigma, s+1) \subseteq \Gamma^L[s+1]$, for every $x \in E(\sigma, s+1)$ choose consistently (see Definition 5) a finite set $\lambda(\sigma^+, y, s+1)$ such that

$$\langle c, \lambda(\sigma^+, c, s+1)\rangle \in \Gamma^s \quad \& \quad \lambda(\sigma^+, y, s+1) \subseteq L[s+1].$$

Finally, let

$$\lambda(\sigma^+, s+1) = \bigcup_{y \in E(\sigma, s+1)} \lambda(\sigma^+, y, s+1).$$

Go and define $\delta_{s+1} \restriction n+2$, if $n+2 \leq s+1$.

5.2.5 σ is an $\mathcal{N}^B$-node

Assume that $\mathcal{R}(\sigma) = \mathcal{N}_k^B$. This case is similar to the case of an $\mathcal{N}^A$-node, but interchanging A with B, while considering the requirement

$$\mathcal{N}_k^B : \qquad B = \Phi_k^L \Rightarrow \overline{K} = \Delta_\sigma^L.$$

5.2.6 σ is an $(\mathcal{N}^B, \mathcal{P})$-node

This case is similar to the case of an $(\mathcal{N}^A, \mathcal{P})$-node, but interchanging A with B and Φ with Ψ, while considering the requirements (assuming that $\mathcal{R}(\sigma) = (\mathcal{N}_k^B, \mathcal{P}_i)$

$$\mathcal{N}_k^B : \qquad B = \Phi_k^L \Rightarrow \overline{K} = \Delta_\sigma^L,$$

and

$$\mathcal{P}_i : \qquad Z_i = \Phi_i^{A\oplus L} = \Psi_i^{B\oplus L} \Rightarrow Z_i = \Gamma_\sigma^L.$$

Notice also that in this case we define finite sets $\alpha(\sigma^+, y, s+1)$ (instead of $\beta(\sigma^+, y, s+1)$) and $\alpha(\sigma^+, s+1)$ (instead of $\beta(\sigma^+, s+1)$), and enumerate into A^{s+1} the elements of $\alpha(\sigma^+, s+1)$.

5.2.7 Final updating

At the end of stage $s+1$ let

$$A^{s+1} = {}^{\delta_{s+1}}A^{s+1}$$

and

$$B^{s+1} = {}^{\delta_{s+1}}B^{s+1}.$$

For every $\sigma \subseteq \delta_{s+1}$, let

$$\begin{aligned} \Gamma_\sigma^{s+1} &= \Gamma_\sigma^s \cup \{\langle x, \lambda\rangle \langle x, \lambda\rangle \searrow \Gamma_\sigma^{s+1}\} \\ \Delta_{A,\sigma}^{s+1} &= \Delta_{A,\sigma}^s \cup \{\langle x, \lambda\rangle \langle x, \lambda\rangle \searrow \Delta_{A,\sigma}^{s+1}\} \\ \Delta_{B,\sigma}^{s+1} &= \Delta_{B,\sigma}^s \cup \{\langle x, \lambda\rangle \langle x, \lambda\rangle \searrow \Delta_{B,\sigma}^{s+1}\}. \end{aligned}$$

6 The verification

We first show

For every n,

(1) $\sigma_n = \liminf_s \delta_s \restriction n$ exists;

(2) for every $i < n$, $\sigma_i \subset \sigma_{i+1}$;

(3) $\lim_s \alpha(\sigma_n, s)$, $\lim_s \beta(\sigma_n, s)$, and $\lim_s \lambda(\sigma_n, s)$ exist and are finite; moreover, if $\lambda(\sigma_n) = \lim_s \lambda(\sigma_n, s)$, then $\lambda(\sigma_n) \subseteq L$.

(4) if $\tau \subset \sigma_n$ is an $\mathcal{N}$-node, and ℓ is the outcome at τ along σ_n, then $\lim_s C(\tau\frown\ell, s)$ exists and is finite.

(5) if $\tau \subset \sigma_n$ is an $(\mathcal{N}, \mathcal{P})$-node, with $\mathcal{R}(\tau) = (\mathcal{N}_k, \mathcal{P}_i)$, and $\pi \subseteq \tau$ such that $\mathcal{R}(\tau) = \mathcal{P}_i$, and h is the outcome at τ along σ_n, then

$$D_h = \{x(\exists s)[x \in H(\tau, s)] \,\&\, x \in \Gamma^L_\pi\}.$$

The proof is by induction on n. For $n = 0$ the claim is trivial, being $\sigma_0 = \emptyset$.

Suppose now that the claim is true of n. Let $\sigma_n = \liminf_s \delta_s \restriction n$, and for every $\tau \preceq \sigma_n$ let $\alpha(\tau) = \lim_s \alpha(\tau, s)$, $\beta(\tau) = \lim_s \beta(\tau, s)$, $\lambda(\tau) = \lim_s \lambda(\tau, s)$.

Moreover,

Let t_{σ_n} be a stage such that, for every $s \geq t_{\sigma_n}$,

- for all $\tau \prec_L \sigma_n$, $\tau \not\subseteq \delta_s$;
- for all $\tau \preceq \sigma_n$

$$\alpha(\tau, s) = \alpha(\tau, t_{\sigma_n}) \,\&\, \beta(\tau, s) = \beta(\tau, t_{\sigma_n}) \,\&\, \lambda(\tau, s) = \lambda(\tau, t_{\sigma_n});$$

- for every $\mathcal{N}$-node $\nu \subset \sigma_n$, if ℓ is the outcome at ν along σ_n, then $C(\nu\frown\ell, s) = C(\nu\frown\ell, t_{\sigma_n})$, and then for every z, t such that $c(\nu, z, t) \leq \ell$, we have that

$$z \in \overline{K} \Leftrightarrow z \in \overline{K}^s;$$

- for every $\mathcal{N}$-node $\nu \subset \sigma_n$, if ℓ is the outcome at ν along σ_n and ℓ is finitary (see Definition 5.2.3), then $\nu \subseteq \delta_s \Rightarrow \nu\frown\ell \subseteq \delta_s$.

We distinguish the following cases, according as σ_n is a $\mathcal{P}$-node, a Γ-node, an $\mathcal{N}$-node, or an $(\mathcal{N}, \mathcal{P})$-node.

Case 1 σ_n is a $\mathcal{P}$-node. Then obviously $\sigma_{n+1} = \liminf_s \delta_s \restriction n + 1 = \sigma_n\frown 0$.

Case 2 σ_n is a Γ-node. Assume that $\mathcal{R}(\sigma_n) = (\mathcal{P}_i, x)$.

Then

1. *either*
$$(\exists^\infty s)[\sigma_n{}^\frown 0 \subseteq \delta_s]:$$
in this case $\sigma_{n+1} = \sigma_n{}^\frown 0$. Notice also that, for every s,
$$\alpha(\sigma_{n+1}, s) = \beta(\sigma_{n+1}, s) = \lambda(\sigma_{n+1}, s) = \emptyset,$$
so part (3) of the claim is trivially verified.

2. *or*
$$\sigma_{n+1} = \sigma_n{}^\frown 1.$$
This case corresponds to the outcome $x \in \Phi_i^{A\oplus L} \cap \Psi_i^{B\oplus L}$ and we are thus able to settle on some $\alpha = \lim_s \alpha(\sigma_{n+1}, s) \subseteq A$, $\beta = \lim_s \beta(\sigma_{n+1}, s) \subseteq B$, and some finite $\lambda^A = \lim_s \lambda^A(\sigma_{n+1}, s)$, $\lambda^B = \lim_s \lambda^B(\sigma_{n+1}, s)$ such that $\lambda^A \cup \lambda^B \subseteq L$ for which $x \in \Phi_i^{\alpha\oplus\lambda^A} \cap \Psi_i^{\beta\oplus\lambda^B}$.

   Clearly, $\alpha(\sigma_{n+1}) = \lim_s \alpha(\sigma_{n+1}, s) = \alpha$, $\beta(\sigma_{n+1}) = \lim_s \beta(\sigma_{n+1}, s) = \beta$, and $\lambda(\sigma_{n+1}) = \lim_s \lambda(\sigma_{n+1}, s) = \lambda^A \cup \lambda^B$.

Case 3 σ_n is an $\mathcal{N}^{\mathcal{A}}$-node. Assume that $\mathcal{R}(\sigma_n) = \mathcal{N}_k$.

The claim easily follows from the following sublemma.

If $\lim_s \ell(\sigma_n, s) = +\infty$, then $\overline{K} =^* \Delta^L_{\sigma_n}$.

First of all notice that, for every τ and number z, if, at some stage t, $c(\tau, z, t)\downarrow$, then
$$(\forall s \geq t)[c(\tau, z, s) = c(\tau, z, t)].$$

Let now
$$D(\sigma_n) = \{c(\sigma_n, z, s) : z, s \in \omega\}$$
and let, for simplicity, $c_z = c(\sigma_n, z, s)$, whenever $c(\sigma_n, z, s)\downarrow$. Assume that there exists some $c \in D(\sigma_n)$ such that $A(c) \neq \Phi_k^L(c)$.

If $c \in A - \Phi_k^L$, then there exists u such that, for every $s \geq u$, $c \in A^s$, and for infinitely many s, $c \notin \Phi_k^L[s]$. Then,

1. *either* there is some least $\langle c', t\rangle < \langle c, u\rangle$ such that $c' \in D(\sigma_n)$ and
$$(\forall s \geq t)[c' \in A^s \,\&\, c' \notin \Phi_k^L]$$
or
$$(\forall s \geq t)[c' \in \Phi_k^L[s] \,\&\, c' \notin A]$$
in which case $\liminf_s \ell(\sigma_n, s) = \langle c', t\rangle$;

2. *or* $\liminf_s \ell(\sigma_n, s) = \langle c, u\rangle$.

Therefore

$$\lim_s \ell(\sigma, s) = +\infty \Rightarrow A = \Phi_k^L.$$

We now show that if $A = \Phi_k^L$ then $\overline{K} =^* \Delta_{\sigma_n}^L$. Clearly, if $z \in \overline{K}$ and $\lim_s \ell(\sigma, s) = +\infty$ then there exists some stage s such that $c(\sigma, z, s)\downarrow$, since there are infinitely many σ_n-expansionary stages, and part (a), Subsection 5.2.3, is performed infinitely often. Since the set $F = \bigcup_{\tau \preceq \sigma_n} \alpha(\tau) \cup \beta(\tau)$ is finite by inductive assumption, there exists some x_0 such that, for every $x \geq x_0$, $c_x \notin F$. We claim that

$$(\forall x \geq x_0)[\overline{K}(x) = \Delta_{\sigma_n}^L(x)].$$

Thus, let $x \geq x_0$.

Assume that $x \in \overline{K}$. If t is a stage such that, for every $s \geq t$, $c_x = c(\sigma, x, s)$ is defined and $\ell(\sigma, s) > x$, then $c_x \in A^s$, for each such stage s (the number $c_x \in \xi_\sigma$ can be extracted only on behalf of requirement $\mathcal{R}(\sigma_n)$, and if $x \in \overline{K}$, then c_x can be extracted at s only if $\ell(\sigma_n, s) \leq x$). Therefore $c_x \in A$, and, thus, $c_x \in \Phi^L$. Let $t' \geq t_{\sigma_n}$ be such that, for every $s \geq t'$, $c_x \in \Phi^L[s]$. At some σ_n-expansionary stage $s' \geq t'$, we are therefore eventually able to appoint some finite set $\lambda = \lambda(\sigma, x, s') = \lim_s \lambda(\sigma, x, s) \subseteq L$, such that $\langle x, \lambda \rangle \in \Delta_{\sigma_n}$: hence $x \in \Delta_{\sigma_n}^L$, as desired.

Assume now that $x \notin \overline{K}$; then, under the assumption that $\lim_s \ell(\sigma_n, s) = +\infty$ and $c_x \notin F$, we extract c_x from A^s, for infinitely many stages s. But then $c_x \notin \Phi_k^L$. Since

$$(\forall \lambda)[\langle x, \lambda \rangle \in \Delta_{\sigma_n} \Rightarrow \langle x, \lambda \rangle \in \Phi_k],$$

it follows that $x \notin \Delta_{\sigma_n}^L$.

Since $\overline{K} \not\leq_e L$, it thus follows that $\ell = \liminf_s \ell(\sigma_n, s)$ is finite. Thus

$$\sigma_{n+1} = \sigma_n{}^\frown \ell.$$

It is left to show that $\lim_s C(\sigma_{n+1}, s)$ exists and is finite: on the other hand, it is clear that $\lim_s C(\sigma_{n+1}, s) = C(\sigma_{n+1})$, where

$$C(\sigma_{n+1}) = \{c \in D(\sigma_n)(\exists u)[\langle c, u \rangle \leq \ell \,\&\, c \in \Phi^L \,\&\, (\forall s \geq u)[c \in A^s]]\}.$$

Moreover, for every $c \in C(\sigma_{n+1})$ we are eventually able to appoint some (consistently chosen) finite set $\lambda(\sigma_{n+1}, c)$ such that $\lambda(\sigma_{n+1}, c) = \lim_s \lambda(\sigma_{n+1}, c, s)$, with $c \in \Phi^{\lambda(\sigma_{n+1}, c)}$ and $\lambda(\sigma_{n+1}, c) \subseteq L$. Therefore $\lambda(\sigma_{n+1}) = \lim_s \lambda(\sigma_{n+1}, s)$ exists and is finite, being $\lambda(\sigma_{n+1}) = \bigcup_{c \in C(\sigma_{n+1})} \lambda(\sigma_{n+1}, c)$. We have also shown that $\lambda(\sigma_{n+1}) \subseteq L$.

Case 4 σ_n is an $(\mathcal{N}^{\mathcal{A}}, \mathcal{P})$-node. Assume that $\mathcal{R}(\sigma) = (\mathcal{N}_k, \mathcal{P}_i)$; let $\pi \subseteq \sigma_n$ be such that $\mathcal{R}(\pi) = \mathcal{P}_i$, and let $\nu = o(\sigma_n)$.

We first show

$\liminf_s h(\sigma_n, s)$ exists and is finite. In fact $\liminf h(\sigma_n, s) = h$, where

$$D_h = \{x(\exists s)[x \in H(\sigma_n, s)] \,\&\, x \in \Gamma_\pi^L\}.$$

First we observe:

Claim The set D, where

$$D = \{x(\exists s)[x \in H(\sigma_n, s)] \,\&\, x \in \Gamma_\pi^L\},$$

is finite.

Indeed, clearly D contains only numbers x, such that we enumerate an axiom $\langle x, \lambda\rangle \in \Gamma_\pi^s$, while acting at some stage s at some Γ-node $\pi' \supseteq \pi$, with $\mathcal{R}(\pi') = (\mathcal{R}(\pi), x)$.

Consider all Γ-node $\pi' \supseteq \pi$, with $\mathcal{R}(\pi') = (\mathcal{R}(\pi), x)$, for which we define axioms $\langle x, \lambda\rangle \in \Gamma_\pi^s$ *only* at stages $s \geq t_{\sigma_n}$.

We distinguish the following two cases.

Case 1 $\sigma_n \prec_L \pi'$. In this case there exists some longest τ such that $\pi \subseteq \tau \subset \sigma_n$, and the outcome o at τ along π' is such that $\sigma_n \prec_L \tau\hat{\ }o$.

(a) τ is a Γ-node. Then $o = 1$, i.e. $\tau\hat{\ }1 \subseteq \pi'$ and $\tau\hat{\ }0 \subseteq \sigma_n$. Suppose that $s \geq t_{\sigma_n}$ is a stage such that $\tau\hat{\ }1 \subseteq \delta_s$. Then there exist finite sets $\alpha = \alpha(\tau\hat{\ }1, s)$, $\beta = \beta(\tau\hat{\ }1, s)$, $\lambda^A = \lambda^A(\tau\hat{\ }1, s)$, $\lambda^B = \lambda^B(\tau\hat{\ }1, s)$ such that $x \in \Phi_\pi^{\alpha\oplus\lambda^A} \cap \Psi_\pi^{\beta\oplus\lambda^B}$, and $\lambda^A \cup \lambda^B \subseteq L[s]$. Thus if $\langle x, \lambda\rangle \in \Gamma_\pi$ is the axiom we define at s at π' then we have that $\lambda^A \cup \lambda^B \subseteq \lambda$. Since $\liminf_t \delta_t|\tau| + 1 = \tau\hat{\ }0$, we must conclude that $\lambda^A \cup \lambda^B \not\subseteq L$, thus $\lambda \not\subseteq L$.

(b) τ is a $\mathcal{N}$-node. Let $\ell = \langle c, u\rangle$ be the outcome at τ along σ_n. Thus there exists ℓ' such that $\ell < \ell'$ and $\tau\hat{\ }\ell' \subseteq \pi' \subseteq \delta_s$. By definition of t_{σ_n} and since we assume to take action at a stage $s \geq t_{\sigma_n}$, we conclude that ℓ is not finitary (see Definition 6). Therefore $c \notin \Phi_\pi^L$, but, at stage s, we have that $c \in \Phi_\pi^L[s]$, and $c \in C(\pi\hat{\ }\ell', s)$; thus if $\langle x, \lambda\rangle \in \Gamma_\pi^s$ is the axiom we define at s at π', then we have that $\lambda(\tau, c, s) \subseteq \lambda$. Since $\lambda(\tau, c, s) \not\subseteq L$, we have that $\lambda \not\subseteq L$.

(c) τ is an $(\mathcal{N}^A, \mathcal{P})$-node. Let h be the outcome of τ along σ_n, and let $s \geq t_{\sigma_n}$. Thus there exists h', with $h < h'$ such that $\tau\hat{\ }h' \subseteq \pi' \subseteq \delta_s$. It follows by induction that

$$D_h = \{x(\exists t)[x \in H(\sigma, t)] \,\&\, x \in \Gamma_\pi^L\}.$$

Since $\liminf_s \delta_s|\tau|+1 = \tau^\frown h$, it follows that there must exist $x \in D_{h'}$ and a finite set $\lambda(\tau,x,s)$ such that $x \notin \Gamma_\pi^L$, and $x \in \Gamma_\pi^{\lambda(\tau,x,s)}$ and $\lambda(\tau,x,s) \subseteq L[s]$. The construction ensures that if $\langle x,\lambda\rangle \in \Gamma_\pi^s$ is the axiom we define at s at π', then $\lambda(\tau,x,s) \subseteq \lambda$: but $\lambda(\tau,x,s) \nsubseteq L$, therefore $\lambda \nsubseteq L$.

(d) τ is an $(\mathcal{N}^B,\mathcal{P})$-node. Similar to (c).

Case 2 $\sigma_n \subset \pi'$. Let $\ell = \langle c,u\rangle$ be the outcome at σ_n along π'.

Given any stage t, let us say that $x \in H(\sigma_n,t)$ *because of* π', if there is an axiom $\langle x,\lambda\rangle \in \Gamma_\pi$ appointed at π' such that, letting

$$V_t^A = \bigcup_{\nu' \subseteq \nu, (\nu')^- \in T^{\mathbb{N}^A}} V(\nu,t),$$

we have that

$$\lambda \subseteq L[t] \,\&\, x \notin \Phi_\pi^{(\omega - V_t^A)\oplus\omega}[t].$$

We claim that there is no t such that $x \in H(\sigma_n,t)$, becuase of π': assume for a contradiction otherwise, and let t be a stage such that, Since we assume that we appoint axioms at π' only at stages $s \geq t_{\sigma_n}$, we may assume that $t \geq t_{\sigma_n}$. Thus, there must exist an axiom $\langle x,\lambda\rangle \in \Gamma_\pi^s$, appointed at some stage $s \geq t_{\sigma_n}$; hence $x \in \Phi_\pi^{A\oplus L}[s]$, and thus there exists a finite set α such that $x \in \Phi_\pi^{\alpha\oplus\lambda}$ and $\alpha \oplus \lambda \subseteq A \oplus L[s]$. Since $\sigma_n = \liminf_s \delta_s n$ and vy our choice of σ_n, it easily follows from the construction that $V_t^A \subseteq V_{t'}^A$ (from which it follows that $V_t^A \cap A^{t'} = \emptyset$), for every $t' \geq t$, such that $\sigma_n \subseteq \delta_{t'}$. It follows that $s < t$, but then $V_t^A \cap \alpha = \emptyset$, since at stages $u > s$ only new numbers (thus numbers not in α) can be appointed as ν'-followers (with $\nu' \subseteq \nu$ and $(\nu')^- \in T^{\mathbb{N}^A}$) and subsequently enter V_t^A. Since this holds of every possible axiom appointed at π' at any s such that $t_{\sigma_n} \leq s < t$, we have a contradiction.

We have thus shown that the set D, where

$$D = \{x (\exists s)[x \in H(\sigma_n,s)] \,\&\, x \in \Gamma_\pi^L\}$$

is finite, since this set can contain only numbers x such that either

- axioms $\langle x,\lambda\rangle \in \Gamma_\pi^s$ are appointed at stages $s \leq t_{\sigma_n}$, or
- axioms $\langle x,\lambda\rangle \in \Gamma_\pi^s$ are appointed at Γ-nodes π' (with $\mathcal{R}(\pi') = (\mathcal{R}(\pi),x)$) such that $\pi \subseteq \pi' \subseteq \sigma_n$.

Let h be the canonical index of D. We are now in a position to show that

$$\sigma_{n+1} = \liminf_s \delta_s n + 1 = \sigma_n {}^\frown h.$$

Clearly there exists a stage $s_0 \geq t_{\sigma_n}$ such that

$$(\forall s \geq s_0)(\forall y \in D_h)[y \in H(\sigma, s) \,\&\, (\forall x \in H(\sigma, s))[y \preceq^s_{\sigma_n} x]]$$

and $\preceq_{\sigma_n} = \lim_s \preceq^s_{\sigma_n}$ exists on D_h, where, for any $y, y' \in D_h$, we have $y \prec_{\sigma_n} y'$ if and only if

$$(\exists t_0)(\forall s \geq t)[y \in \Gamma^L_\pi[s] \,\&\, (\exists t_1 \geq t_0)[y' \notin \Gamma^L_\pi[t_1]]].$$

It is also clear that $D_h \subseteq D_{h(\sigma_n,s)}$, for every $s \geq s_0$.

To show that there are infinitely many stages s such that $h(\sigma_n, s) = h$, we show that for every $t \geq s_0$ there exists $s > t$ such that $h(\sigma_n, s) = h$. To this end, let $t \geq s_0$. Suppose that $s' \geq t$ is such that $\sigma_n \subseteq \delta_{s'}$: then $D_h \subseteq D_{h(\sigma_n,s')}$. Let us assume that $x \in D_{h(\sigma_n,s')} - D_h$, and x is the $\prec^{s'}_{\sigma_n}$-least such element; clearly $x \notin \Gamma^L_\pi$. Thus, for every $y \in D_h$, $y \prec^{s'}_{\sigma_n} x$. It follows that at the least stage $s > s'$ such that $\sigma_n \subseteq \delta_s$ such that $x \notin \Gamma^L_\pi[s]$, we define $E(\sigma_n, s) = D_h$, hence $h(\sigma_n, s) = h$.

It follows that we eventually appoint some consistently chosen finite sets $\beta(\sigma_{n+1}, y) = \lim_s \beta(\sigma_{n+1}, y, s)$, $\lambda(\sigma_{n+1}, y) = \lim_s \lambda(\sigma_{n+1}, y, s)$ for every $y \in D_h$, such that $y \in \Psi^{\beta(\sigma_{n+1},y) \oplus \lambda(\sigma_{n+1},y)}$, and $\lambda(\sigma_{n+1}, y) \subseteq L$, and thus the set $\beta(\sigma_{n+1}) = \liminf_s \beta(\sigma_{n+1}, s)$ exists and is finite, being $\beta(\sigma_{n+1}) = \bigcup_{y \in D_h} \beta(\sigma_{n+1}, y)$. Finally we observe that $\lim_s \lambda(\sigma_{n+1}, s) = \lambda(\sigma_{n+1})$, where $\lambda(\sigma_{n+1}) = \bigcup_{y \in D_h}$, and $\lambda(\sigma_{n+1}) \subseteq L$.

Case 5 σ_n is an $\mathcal{N}^{\mathcal{B}}$-node. The verification is similar to Case 3, but interchanging A with B.

Case 6 σ_n is an $(\mathcal{N}^{\mathcal{B}}, \mathcal{P})$-node. The verification is similar to Case 4, but interchanging A with B, and Φ with Ψ.

By Lemma 6, let f the infinite path through T such that, for every n, $f \restriction n = \sigma_n$. f is called the *true path*.

For every k, the requirements $\mathcal{N}^A_k$ and $\mathcal{N}^B_k$ are satisfied.

Assume that n is such that $\mathcal{R}(f \restriction n) = \mathcal{N}^A_k$. Then by Lemma 6, Case 3, $\liminf_s \ell(f \restriction n, s)$ exists, and the proof of Sublemma 6 shows that in fact $A \neq \Phi^A_k$.

A similar argument applies if $f \restriction n$ is an $\mathcal{N}^{\mathcal{B}}$-node.

For every i the requirement $\mathcal{P}_i$ is satisfied.

Given i, we want to show that

$$Z_i = \Phi_i^{A\oplus L} = \Psi_i^{B\oplus L} \Rightarrow Z_i =^* \Gamma^L$$

where $\Gamma = \Gamma_\sigma$ is the e-operator that we construct at nodes $\tau \supseteq \sigma$, with $\sigma \subset f$ such that $\mathcal{R}(\sigma) = \mathcal{P}_i$.

For simplicity, throughout the following proof, we will omit the subscript i.

First assume that $x \in \Phi^{A\oplus L} \cap \Psi^{B\oplus L}$. Then there exists a stage t such that, for every $s \geq t$, $x \in \Phi^{A\oplus L}[s] \cap \Psi^{B\oplus L}[s]$. Let $\tau \subset f$ be the Γ-node, such that $\mathcal{R}(\tau) = (\mathcal{P}_i, x)$, and, by Definition 6, let t_τ be a stage such that for every $s \geq t_\tau$,

- for all $\tau' \prec_L \tau$, $\tau' \not\subseteq \delta_s$.
- for all $\rho \subseteq \tau$, $\lambda(\rho, s) = \lambda(\rho, t_\tau)(= \lambda(\rho)$;
- for every $\mathcal{N}$-node $\nu \subset \tau$, if ℓ is the outcome at ν along τ, then $C(\nu^\frown\ell, s) = C(\nu^\frown\ell, t_\tau)$ and for every z, t such that $c(\nu, z, t) \leq \ell$, we have that
$$z \in \overline{K} \Leftrightarrow x \in \overline{K}^s.$$

Since we act at τ infinitely often, we can eventually find a stage $t_0 \geq t_{\sigma_n}$ such that at t_0 we appoint finite sets

$$\begin{aligned}
\lambda^A(\tau^\frown 1) &= \lim\nolimits_s \lambda^A(\tau^\frown 1, s),\\
\lambda^B(\tau^\frown 1) &= \lim\nolimits_s \lambda^B(\tau^\frown 1, s),\\
\alpha(\tau^\frown 1) &= \lim\nolimits_s \alpha(\tau^\frown 1, s),\\
\beta(\tau^\frown 1) &= \lim\nolimits_s \beta(\tau^\frown 1, s)
\end{aligned}$$

such that

$$\alpha(\tau^\frown 1) \oplus \lambda^A(\tau^\frown 1) \subseteq A \oplus L$$

$$\beta(\tau^\frown 1) \oplus \lambda^B(\tau^\frown 1) \subseteq B \oplus L$$

and $x \in \Phi^{\alpha(\tau^\frown 1)\oplus\lambda^A(\tau^\frown 1)} \cap \Psi^{\beta(\tau^\frown 1)\oplus\lambda^B(\tau^\frown 1)}$, and, for every $s \geq t_0$,

$$\alpha(\tau^\frown 1) \cap \bigcup_{\nu\subseteq\sigma, \nu^- \in T^{\mathbb{N}A}} V(\nu, s) = \emptyset$$

$$\beta(\tau^\frown 1) \cap \bigcup_{\nu\subseteq\sigma, \nu^- \in T^{\mathbb{N}B}} V(\nu, s) = \emptyset.$$

Let $\lambda(\tau^\frown 1) = \lambda^A \cup \lambda^B$, and, for every $\rho \subseteq \sigma_n$, let $\lambda(\rho) = \lim_s \lambda(\rho, s)$.

Then we eventually appoint an axiom

$$\langle x, \lambda(\tau^\frown 1) \cup \bigcup\{\lambda(\rho)\rho \preceq \sigma_n\} \in \Gamma.$$

Therefore $x \in \Gamma^L$, since by Lemma 6 $\lambda(\rho) \subseteq L$, for every $\rho \subseteq \sigma_n$.

Assume now that $\Phi^{A\oplus L} = \Psi^{B\oplus L}$, and let, for a contradiction, $x \in \Gamma^L - Z$, where $Z = \Phi^{A\oplus L} = \Psi^{B\oplus L}$. Assume further that axioms of the form $\langle x, \lambda\rangle \in \Gamma$ are enumerated at stages $s \geq t_\sigma$, where t_σ is as given in Definition 6.

Suppose that $s_0 \geq t_\sigma$ is the least stage such that we appoint at some $\pi' \supseteq \sigma$ finite sets $\alpha = \alpha(\pi', s_0), \beta = \beta(\pi', s_0), \lambda^A(\pi', s_0), \lambda^B(\pi', s_0)$ and we enumerate an axiom $\langle x, \lambda\rangle \in \Gamma$ such that $\lambda^A(\pi', s_0) \cup \lambda^B(\pi', s_0) \subseteq \lambda \subseteq L$.

Then when we visit at stages $s \geq t_\sigma$ the node $\pi \subset f$ such that $\mathcal{R}(\pi) = \langle \mathcal{P}_i, x\rangle$, either we can not find finite sets α', β' such that $\alpha' \cap V_s^A = \emptyset$ and $\beta' \cap V_s^B = \emptyset$ where

$$V_s^A = \bigcup_{\nu\subseteq\sigma, \nu^- \in T^{\mathbb{N}^A}} V(\nu, s),$$

$$V_s^B = \bigcup_{\nu\subseteq\sigma, \nu^- \in T^{\mathbb{N}^B}} V(\nu, s),$$

or we can find such sets, but, subsequently, our extracting activity on behalf of the $\mathcal{N}$-requirements located at $\mathcal{N}$-nodes entails $\alpha' \not\subseteq A$ or $\beta' \not\subseteq B$.

By our choice of s_0, this can happen only if at some $\mathcal{N}$-node ν such that $\sigma \subseteq \nu \subset \pi$, we extract elements of α from A or elements of β from B. Let ν be the least $\mathcal{N}$-node for which this happen, and assume that ν is an $\mathcal{N}^{\mathcal{A}}$-node (similar arguments apply if ν is an $\mathcal{N}^{\mathcal{B}}$-node). Thus

$$\alpha \cap \{c(\exists^\infty s)[c \in V(\nu^\frown \ell, s) \,\&\, \nu^\frown \ell \subseteq \delta_s]\} \neq \emptyset$$

(where ℓ is the outcome of ν along τ), and $x \notin \Phi^{A\oplus L}$.

Now, let $c = c(\nu, z, s')$ be such that $c \in \alpha$ and $(\exists^\infty s)[c \in V(\nu^\frown \ell, s) \,\&\, \nu^\frown \ell \subseteq \delta_s]$, where s' is the stage at which we appoint c (notice that $s' \leq s_0$, because we always appoint new numbers as ν-followers). We may also assume that we can not restore $x \in \Phi^{A\oplus L}$ at π at any stage $t \geq t_{\nu^\frown \ell}$.

Let $\tau \subset f$ be the $(\mathcal{N}, \mathcal{P}_i)$-node immediately following ν on the true path. It follows from Lemma 6(5) that $x \in D_h$, where h is the outcome at τ along f. By assumptions, there is no $\mathcal{N}^B$-node ν' such that $\sigma \subseteq \nu' \subseteq \nu$ such that the extracting activity demanded by $\mathcal{R}(\nu')$ interferes with restraining some finite set $\beta \subseteq B$ to get $x \in \Psi^{B\oplus L}$. Therefore, we can eventually restrain a finite set $\beta(\pi^\frown h, x) = \lim_s \beta(\pi^\frown h, x, s) \subseteq B$ such that $x \in \Psi^{\beta(\pi^\frown h, x)\oplus L}$.

This shows that $x \in \Psi^{B\oplus L} - \Phi^{A\oplus L}$, contradicting the hypothesis that $\Phi^{A\oplus L} = \Psi^{B\oplus L}$.

This concludes the proof of the theorem.

7 Simultaneous branching

Let us now go back to Theorem 1.

Let L_1, L_2 be Σ^0_2 sets such that $L_1, L_2 <_e \overline{K}$. To prove the theorem we need to build Σ^0_2 sets A, B satisfying the following requirements: for every $i, j, k \in \omega$.

$$\begin{array}{ll}
\mathcal{P}^{L_1}_i: & Z^{L_1}_i = \Phi^{A\oplus L_1}_i = \Psi^{B\oplus L_1}_i \Rightarrow Z^{L_1}_i = \Gamma^{L_1}_{L_1,i} \\
\mathcal{P}^{L_2}_i: & Z^{L_2}_i = \Phi^{A\oplus L_2}_i = \Psi^{B\oplus L_2}_i \Rightarrow Z^{L_2}_i = \Gamma^{L_2}_{L_2,i} \\
\mathcal{N}^{A,L_1}_k: & A = \Phi^{L_1}_k \Rightarrow \overline{K} = \Delta^{L_1}_{A,L_1,k} \\
\mathcal{N}^{A,L_2}_k: & A = \Phi^{L_2}_k \Rightarrow \overline{K} = \Delta^{L_2}_{A,L_2,k} \\
\mathcal{N}^{B,L_1}_k: & B = \Phi^{L_1}_k \Rightarrow \overline{K} = \Delta^{L_1}_{B,L_1,k} \\
\mathcal{N}^{B,L_2}_k: & B = \Phi^{L_2}_k \Rightarrow \overline{K} = \Delta^{L_2}_{B,L_2,k}
\end{array}$$

where $\Gamma_{L_1,i}, \Delta_{A,L_1,k}, \Delta_{B,L_1,k}, \Gamma_{L_2,i}, \Delta_{A,L_2,k}, \Delta_{B,L_2,k}$ are e-operators to be constructed.

The requirements are given the priority ordering

$$\ldots \mathcal{P}^{L_1}_i < \mathcal{P}^{L_2}_i < \mathcal{N}^{A,L_1}_i < \mathcal{N}^{A,L_2}_i < \mathcal{N}^{B,L_1}_i < \mathcal{N}^{B,L_2}_i < \\ \mathcal{P}^{L_1}_{i+1} < \mathcal{P}^{L_2}_{i+1} < \mathcal{N}^{A,L_1}_{i+1} < \mathcal{N}^{A,L_2}_{i+1} < \mathcal{N}^{B,L_1}_{i+1} < \mathcal{N}^{B,L_2}_{i+1}, \ldots$$

The construction is virtually the same as in the proof of Theorem 1. We use a tree of outcomes in which we distinguish $\mathcal{N}$-nodes (partitioned into the following classes of nodes: $\mathcal{N}^{A,L_1}-, \mathcal{N}^{A,L_2}-, \mathcal{N}^{B,L_1}-$, and $\mathcal{N}^{A,L_2}-$nodes), $\mathcal{P}$-nodes ($\mathcal{P}^{L_1}-$ and $\mathcal{P}^{L_2}-$nodes), and Γ-nodes. Each $\mathcal{N}$-node σ is followed by a finite sequence of $(\mathcal{N}, \mathcal{P})$-nodes, recording the effects on the strategy for $\mathcal{R}(\sigma)$ on higher priority $\mathcal{P}$-requirements.

For every $\mathbf{c} < \mathbf{0}'_e$ there exists a minimal pair $\mathbf{a}, \mathbf{b}$ such that $\mathbf{c} < \mathbf{c} \cup \mathbf{a}$, $\mathbf{c} < \mathbf{c} \cup \mathbf{b}$ and $\mathbf{c} = (\mathbf{c} \cup \mathbf{a}) \cap (\mathbf{c} \cup \mathbf{b})$.