Relativization of Structures Arising from Computability Theory

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We describe a general method to separate relativizations of structures arising from computability theory. The method is applied to the lattice of r.e. sets, and the partial orders of r.e. m–degrees and T–degrees. We also consider classes of oracles where all relativizations are elementarily equivalent. We hope that the paper can serve as well as an introduction to coding in these structures.

1. Introduction The relativization of a concept from computability theory to an oracle set Z is obtained by expanding the underlying concept of computation in a way such that, at any step of the computation procedure, tests of the form "$n \in Z$", where n is some number obtained previously in the computation, are allowed. For instance, the relativization of the concept of r.e. sets to Z is "set r.e. in Z". In this paper, we study to what extent the isomorphism type and the theory of the relativization A^Z of a structure A from computability theory depend on the oracle set Z. We consider mainly the case that A is the structure E of r.e. sets under inclusion or a degree structure on r.e. sets, but first discuss the case that A is the structure of D_T all T–degrees or D_m of all m–degrees. In this case, D_m^Z is the structure of degrees of subsets of ω under many–one reductions via (total) functions recursive in Z, while D_T^Z is simply the upper cone of D_T above the T–degree of Z.

It is a common phenomenon in computability theory that the proof of a result is actually a proof of all relativized forms of the result. Thus, the proof that there is a minimal T–degree below $0''$ actually shows that each degree z has a minimal cover below z'', and the construction of a maximal r.e. set actually gives an index i such that (W_i^Z) is a coatom in $(E^Z)^*$.

This observation led to the "strong homogeneity conjecture" [Rogers 67] that, for each Z, $D_T^Z \cong D_T$. Yates [Ya 70] speculated, based on results of Martin, that the conjecture and also its weaker form asserting that D_T^Z is elementarily equivalent to D_T for each Z is independent of ZFC. Even the weaker form of the conjecture was refuted by Shore [Sh 82]: if $D_T \equiv D_T^Z$, then Z must be of arithmetical degree. Here already some of the ideas occur which will be exploited in the present paper.

Surprisingly, the analog of the homogeneity conjecture holds for D_m. Ershov [Er 75], with an addendum by Paliutin gave a characterization of D_m which is purely algebraic: D_m is the only distributive upper semilattice with 0 that has cardinality 2^ω, the countable predecessor property and a certain extension property for ideals of cardinality $< 2^\omega$. Relativizations of the proofs that these properties hold give exactly the same properties for D_m^Z, so $D_m^Z \cong D_m$.

There are several reasons to study relativizations of structures. One is that, as mentioned above, relativized versions of results are often already implicitly obtained. Moreover, in some cases the relativized structures arise naturally in some other way. For instance, if $Z = \emptyset^{(n-1)}$, then E^Z is the lattice of Σ_n^0–sets, and for any Z, if $z = \deg_T(Z)$, the relativization of the Δ_2^0–Turing degrees to Z is the interval $[z, z']$.

The way to prove $A^Z \not\cong A^W$ if Z, W are sufficiently different oracle sets is to show that, to some accuracy, the complexity of the oracle set X can be recovered from the isomorphism type of A^X. To make this precise, we need the notion of (uniform) coding of *extended standard models of arithmetic* (extended SMA). An extended SMA is a structure (M, U), where $M \cong I\!N$ and $U \subseteq M$. In general, a coding with parameters of a relational structure C of finite signature in a structure D is given by a scheme S of formulas $\varphi_S(x, \overline{p})$ and $\varphi_R(x_1, \ldots, x_n; \overline{p})$ for each n–ary relation symbol R in the language of C (including equality) such that, for an appropriate list $\overline{d}$ of parameters in D, $\varphi_=$ defines an equivalence relation on $\{x : D \models \varphi_S(x, \overline{d})\}$ and the structure defined on equivalence classes by the remaining formulas φ_R is isomorphic to C.

¿From now on, we focus on arithmetical structures A of finite signature. Such a structure is determined by a scheme of arithmetical formulas without parameters, which gives a representation of A in terms of natural numbers ("indices"). For instance, the scheme for E contains a Π_2^0 formula defining $\{\langle i, j\rangle : W_i \subseteq W_j\}$. Suppose the ground level Δ_1^0 of the arithmetical hierarchy is defined in terms of the Kleene T–predicate. Then we obtain relativizations of each arithmetical formula to an "oracle predicate" Z by replacing the computations the definition of T is based on by oracle computations.

In the terminology of Hodges [Ho 94], there is an interpretation Γ of structures in the language of A in the extended SMA $(I\!N, Z)$ and A^Z can be defined as $\Gamma(I\!N, Z)$. We call the least number r such that each arithmetical formula needed in defining A is a boolean combination of Σ_r–formulas the *arithmetical complexity* of A. This complexity is 2 for E, 3 for E^* and R_m, and 4 for R_T.

Note that, for each Z, there is a representation of the diagram of A^Z which is recursive in $Z^{(r)}$.

Now suppose that in the converse direction there is a coding scheme S for coding the extended SMA (M, X) in A^X with parameters. This coding condition, which is satisfied e.g. by R_m, R_T and E (see below), is a crude form of expressing that the complexity of the oracle X is reflected in A^X, the isomorphism type of A^X. We abbreviate the coding condition by CO ("coded oracle").

We will always assume that, if (M, X) is coded by a certain list of parameters $\overline{p}$, M is a model of a finitely axiomatized fragment PA^- of Peano arithmetic (say Robinson arithmetic R) which implies M is an end extension of $I\!N$. This can be expressed by a first order condition on $\overline{p}$.

Whenever an extended SMA (M, V) is coded in A^X, then by combining this coding with the coding of A^X in $(I\!N, X)$ we obtain that V is $\Sigma^0_d(X)$ for some natural number d. Thus, if A satisfies CO, then $A^Z \cong A^W$ implies that Z, W have the same arithmetical degree. If we can in addition recognize standardness of coded models M by a first–order condition on parameters (call this coding condition CO_{st}), then we obtain an elementary difference between A and A^Z, for $Z \notin \Sigma^0_d$: the first–order sentence expressing

$$\text{“}Whenever\ (M, V)\ isacodedextendedSMA, then\ V(asasubsetof\ M)\ is\ \Sigma^0_d\text{''}$$

$*holdsin$A$, butnotin$A^Z.

In Section 2, we use a still stronger coding condition $\text{CO}_{st}(k)$, which depends on $k \geq 1$, to refine these separations of isomorphism types and of theories. (A somewhat similar idea was used first in [Sh 81] for the special case $A = D_T(\leq 0')$.) In the central Section 3, we explain why such a coding condition is satisfied for R_m, R_T and E. For R_T and E, the full proofs are in [Ha, N ta] and [N, Sh, Sl ta], respectively, and we review them here in survey style.

In Section 4 we discuss "large" classes of oracles where relativizations of A are all elementarily equivalent. Finally, in Section 5, we show that the subset of $Th(A)$ of relativizing sentences is much more complex than $Th(A)$, assuming CO_{st}. This fact was obtained in collaboration with T. Slaman. It implies that there is no way to give an effective relativizability criterion C such that $Th(A) \cap C$ is the set of relativizable sentences, i.e. the sentences which hold in every relativization of A. In other words, it is not possible to distinguish, say, in R_m a relativizing sentence like "each incomplete degree has a minimal cover" from a sentence like $(*)$ above (assuming the sentences are true). (For the *particular* sentence $(*)$ it is easy to grasp why it does not relativize: to say that V as a subset of M is Σ^0_d keeps the same meaning in all relativizations.)

2. Separating relativizations We first list the hierarchy of coding conditions used. In saying A satisfies a certain coding condition, we view A as an interpretation in extended SMA's.

CO — In a uniform way, it is possible to code $(I\!N, X)$ in A^X.

CO_{st} — In the underlying scheme s to code structures M, $M \models PA^-$, in a relativization A^X, one can recognize standardness of M by a fixed first order condition on parameters.

CO(k) — $(k \geq 1)$ Suppose the arithmetical complexity of A is r, and let $c = r + k - 1$. The extended SMA $(I\!N, \overline{X^{(c)}})$ can be uniformly coded in A^X using a scheme of Σ_k-formulas with parameters.

$\text{CO}_{st}(k)$ — CO(k), and (as in the conditions CO_{st}) standardness can be recognized.

2.1 Separation Theorem Suppose A **satisfies CO(k).** *Then, if* $Z^{(c)} \not\equiv_T W^{(c)}$, $A^Z \not\cong A^W$, *where* $c = r + k - 1$ *and* r *is the arithmetical complexity of* A.

Proof: If M is a model of PA^- coded in A^X via the scheme s, then there is an $f \leq_T X^{(c)}$ such that $f(n)$ is an index of n^M in the canonical representation of A^X. For, the successor relation S of M, viewed as a relation on indices, is r.e. in the $(k-1)$–th jump of the atomic diagram of A^X (since the scheme is Σ_k), so it is r.e. in $X^{(c)}$. To compute f inductively, let $f(0)$ be an index of 0^M, and let $f(n+1)$ be an index j such that $Sf(n)j$ holds. Then, since M is an end extension of $I\!N$, f is total, and f is recursive in $X^{(c)}$.

Now we can obtain an upper bound on the complexity of U, for an extended SMA (M,U) coded in A^X: U is r.e. in $X^{(c)}$ via the enumeration procedure which enumerates n into U iff the Σ_k–formula defining U (with a fixed list of parameters in A^X) holds for $f(n)$.

Suppose $Z^{(c)} \not\leq_T W^{(c)}$. Then $Z^{(r+k)}$ is not r.e. in $W^{(r+k-1)}$. By hypothesis, the extended SMA $(M, Z^{(c)})$ can be coded in A^Z. But if $(M, \overline{Z^{(c)}})$ can be coded in A^W, then $Z^{(r+k)}$ is r.e. in $W^{(c)}$, contradiction. So $A^Z \not\cong A^W$.

Recall that a set $U \subseteq I\!N$ is *implicitly definable in arithmetic* (i.d.) if there is a first–order description ψ in the language of extended SMA's such that $(I\!N, X) \models \psi \Leftrightarrow X = U$. For instance, all recursive jumps $\emptyset^{(\alpha)}$, $\alpha < \omega_1^{CK}$, are i.d. Implicit definability of U only depends on the arithmetical degree of U, and can only hold if U is hyperarithmetical.

2.2 Theorem (Separation Theorem for Elementary Equivalence) Suppose $\mathbf{CO}_{st}(k)$ is satisfied for A. *Let $c = r+k-1$. If $Z^{(c)} \not\leq_T W^{(c)}$ and Z or W is implicitly definable in arithmetic, then $A^Z \not\equiv A^W$.*

Note that this includes the case that $W = \emptyset$ and $Z \notin Low_c$. Thus for sufficiently complex Z, the theory of the relativization to Z differs from the theory of the unrelativized structure.

Proof: We attempt to express the fact which led to $A^Z \not\cong A^W$ in the first–order language of A. First suppose that Z is implicitly definable. Then the statement

"Some (M,U) can be coded such that M is standard and there is $e \in M$, where $\{e\}^U$ satisfies the description of Z, such that $\overline{U} = (\{e\}^U)^{(c)}$"

is expressible in that language, holds in A^Z but fails in A^W.

If W is implicitly definable, we distinguish two cases. If $Z^{(c)} \not\geq_T W^{(c)}$, then $A^Z \not\equiv A^W$ by the argument above. Else $Z^{(c)} >_T W^{(c)}$, and there is an index e such that $\{e\}(\overline{Z^{(c)}}) = W$. So the first–order sentence expressing

"there is a coded extended SMA(M,U) and an $e \in M$ such that $\{e\}^U$ satisfies the description of W and $U \notin \Sigma^0_{c+1}(\{e\}^U)$"

is true in A^Z via a coding of $(M, \overline{Z^{(c)}})$, but not in A^W.

3. The structures R_m, E and R_T We sketch proofs that R_m and E satisfy the condition $\mathrm{CO}_{st}(k)$ used in the Separation Theorem for elementary equivalence. In the cases R_m and R_T the coding condition holds with $k=1$. The full proofs for R_T and E are implicit in the results in [N, Sh, Sl ta] and [Ha, N ta], respectively. In all proofs, it is sufficient to consider the unrelativized structure and note the relativizability of the proof techniques used.

3.1 R.e. many–one degrees.

In five steps, we build up a coding scheme of Σ_1-formulas for coding an extended SMA $(M, \overline{X^{(3)}})$ in R_m^X with parameters. This proves the condition CO(1), since $r = 3$ for R_m.

We use two auxiliary structures: first a bipartite graph and then a distributive lattice. This makes it necessary to apply a transitive version of coding with Σ_1–formula: as in [N ta1], a relational structure A is Σ_1–e.d.(p) in a structure B if there is a coding scheme of Σ_1 formulas (with parameters) for defining the universe of A, the relations of A *and their complements.*

Step 1. $I\!N$ is Σ_1–e.d. in a recursive bi–partite graph $G = (Le, Ri, E)$, using the coding given in the proof of Theorem 4.2 in [N ta1]. The class of vertices representing numbers is a recursive Σ_1–definable subset of the left domain Le of G.

Step 2. G is Σ_1–e.d.p. in a recursive distributive lattice L_G, viewed as a p.o. This step is carried out in [N ta1] for finite bipartive graphs, in order to show that the Π_3–theory of the class of finite distributive lattices (as p.o.) is hereditarily undecidable. An obvious modification of the proof yields L_G. For instance, to define a sequence of infinitely many independent elements A_i (representing the left domain of G) in an appropriate recursive distributive lattice L by a quantifier free formulas with one parameter, consider copies B_1, B_2 of the boolean algebra of finite or cofinite subsets of CO, put B_2 on top of B_1 where $P :=$ greatest element of

B_1 = least element of B_2. For each i, insert the new element A_i between the i–th coatom of B_1 and the i–th atom of B_2. In this way, obtain L. Now

$$\{A_i : i \in \omega\} = \{X \in L : X, P \textit{ incomparable}\},$$

and no A_i is below a finite supremum $j \in F \rightarrow \bigvee A_i$ for $i \notin F$ (i.e., the elements A_i are independent).

The coding of G in a lattice L_G is obtained by an extension of this: take another copy of L, such that elements $\{\widetilde{A}_j : j \in \omega\}$ represent the right domain, and add further parameters C_E, $C_{\overline{E}}$ such that $Eij \Leftrightarrow C_E \not\leq A_i \vee \widetilde{A}_j$, and similarly for $\overline{E}$ and $C_{\overline{E}}$. In [N ta1] it is described how the further parameters can be introduced without interfering with the Σ_1–definition of the set of elements representing the left and right domain.

Step 3. By a theorem of Lachlan [La 72], $L_G \cong [0, a]$ for some $a \in R_m$, by an effective map on indices. This gives a scheme S_M with parameters $\overline{p}$ (including the upper bound a) to code SMA's M. Note that, by effectivity, for the particular a above, there is a uniformly r.e. sequence (c_i) of m–degrees such that c_i represents i^M: the sequence (c_i) is a subsequence of the degrees representing the elements A_i of L_G. Thus, also (c_i) is an independent sequence.

Step 4. Given a Π^0_3–complete (or in fact, any Π^0_3–) set S, by the Exact Degree Theorem for structures of arithmetical complexity 3 in [N ta2], there is a $b \in R_m$ such that $i \in S \Leftrightarrow c_i \not\leq b$. Including b as a parameter, we obtain the desired scheme S in the unrelativized case.

Step 5. Since all the proof techniques used are relativizable, via the same scheme, CO(1) is satisfied: for each X, there is a list of parameters in R^X_m coding $(\mathbb{N}, \overline{X^{(3)}})$ via S.

To recognize standardness, we argue as in [N 94], where an interpretation of true arithmetic in $Th(R_m)$ is given. For any model M coded in R^X_m by the scheme s_M, if M satisfies PA^-, then $\{i : \deg(W^X_i)$ *is a standard number of* $M\}$ is $\Sigma^0_k(X)$ for some fixed k and bounded from above by a. By the relativized form of the Definability Lemma in [N 94] we can quantify over such sets in the first order language of R_m and therefore, we can express that M is standard.

Applying 2.2, we now obtain the following result:

3.1 Theorem If $Z^{(3)} \not\leq_T W^{(3)}$ and Z or W is implicitly definable in arithmetic, then $R^Z_m \not\equiv R^W_m$.

3.2 The lattice of r.e. sets.

We review the necessary facts about E to prove that E satisfies $\mathrm{CO}_{st}(k)$ for some k. As in [Ha ta] and [Ha, N ta], for any r.e. set E, $B(E)$ is the boolean algebra of r.e. subsets X of E such that $E - X$ is r.e. and $R(E)$ is the ideal of recursive subsets of E. *The variables R, S range over recursive sets.* If $X \in B(E)$, we write XE. An ideal I of $B(E)$ is *k–acceptable* if $R(E) \subseteq I$ and I has a Σ^0_k index set. I is *acceptable* if it is k–acceptable for some k.

A class C of subsets of a structure S is *uniformly definable* if, for some formula $\varphi(x; \overline{p})$, C is the class of sets defined by this formula as $\overline{p}$ varies over tuples of parameters in S. (Sometimes in the literature it is only required that C be included in such a class, e.g., in [N 94].)

Ideal Definability Lemma [Ha ta] **For each nonrecursive r.e. set E and each $n \geq 1$, the class of $2n+1$–acceptable ideals of $B(E)$ is uniformly definable by a formula φ_{2n+1}.**

The formula used for the 3–acceptable ideals is

$$\varphi_3(X; E, C) \equiv XE \wedge (\exists R \subseteq E)[X \subseteq C \cup R]$$

which clearly can only define 3–acceptable ideals. The formula φ_{2n+3} for $2n+3$–acceptable ideals has an $\exists\forall$ quantifier prefix in front of an instance of φ_{2n+1} with different parameters and therefore only defines Σ_{2n+3}–ideals. More precisely,

$$\varphi_{2n+3}(X; E, \overline{C}, C_n) \equiv XE \;\wedge\; (\exists R \subseteq E)(\forall S \subseteq E - R)[\varphi_{2n+1}(X \cap S \cap C_n; \overline{C})].$$

The general framework to use induction on k in this way for obtaining uniform definability of objects with Σ_k–index set is adapted from [N 94]. Note that φ_{2n+1} is a Σ_{2n-1}–formula in the language of E, as a lattice with least and greatest element.

In [Ha, N ta], we use the Ideal Definability Lemma to establish the hypothesis $\mathrm{CO}_{st}(k)$ (some k) of the Separation Theorem for elementary equivalence. As for R_m, here we describe the coding process in several steps.

Step 1. If E is an r.e. nonrecursive set, let $(P_k)_{k\in\omega}$ be any u.r.e. partition of E into nonrecursive sets. Such a partition can be obtained by the method of the Friedberg Splitting Theorem (see [So 87]). Modulo some ideal I, the sets P_k will be the elements of the SMA to be coded.

Step 2. For each r.e. nonrecursive set D, one can obtain uniformly in an index of D a maximal ideal $I(D)$ of $B(D)$ which contains $R(D)$ and has Δ^0_4 index set. Apply this process to each P_k, and let $I = \{XA : (\forall k)[X \cap P_k \in I(P_k)]\}$. Then $(P_k/I)_{k\in\omega}$ is a uniformly r.e. listing of the atoms in $B(E)/I$ without repetitions.

Step 3. To be able to code ternary relations corresponding to the arithmetical operations $+$, $\times$ on the atoms of $B(E)/I$ for an appropriate I, we require that E is hh–simple, where the lattice $L^*(E)$ is isomorphic to the boolean algebra of finite and cofinite subsets of ω. Then, with the right choice of the ideals $I(P_k)$, the atoms of $L^*(E)$ can be used to represent 3–tuples of atoms. (This, however, increases the arithmetical complexity of the index set of the ideals $I(P_k)$ and hence of I.) Then any recursive ternary relation on the atoms P_k/I can be defined in terms of three further acceptable ideals.

Thus we obtain a scheme with parameters to code a SMA M in E.

Step 4. To be able to uniformly define subsets S of the standard part of a model of PA^- M coded by this scheme which have an arithmetical index set, we first proceed as in the proof of the Separation Theorem: for some fixed c depending only on the coding formulas, there is an $f \leq \emptyset^{(c)}$, such that

$$i^M = W_{f(i)}/I$$

for each $i \in \omega$.

Moreover, since atoms of a boolean algebra are independent, S can be recovered from the ideal of $B(E)$ it generates: let P, Q range over $\{XE : X/I \text{ atom in } B(E)/I\}$. If P/I is a standard number of M, then

$$P/I \in S \Leftrightarrow P \in I_S,$$

where I_S is the ideal generated by I and those Q such that $Q/I \in S$.

Clearly I_S is acceptable if S has an arithmetical index set (in the sense that $\{Q : Q/I \in S\}$ has one). Then, since the standard part of M is such a set, we can quantify over the possible subsets of M which can be the standard part and thus express that M is standard.

The same first–order condition for recognizing standardness works in every relativization E^Z, since the proof of the Ideal Definability Lemma relativizes.

Step 5. To define extended SMA's of the type required to satisfy $CO_{st}(k)$, note that, using the function $f \leq \emptyset^{(c)}$ obtained in Step 4, if M is standard then, for some sufficiently large odd $d > c$,

$$S \subseteq M \ is \ \Sigma^0_d \ asasubsetof \ M \Leftrightarrow S \ has \ \Sigma^0_d \ indexset \Leftrightarrow I_S \ has \ \Sigma^0_d \ indexset.$$

Then, for d large enough, because of the remarks following the Ideal Definability Lemma, an extended SMA (M, S) can be coded in E using a scheme of Σ_{d-2}–formulas (in the language of lattices with 0, 1), for any Σ^0_d–set S.

Moreover, by relativizability of the proof techniques, the same scheme can be used to code (M, S) in E^X, if S is $\Sigma^0_d(X)$. We can conclude that $\mathrm{CO}_{st}(d-2)$ is satisfied: recall that $r = 2$ for E, so with the value $k = d - 2$ there is a scheme of Σ_k–formulas such that each $\Sigma^0_{r+k}(X)$ set and hence $\overline{X^{(c)}}$ is coded, where $c = d - 1$. We have obtained the following.

3.2 Theorem [Ha, N ta] For some c, if $Z^{(c)} \not\leq_T W^{(c)}$ and Z or W is implicitly definable, then *$E^Z \not\equiv E^W$. In particular, if $Z \notin Low_c$ then $E^Z \not\equiv E$.*

3.3 R.e. T–degrees, and r.e. m–degrees revisited.

The coding methods developed in [N, Sh, Sl ta] suffice to satisfy $\mathrm{CO}_{st}(1)$ for R_T, viewed as an u.s.l. Since $r = 4$ for $(R_T, \vee)$, this gives the separation of $Th(R_T^Z)$ and $Th(R_T^W)$ for $Z^{(4)} \not\leq_T W^{(4)}$ if Z or W is implicitly definable.

We now describe a way to give, for both R_m and R_T, a first–order condition $R(\overline{p})$ on parameters $\overline{p}$ coding an extended SMA (M, U) which, in each relativization R_T^X, holds only if $U^{(3)} \equiv_T X^{(3)}$. Since some parameters will satisfy the condition, this can be interpreted by saying that we can, in a uniform first order way, recover the T–degree of $X^{(3)}$ from R_T^X and R_m^X. Then, if Z is implicitly definable, there is a formula φ which holds in R_m^X (R_T^X) iff $Z^{(3)} \equiv_T X^{(3)}$.

We use that, with a suitable scheme s_M, R_m and R_T satisfy the coding condition

"for each M_1, M_2, the isomorphism between the standard parts of M_1, M_2 is uniformly definable",

i.e., there is a formula $\varphi(x, y, \overline{q})$ which, uniformly with parameters defines all these isomorphisms. This coding condition makes it possible to recognize standardness, and to code a SMA in the degree structure without parameters. For R_m, we can use the scheme s_M introduced in 3.1. So it is part of a scheme for defining extended SMA's such that in R_m^X , an extended SMA $(M, \overline{X^{(3)}})$ can be coded with appropriate parameters. Note that, if an extended SMA (M, Z) is coded, we can express that $Z = \overline{U^{(3)}}$ for some U, since any such U must satisfy $U = \{e\}^Z$ for some e, so U is represented within (M, Z). Now consider the property of a parameter list

"$\overline{p}$ codes an extended SMA $(M, \overline{U^{(3)}})$ such that, for each coded extended SMA $(N, \overline{V^3})$, $V^{(3)} \leq_T U^{(3)}$.

By the uniform definability of the isomorphism $h : M \leftrightarrow N$ and the remark above, this property is equivalent to a first–order property $R(\overline{p})$, since we can compare the T–degrees of $V^{(3)}$ and $h(U^{(3)})$ inside N.

It was proved above that an extended SMA $(M, \overline{X^{(3)}})$ can be coded in R_m^X. Now, whenever $(N, \overline{V^3})$ is coded, then $V^3 \leq_T X^3$, because $\overline{V^{(3)}}$ is r.e. in $X^{(3)}$ by the argument used in the proof of the Separation Theorem. So the property $R(\overline{p})$ holds in R_m^X for any list of parameters coding an extended SMA $(M, \overline{X^{(3)}})$.

In R_T one can argue similarly to decode the degree of $X^{(4)}$ from R_T^X. For decoding the degree of $X^{(3)}$, we use the fact, proved in [N, Sh, Sl ta] that

for each r.e. nonrecursive A, if $a = \deg_T(A)$, the extended SMA $(M, \overline{A^{(3)}})$ is coded in the u.s.l. $[0, a]$ using a fixed scheme of Σ_1–formulas.

Now consider the first–order property $R(\overline{p})$ expressing

"$\overline{p}$ codes a model $(M, \overline{U^{(3)}})$, M standard, such that $U^{(3)}$ is $\leq_T$–maximal with respect to the property that in each u.s.l. $[0, a]$, $a \neq 0$, a structure $(M, \overline{V^{(3)}})$ is coded such that $V^{(3)} \equiv_T U^{(3)}$"

If this property holds in R_T^X for $\overline{p}$, then $\overline{p}$ codes $(M, \overline{U^{(3)}})$, $U^{(3)} \equiv_T X^{(3)}$, since there is a nonzero $a \in R_T^X$ such that $a' = X'$. The following theorem is now almost immediate.

Theorem Suppose $A = R_m$ or $A = R_T$.

(i) If $Z^{(3)} \not\equiv_T W^{(3)}$, then $A^Z \not\equiv A^W$.

(ii) If Z is implicitly definable in arithmetic, then there is a sentence φ such that, for each W,

$$A^W \models \varphi \Leftrightarrow W^{(3)} \equiv_T Z^{(3)}.$$

In particular, there is φ which holds precisely in the relativizations to Low_3 oracles.

4. Elementarily equivalent relativizations We consider several results of the form

$$Z, W \in C \Rightarrow A^Z \equiv A^W,$$

$*whereCisaninsomesensealargeclassofsubsetsof\omega$ (reals). It is reasonable to assume that the isomorphism type of A^Z depends only on the T–degree of Z. Then, for any φ in the language of A

$$\{Z : A^Z \models \varphi\}$$

is an arithmetical class of reals closed under $\equiv_T$.

By arithmetic determinacy, $(*)$ holds for a class $C = \{Z : Z \geq_T F\}$, for some real F which is an upper bound for sets encoding winning strategies. This was observed in [Sh 81] for $A = D_T(\leq 0')$.

In the following, we derive $(*)$ for the classes of ω–generic and ω–random sets Z. Recall that Z is ω–generic iff Z is in every comeager arithmetical class of reals, and Z is ω–random iff Z is in every arithmetical class of measure 1. Both classes can be defined in terms of forcing notions — see [Od ta] for the first and [Kau 91] for the second.

4.1 Proposition If Z, W are ω–generic, then $A^Z \equiv A^W$. *Thus $(*)$ holds for a comeager class.*

Proof: Since $A^Z \models \varphi$ does not depend on finite variations of Z, the following equivalences hold:

$$A^Z \models \varphi\, forsome\ \omega--generic\ Z \Leftrightarrow (\exists\theta) \subseteq Z\, [\theta``A^X \models \varphi''],\ i.e., forall\ \omega--generic\ X \supset \theta, A^X \models \varphi \Leftrightarrow A^X \models \varphi\, forall\ \omega-$$

Note that the class of ω–generic reals G is radically different from the class of implicitly definable reals Z considered in Sections 2 and 3: If Z is arithmetical in G, then Z is an arithmetical set.

4.2 Proposition If Z, W are ω–random, then $A^Z \equiv A^W$. *Thus $(*)$ holds for a class of measure 1.*

Proof: It follows from Kolmogorov's 0–1 law that each measurable degree–invariant (or even $=^*$–invariant) class of reals has measure 0 or 1. Thus, for Z, W ω–random

$$A^Z \models \varphi \Leftrightarrow \{X : A^X \models \varphi\}\ hasmeasure\ 1 \Leftrightarrow A^W \models \varphi.$$

For the rest of this section we assume that A satisfies the coding conditions CO_{st}.

Let G be some ω–generic and R some ω–random set.

4.3 Proposition $Th(A^G) \equiv_T Th(A^R) \equiv_T \emptyset^{(\omega)}$.

Proof: We can assume $R, G \leq_T \phi^{(\omega)}$. By the hypothesis CO_{st}, true arithmetic can be interpreted in both theories. Conversely, to obtain $\emptyset^{(\omega)}$ as an upper bound, first note that, for any X, $Th(A^X) \leq_T X^{(\omega)}$. But for $X = G$ and $X = R$, $X^{(n)} = X \oplus \emptyset^{(n)}$ where the T–reductions are obtained uniformly in n, by results in [Kur 81] and [Kau 91], respectively, so $X^{(\omega)} \equiv_T X \oplus \emptyset^{(\omega)} \equiv_T \emptyset^{(\omega)}$.

We now show that, assuming CO_{st} the three theories $Th(A)$, $Th(A^G)$ and $Th(A^R)$ are all different. Thus the theory of the unrelativized structure behaves typically neither in the sense of category nor in the sense of measure.

4.4 Theorem The theories $Th(A)$, $Th(A^G)$ and $Th(A^R)$ are pairwise distinct.

Proof: We first prove that the structures are nonisomorphic. If $A \cong A^X$ for $X = G$ or $X = R$, then an extended SMA (M, X) is coded in A. Hence X is Σ^0_c for some c, which is impossible.

If $A^R \cong A^G$, then an extended SMA (M, R) can be coded in A^G, so R is in $\Sigma^0_c(G)$ for some sufficiently large c, and $R \leq_T \emptyset^{(c+1)} \oplus G$. This is impossible by the following

Fact: If R is $n+1$–random and G is $n+1$–generic, then $R \not\leq_T \emptyset^{(n)} + G$. (See [Kau 91] for definitions of k–random and k–generic. Here it is enough to know that these are arithmetical classes of reals whose intersection is the class of ω–random respectively ω–generic reals.) The proof of this fact is obtained in a straightforward way by adapting Kurtz's proof (Theorem 4.2 in [Kur 81]) that the downward closure of the class of 1–generic degrees has measure 0.

We now obtain the stronger facts that the structures are not elementarily equivalent: for $X = R$, G, A^X, but not A, satisfies

"an extended SMA (M, U) can be coded such that U is not Σ^0_c".

Moreover, A^R, but not A^G, satisfies

"an extended SMA (M, U) can be coded such that U is $c+2$–random."

For the second, use the above fact for $n = c+1$, together with Kautz's result [Kau 91] that $R^{(k)} \equiv_T R + \emptyset^{(k)}$ for $k+1$–random R.

5. The set of relativizable sentences Assuming the coding conditions CO_{st} on A as in the preceding section, we investigate the theory

$$T = \bigcap_{X \subseteq \omega} Th(A^X),$$

i.e., the class of sentences which hold in all relativizations of A. Note that it suffices to take the intersection over all hyperarithmetical X. Both of the facts we prove show that T is complicated in some sense.

5.1 Proposition If S is a consistent theory containing T, then $\emptyset^{(\omega)} \leq_m S$.

Proof: Given $\varphi \in L(+, \times)$, let $F(\varphi)$ be the sentence expressing "φ holds in some coded SMA". Then $\varphi \in Th(I\!N)$ implies $\widetilde{\varphi} \in T$, so $F(\varphi) \in S$, and $\neg\varphi \in Th(I\!N)$ implies $F(\neg\varphi) \in S$, so $F(\varphi) \notin S$ since S is consistent. So $\emptyset^{(\omega)} \leq_m S$.

5.2 Proposition T is Π^1_1–complete.

Proof: Since A^X is given as $\Gamma(I\!N, X)$ for an interpretation Γ, there is a fixed recursive function f such that $Th(A^X) \leq_m X^{(\omega)}$ via f. Then

$$T = \{\varphi : (\forall X)(\forall Y)[Y = X^{(\omega)} \Rightarrow f(\varphi) \in Y]\}.$$

The matrix of this expression is arithmetical so T is Π^1_1.

To show completeness, we give a reduction of the Π^1_1–complete set

$$\{\psi \in L(+, \times, U) : \ noextendedSMA\ satisfies\ \psi\}.$$

Let $g(\psi)$ be the negation nof the sentence in the language of A expressing

"for some extended SMA (M, U), $M \models \psi(U)$".

Then, if no extended SMA satisfies ψ, $g(\psi) \in T$, and if some extended SMA does, then $g(\psi)$ fails in any A^U such that $(I\!N, U) \models \psi$ holds, so $g(\psi) \notin T$.

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