A lower cone in the wtt degrees of non-integral effective dimension

André Nies

Department of Computer Science
University of Auckland, New Zealand

Jan Reimann

Institut für Informatik
Ruprecht-Karls-Universität Heidelberg, Germany

For any rational number r, we show that there exists a set A (weak truth-table reducible to the halting problem) such that any set B weak truth-table reducible to it has effective Hausdorff dimension at most r, where A itself has dimension at least r. This implies, for any rational r, the existence of a wtt-lower cone of effective dimension r.

1. Introduction

Since the introduction of effective dimension concepts by Lutz [8, 9], considerable effort has been put into studying the effective or resource-bounded dimension of objects occurring in computability or complexity theory. However, up to now there are basically only three types of examples known for individual sets of non-integral effective dimension: The first consists of sets obtained by 'diluting' a Martin-Löf random set with zeroes (or any other computable set). The second example comprises all sets which are random with respect to a Bernoulli distribution on Cantor space. Here Lutz transferred a classic result by Eggleston [1] to show that if μ is a (generalized) Bernoulli measure, then the effective dimension of a Martin-Löf μ-random set coincides with the entropy $\mathrm{H}(\mu)$ of the measure μ. Finally, the third example is a parameterized version of Chaitin's Ω introduced by Tadaki [23].

An obvious question is whether there exist examples of non-integral effective dimension among classes of central interest to computability theory, such as cones or degrees. It is interesting to note that all the examples mentioned above actually produce sets which are Turing equivalent to a Martin-Löf random set. Therefore, one cannot use them to obtain Turing cones of non-integral dimension.

However, when restricted to many-one reducibility, Reimann and Terwijn [15] showed that the lower cone of a Bernoulli random set cannot contain a set of higher dimension than the random set it reduces to, thereby obtaining many-one lower cones of non-integral effective dimension. But the proof does not transfer to weaker reducibilities. Using a different approach, Stephan [22] was able to construct an oracle relative to which there exists a wtt-lower cone of positive effective dimension at most $1/2$.

In this paper we construct, for an arbitrary rational number r, a wtt-lower cone of effective Hausdorff dimension r. This result was independently announced by Hirschfeldt and Miller [3]. The case of Turing reducibility seems much more difficult and remains a major open problem in the field (see also [12]).

Notation. Our notation is fairly standard. 2^ω denotes Cantor space, the set of all infinite binary sequences. We identify elements of 2^ω with subsets of the natural numbers $\mathbb{N}$ by means of the characteristic function, thus elements of 2^ω are generally called *sets*, whereas subsets of 2^ω are called *classes*. Sets will be denoted by upper case letters like A, B, C, or X, Y, Z, classes by calligraphic upper case letters $\mathcal{A}, \mathcal{B}, \dots$.

Strings are finite initial segments of sets will be denoted by lower case latin or Greek letters such as u, v, w, x, y, z or σ, τ. $2^{<\omega}$ will denote the set of all strings. The *initial segment of length* n, $A \restriction n$, of a set A is the string of length n corresponding to the first n bits of A.

Given two strings v, w, v is called a *prefix* of w, written $v \preceq w$, if there exists a string x such that $vx = w$, where vx is the concatenation of v and x. If w is strictly longer than v, we write $v \prec w$. This extends in a natural way to hold between strings and sets. A set of strings is called *prefix free* if no element has a prefix (other than itself) in the set.

Initial segments induce a standard topology on 2^ω. The basis of the topology is formed by the *basic open cylinders* (or just *cylinders*, for short). Given a string $w = w_0 \dots w_{n-1}$ of length n, these are defined as

$$[w] \;=\; \{A \in 2^\omega : \; A \restriction n = w\}.$$

Imposing this topology turns 2^ω into a totally disconnected Polish space. A class is clopen in 2^ω if and only it is the union of finitely many cylinders.

Finally, λ denotes Lebesgue measure on 2^ω, generated by setting $\lambda[\sigma] = 2^{-|\sigma|}$ for every string σ. For each measurable $\mathcal{C} \subseteq 2^\omega$, recall that the conditional probability is

$$\lambda(\mathcal{C} \mid \sigma) = \lambda(\mathcal{C} \cap [\sigma])2^{|\sigma|}.$$

For all unexplained notions from computability theory we refer to any standard textbook such as [14] or [19], for details on Kolmogorov complexity, the reader may consult [7]; [13]will provide background on the use of measure theory, especially martingales, in the theory of algorithmic randomness.

In the proof of our main result we will use so-called *Kraft-Chaitin sets.* A Kraft-Chaitin set L is a c.e. set of pairs $\langle l, x\rangle$ (called *requests*), where l is a natural number and x is a string, and $\sum_L 2^{-l} \leq 1$. It is a fundamental result in algorithmic randomness that if L is a Kraft-Chaitin set, then $K(x) \leq^+ l$ if $\langle l, x\rangle \in L$.

2. Effective Dimension

In this section we briefly introduce the concept of effective Hausdorff dimension. As we deal exclusively with Hausdorff dimension, we shall in the following often suppress "Hausdorff" and speak simply of *effective dimension.* For a more detailed account of effective dimension notions we refer to [15].

Hausdorff dimension is based on *Hausdorff measures*, which can be effectivized in the same way Martin-Löf tests effectivize Lebesgue measure on Cantor space.

Definition 2.1: Let $0 \leq s \leq 1$ be a rational number. A class $\mathcal{X} \subseteq 2^\omega$ has *effective s-dimensional Hausdorff measure* 0 (or simply is *effectively $\mathcal{H}^s$-null*) if there is a uniformly computably enumerable sequence $\{C_n\}_{n\in\mathbb{N}}$ of sets of strings such that for every $n \in \mathbb{N}$,

$$\mathcal{X} \subseteq \bigcup_{\sigma \in C_n} [\sigma] \quad \text{and} \quad \sum_{w \in C_n} 2^{-s|w|} \leq 2^{-n}.$$

It is obvious that if $\mathcal{X}$ is effectively $\mathcal{H}^s$-null for some rational $s \geq 0$, then it is also effectively $\mathcal{H}^t$-null for any rational $t > s$. This justifies the following definition.

Definition 2.2: The *effective Hausdorff dimension* $\dim^1_{\mathrm{H}} \mathcal{X}$ of a class $\mathcal{X} \subseteq 2^\omega$ is defined as

$$\dim^1_{\mathrm{H}} \mathcal{X} = \inf\{s \in \mathbb{Q}^+ : \mathcal{X} \text{ is effectively } \mathcal{H}^s\text{-null}\}$$

The classical (i.e. non-effective) notion of Hausdorff dimension can be interpreted as the right "scaling factor" of $\mathcal{X}$ with respect to Lebesgue measure. The effective theory, however, allows for an interpretation in terms of *algorithmic randomness*. There exist singleton classes, i.e. sets, of positive dimension (whereas in the classical setting every countable class is of Hausdorff dimension zero). In fact, effective dimension has a strong stability property [9]: For any class $\mathcal{X}$ it holds that

$$\dim^1_{\mathrm{H}} \mathcal{X} = \sup\{\dim^1_{\mathrm{H}}\{A\} : A \in \mathcal{X}\}. \tag{2.1}$$

That is, the effective dimension of a class is completely determined by the dimension of its members (viewed as singleton classes). We simplify notation by writing $\dim^1_{\mathrm{H}} A$ in place of $\dim^1_{\mathrm{H}}\{A\}$. The effective dimension of a set can be regarded as an indicator of its *degree of randomness*. This is reflected in the following theorem.

Theorem 2.3: *For any set $A \in 2^\omega$ it holds that*

$$\dim^1_{\mathrm{H}} A = \liminf_{n\to\infty} \frac{K(A \restriction n)}{n}.$$

In other words, the effective dimension of an individual set equals its lower asymptotic entropy. In the following, we will use $\underline{K}(A)$ to denote $\liminf K(A \restriction n)/n$. Theorem 2.3 was first explicitly proved in [11], but much of it is already present in earlier works on Kolmogorov complexity and Hausdorff dimension, such as [17] or [20]. The result can be derived quite easily from the existence of a universal semimeasure (discrete or continuous) by using the *coding theorem*, as observed by Reimann [15] and Staiger [21].

Examples for effective dimension. As mentioned in the introduction, there are mainly three types of examples of sets of non-integral effective dimension.

(1) If $0 < r < 1$ is rational, let $Z_r = \{\lfloor n/r \rfloor : n \in \mathbb{N}\}$. Given a Martin-Löf random set X, define X_r by

$$X_r(m) = \begin{cases} X(n) & \text{if } m = \lfloor n/r \rfloor, \\ 0 & \text{otherwise.} \end{cases}$$

Then, using Theorem 2.3, it is easy to see that

$$\dim^1_{\mathrm{H}} X_r = r.$$

This technique can be refined to obtain sets of effective dimension s, where $0 \le s \le 1$ is any Δ^0_2-computable real number (see e.g. [10]).

(2) Given a Bernoulli measure μ_p with bias $p \in \mathbb{Q} \cap [0,1]$, the effective dimension of any set that is Martin-Löf random with respect to μ_p equals the entropy of the measure $\mathrm{H}(\mu_p) = -[p \log p + (1-p)\log(1-p)]$ [8]. This is an effective version of a classical theorem due to Eggleston [1].

(3) Let U be a universal, prefix-free machine. Given a computable real number $0 < s \leq 1$, the binary expansion of the real number

$$\Omega^{(s)} = \sum_{\sigma \in \mathrm{dom}(U)} 2^{-\frac{|\sigma|}{s}}$$

has effective dimension s. This was shown by Tadaki [23]. Note that $\Omega^{(1)}$ is just Chaitin's Ω.

2.1. *Effective Dimension of cones and degrees*

Fundamental results by Gacs [2] and Kučera [6] showed that every set is Turing reducible to a Martin-Löf random one. Since a Martin-Löf random set has effective dimension 1, it follows from (2.1) that every Turing upper cone is of effective dimension 1. Even more, Reimann [15] was able to show that every many-one upper cone has classical Hausdorff dimension 1 (and hence effective dimension 1, too). This contrasts a classical result by Sacks [18] which shows that the Turing upper cone of a set has Lebesgue measure zero unless the set is recursive.

As regards lower cones and degrees, the situation is different. First, using coding at very sparse locations along with symmetry of algorithmic information, one can show that effective dimension is closed upwards in the weak truth-table degrees, that is, for any sets $A \leq_{wtt} B$, the weak truth table-degree of B contains a set C of dimension $\dim^1_{\mathrm{H}} A$. It is sufficient to choose a computable set R of density $\lim_n |R \cap \{0, \ldots, n-1\}|/n = 1$, and let C equal A on R and B on the complement of R. It follows that the dimension of the weak truth-table degree and the weak truth-table lower cone of a set coincide. The same holds for Turing reducibility.

All three types of examples mentioned above compute a Martin-Löf random set, albeit for different reasons.

It is obvious that any diluted set X_r computes a Martin-Löf random sequence. Furthermore, Levin [25] and independently Kautz [4] showed that any sequence which is random with respect to a computable probability measure on 2^ω (which includes the Bernoulli measures μ_p with rational bias) computes a Martin-Löf random set. Finally, for every rational s, the set Ω_s is a left-c.e. real number. Furthermore, it is not hard to see that it computes a fixed-point free (fpf) function. Hence it follows from the Arslanov completeness criterion that Ω_s is Turing complete and therefore computes a Martin-Löf random set as well.

Regarding stronger reducibilities, Reimann and Terwijn [15] showed that a many-one reduction cannot increase the entropy of a set random with respect to a Bernoulli measure μ_p, p rational. It follows that the many-one lower cone of such a set has effective dimension $\mathrm{H}(\mu_p)$.

However, this result does not extend to weaker reducibilities such as truth-table reducibility, since for such measures the Levin-Kautz result holds for a total Turing reduction.

Recently, using a different approach, Stephan [22] was able to construct an oracle relative to which there exists a wtt-lower cone of positive effective dimension at most 1/2. In the next section we improve this by showing that, for an arbitrary rational number r, there exists an (unrelativized) weak truth-table lower cone of effective Hausdorff dimension at most r.

3. The Main Result

Theorem 3.1: *For each rational α, $0 \leq \alpha \leq 1$, there is a set $A \leq_{\mathrm{wtt}} \emptyset'$ such that $\underline{K}(A) = \alpha$ and $\underline{K}(Z) \leq \alpha$ for each $Z \leq_{\mathrm{wtt}} A$.*

Proof. Let $\mathcal{P}$ be the Π^0_1-class given by

$$\mathcal{P} = \{Z : \forall n \geq n_0\, K(Z \upharpoonright n) \geq \lfloor \alpha n \rfloor\},$$

where n_0 is chosen so that $\lambda\mathcal{P} \geq 1/2$. Recall that each Π^0_1-class comes with an effective clopen approximation, so we assume that there exists an effective sequence (P_s) of finite sets of strings such that $\mathcal{P} = \bigcap_s P_s$. To facilitate readability we mostly identify finite sets of strings with the clopen class they induce. (If we want to explicitly denote the clopen class induced by some finite set S of strings, we write $[S]^{\preceq}$.) As usual, it is useful to imagine $\mathcal{P}$ and the P_s as sets of infinite paths through trees.

Lemma 3.2: *Let $\mathcal{C}$ be a clopen class such that $\mathcal{C} \subseteq P_s$ and $\mathcal{C} \cap P_t = \emptyset$ for stages $s < t$. Then $\Omega_t - \Omega_s \geq (\lambda\mathcal{C})^\alpha$.*

Proof. Each minimal string in $\mathcal{C}$ has a substring x that receives a description of length at most $\alpha|x|$ between s and t. Thus there is a prefix free set $\{x_1, \ldots, x_m\}$ such that all $[x_i] \cap P_s \neq \emptyset$, $\mathcal{C} \subseteq \bigcup_i [x_i]$, and $K_t(x_i) \leq \alpha|x_i|$. Then, since the function $y \mapsto y^\alpha$ is concave,

$$\Omega_t - \Omega_s \geq \sum_i 2^{-\alpha|x_i|} \geq (\sum_i 2^{-|x_i|})^\alpha \geq (\lambda\mathcal{C})^\alpha.$$

This proves the lemma.

Now we build A on P, thus $\underline{K}(A) \geq \alpha$. To ensure $\underline{K}(Z) \leq \alpha$ for each $Z \leq_{\mathrm{wtt}} A$, we meet the requirements R_j, for each $j = \langle e, b \rangle > 0$.

$$R_j : Z = \Psi_e(A) \;\Rightarrow\; \exists k \geq j\, K(Z \upharpoonright k) \leq^+ \beta k,$$

where $\beta = \alpha + 2^{-b} < 1$, and $(\Psi_e)_{e \in \mathbb{N}}$ is a uniform listing of wtt reduction procedures, with partial computable use bound g_e, such that

$$\forall k\, (m = g_e(k) \downarrow \Rightarrow\; m \geq \beta k/2).$$

Thus we only consider reductions which do not turn a short oracle string into a long output string. This is sufficient because a short oracle string would be enough to compress an initial segment of Z. More precisely, consider the plain machine S given by $S(0^e 1 \sigma) \simeq \Phi_e^\sigma$ (where that $(\Phi_e)_{e \in \mathbb{N}}$ is a uniform listing of Turing reduction procedures). Using S, we see that $\Phi_e^\sigma = x$ implies $C(x) \leq^+ |\sigma| + e + 1$. Hence $|\sigma| < \beta k/2$ implies $K(x) \leq^+ \beta |x|$.

We let $A = \bigcup_j \sigma_j$ where σ_j is a string of length m_j. Both m_j and σ_j are controlled by R_j for $j > 0$. At any stage s, we have $\sigma_{j-1,s} \prec \sigma_{j,s}$ and

$$\lambda(\mathcal{P} \mid \sigma_j) \geq 2^{-2j-1}. \tag{3.1}$$

We let $m_0 = 0$ and hence $\sigma_0 = \emptyset$, so (3.1) is also true for $j = 0$.

We construct a Kraft-Chaitin set L. Each R_j may enumerate into L in order to ensure $K(Z \upharpoonright k) \leq^+ \beta k$.

The idea behind the construction is as follows. We are playing the following R_j strategy. We define a length k_j where we intend to compress Z, and let m_j be the use bound of Ψ_e, $g_e(k_j)$. We define σ_j of length m_j in a way that, if $x = \Psi_e^{\sigma_j}$ is defined then we compress it down to βk_j, by putting an appropriate request into L. The opponent's answer could be to remove σ_j from $\mathcal{P}$. But in that case, the measure he spent for this removal exceeds what we spent for our request, so we can account ours against his. Of course, usually σ_j is much longer than x. So we will only compress x when the measure of oracle strings computing it is large, and use Lemma 3.2.

For each $j, m \in \mathbb{N}$ and each stage t, let

$$G_{j,m,t} = \{\sigma : |\sigma| = m \;\&\; \lambda(P_t \mid \sigma) \geq 2^{-2j}\}.$$

Informally, let us call a string σ of length m_j *good for R_j at stage t* if $\sigma \succ \sigma_{j-1,t}$ and $\sigma \in G_{j,m_j,t}$. These are the only oracle strings R_j looks at. The reason to allow the conditional measure to drop from 2^{-2j+1} at σ_{j-1} down to 2^{-2j} is that we want a sufficiently large measure of them.

Lemma 3.3: *There is an effective sequence (u_j) of natural numbers such that the following holds. Whenever ρ is a string such that $\lambda(P_t \cap [\rho]) \geq 2^{-(2j-1)-|\rho|}$, then for each $m > |\rho|$,*

$$\lambda(G_{j,m,t} \cap [\rho]) \geq 2^{-u_j-|\rho|}.$$

Proof. For each measurable $\mathcal{C}$, one obtains a martingale by letting

$$M_{\mathcal{C}}(\sigma) = \lambda(\mathcal{C} \mid \sigma) = \lambda(\mathcal{C} \cap [\sigma])2^{|\sigma|}.$$

Now let $\mathcal{C} = 2^\omega - \mathcal{P}$. Let $d = 2j-1$. By hypothesis $M_{\mathcal{C}}(\rho) \leq 1 - 2^{-d}$, so we may apply the so-called Kolmogorov inequality, which bounds the measure of strings $\sigma \succ \rho$ where M can reach $1 - 2^{-(d+1)}$:

$$\lambda(\{\sigma : |\sigma| = m \;\&\; M_{\mathcal{C}}(\sigma) \geq 1 - 2^{-(d+1)}\} \mid \rho) \leq \frac{1-2^{-d}}{1-2^{-(d+1)}}.$$

Now it suffices to determine u_j such that $1 - 2^{-u_j} \geq (1-2^{-d})/(1 - 2^{-(d+1)})$, since then $\lambda(G_{j,m,t} \mid \rho) \leq 2^{-u_j}$. This proves the lemma.

R_j compresses x when the measure of good strings computing it is large (4. below). If each good string σ computing x later becomes very bad, in the sense that the conditional measure $\lambda(\mathcal{P} \mid \sigma)$ dropped down to half, then we can carry out the accounting argument mentioned above and therefore choose a new x (5. below).

The *construction* at stage $s > 0$ consists in letting the requirements R_j, for $j = 0 \ldots s$, carry out one step of their strategy. Each time σ_j is newly defined, all the strategies R_l, $l > j$, are initialized. Suppose $j > 0$, that $\rho = \sigma_{j-1}$ is defined already, (3.1) holds for $j-1$ and $m_{j-1} = |\rho|$. Let

$$\mathsf{init}(j) = j + 2 + \text{the number of times } R_j \text{ has been initialized.}$$

1. *Let k_j be so large that $\beta k_j \geq \max(\mathit{init}(j), 2m_{j-1})$, and, where $r_j = u_j + m_{j-1} + k_j$,*
$$\alpha(r_j + 2j + 1) \leq \beta k_j - \mathsf{init}(j). \tag{3.2}$$
2. *While $g_e(k_j)$ is undefined, let $m_j = m_{j-1}+1$, let $\sigma_{j,s}$ be the leftmost extension of σ_{j-1} of length m_j such that $\lambda(P_s \mid \sigma_{j,s}) \geq 2^{-2j}$, and stay at 2. Else let $m_j = g_e(k_j)$ and go to 3.*
3. *Let $G_s = G_{j,m_j,s} \cap [\rho]$. While there exists, let $\sigma_{j,s}$ be the leftmost string $\sigma \in G_s$ such that $\Psi_e^\sigma \upharpoonright k_j \uparrow$, and stay at 3. Else, for each string y of length k_j, let*
$$S_y = \{\sigma \succ \rho : |\sigma| = m_j \;\&\; \Psi^\sigma_{e,s} = y\},$$

and go to 4.

4. *Let x be the leftmost string of length k_j such that $\lambda(G_s \cap S_x) \geq 2^{-r_j}$ where $r_j = u_j + m_{j-1} + k_j$ as above. Put a request*
$$\langle \lceil \beta k_j \rceil, x \rangle$$
into L. Note that x exists since $\lambda G_s \geq 2^{-u_j - m_{j-1}}$ by Lemma 3.3, and $(G_s \cap S_y)_{|y|=k_j}$ is a partition of G_s into at most 2^{k_j} sets (because we passed 3). Of course, $G_s \cap S_x$ may shrink later, in which case we have to try a new x. Eventually we will find the right one.
5. *From now on, let $\sigma_{j,s}$ be the leftmost string σ in S_x that satisfies (3.1), namely, $\lambda(P_s \mid \sigma) \geq 2^{-2j-1}$. If there is no such σ, then we need to pick a new x, so go back to 4.* (We had picked the wrong x. As indicated earlier, we will have to verify that this change is allowed, i.e., the contribution to L does no become too large. The reason is that the opponent had to add at least a measure $2^{-\alpha(r_j+2j+1)}$ of new descriptions to the universal prefix free machine in order to make $G_s \cap S_x$ shrink sufficiently, and we will account the cost of our request against the measure of his descriptions.)

Verification.

Claim 3.1: For each j, $\sigma_j = \lim_s \sigma_{j,s}$ exists.

This is trivial for $j = 0$. Suppose $j > 0$ and inductively the claim holds for $j - 1$. Once σ_{j-1} is stable at stage s_0, we define a final value k_j in 1. If $m_j = g_e(k_j)$ is undefined then $|\sigma_j| = |\sigma_{j-1}| + 1$ and σ_j can change at most once after s_0. Otherwise σ_j can change at most 2^{m_j} times till we reach 4. As remarked above, we always can choose some x in 4. If we cannot find σ in 5. any longer, then $\lambda(G_s \cap S_x) < 2^{-r_j}$, so we will discard this x and not pick it in 4. any more. Since there is an x such that $\lambda(G_t \cap S_x) \geq 2^{-r_j}$ for all $t \geq s_0$, eventually we will stay at 5. Since all strings σ_j we try here have length m_j, eventually σ_j stabilizes. This proves the claim.

Note that $|\sigma_j| > |\sigma_{j-1}|$, so $A = \bigcup_j \sigma_j$ defines a set. Let $A_s = \bigcup_j \sigma_{j,s}$. It is easy to verify that for each x, the number of changes of $A_s(x)$ is computably bounded in x, since the values $m_{j,s}$ are nondecreasing in s, and in 3. and 5., we only move to the right. Thus $A \leq_{\mathrm{wtt}} \emptyset'$.

Claim 3.2: L is Kraft-Chaitin set.

By the definition of $\mathsf{init}(j)$, it suffices to verify that for each value $v = \mathsf{init}(j)$, the weight of the contributions of R_j to L is at most 2^{-v+1}. When a request $\langle r, x \rangle$ is enumerated by R_j at stage s, we distinguish two cases.

Case 1. The strategy stays at 5. after s or R_j is initialized. Then this was the last contribution, and it weighs at most 2^{-v} since we chose k_j in a way that $\beta k_j \geq v$.

Case 2. Otherwise, that is, the strategy gets back to 4. at a stage $t > s$. Then, for each $\sigma \in G_s \cap S_x$, $\lambda(P_t \mid \sigma) < 2^{-2j-1}$ (while, by the definition of G_s, we had $\lambda(P_s \mid \sigma) \geq 2^{-2j}$). Now consider the clopen class

$$\mathcal{C} = P_s \cap [G_s \cap S_x]^{\preceq} - P_t.$$

Since $\lambda(G_s \cap S_x) \geq 2^{-r_j}$,

$$\begin{aligned} \lambda\mathcal{C} &\geq \sum_{\sigma \in G_s \cap S_x} 2^{-|\sigma|}(\lambda(P_s|\sigma) - \lambda(P_t|\sigma)) \\ &\geq \sum_{\sigma \in G_s \cap S_x} 2^{-|\sigma|}(2^{-2j} - 2^{-2j-1}) = \sum_{\sigma \in G_s \cap S_x} 2^{-|\sigma|}2^{-2j-1} \geq 2^{-r_j-2j-1}. \end{aligned}$$

Clearly $\mathcal{C} \subseteq P_s$ and $\mathcal{C} \cap P_t = \emptyset$, so by Lemma 3.2 and (3.2),

$$\Omega_t - \Omega_s \geq 2^{-\alpha(r_j+2j+1)} \geq 2^{-\beta k_j + v},$$

hence the total contributions in Case 2 weigh at most $2^{-v}\Omega$. Together, the contribution is at most 2^{-v+1}.

Claim 3.3: If $Z \leq_{\mathrm{wtt}} A$ then $\underline{K}(Z) \leq \alpha$.

It suffices to show that each requirement R_j is met. For then, if $Z \leq_{\mathrm{wtt}} A$ either, the reduction has small use (see remarks at the beginning of the proof) or it is included in the list (Ψ_e). In the latter case, meeting $R_{\langle e,b\rangle}$ for each b ensures $\underline{K}(Z) \leq \alpha$.

By the first claim, σ_j and k_j reach a final value at some stage s_0. If $\Psi_e^A \upharpoonright k_j \uparrow$ then R_j is met. Otherwise, the strategy forR_j gets to 4. after s_0, and enumerates a request into L which ensures $K(Z \upharpoonright k_j) \leq^+ \beta k_j$.

This concludes the proof of Theorem 3.1.

4. Concluding Remarks

It remains an open problem whether there exists a Turing lower cone of non-integral effective dimension (see [12]). This case appears to be much harder. It is, for instance, not even known whether there exists a set of non-integral dimension which does not compute a Martin-Löf random set.

The best known result in this direction is that there exists a computable, non-decreasing, unbounded function f and a set A such that

$K(A \upharpoonright n) \geq f(n)$ and A does not compute a Martin-Löf random set. This has been independently proved by Reimann and Slaman [16] and Kjos-Hanssen, Merkle, and Stephan [5].

References

1. H. G. Eggleston. The fractional dimension of a set defined by decimal properties. *Quart. J. Math., Oxford Ser.*, 20:31–36, 1949.
2. P. Gács. Every sequence is reducible to a random one. *Inform. and Control*, 70(2-3):186–192, 1986.
3. D. R. Hirschfeldt and J. S. Miller. personal communication, 2005.
4. S. M. Kautz. *Degrees of Random sequences*. PhD thesis, Cornell University, 1991.
5. B. Kjos-Hanssen, W. Merkle, and F. Stephan. Kolmogorov complexity and the Recursion Theorem. Manuscript, submitted for publication, 2005.
6. A. Kučera. Measure, Π^0_1-classes and complete extensions of PA. In *Recursion theory week (Oberwolfach, 1984)*, volume 1141 of *Lecture Notes in Math.*, pages 245–259. Springer, Berlin, 1985.
7. M. Li and P. Vitányi. *An introduction to Kolmogorov complexity and its applications*. Graduate Texts in Computer Science. Springer-Verlag, New York, 1997.
8. J. H. Lutz. Dimension in complexity classes. In *Proceedings of the Fifteenth Annual IEEE Conference on Computational Complexity*, pp. 158–169. IEEE Computer Society, 2000.
9. J. H. Lutz. Gales and the constructive dimension of individual sequences. In *Automata, languages and programming (Geneva, 2000)*, pp. 902–913. Springer, Berlin, 2000.
10. J. H. Lutz. The dimensions of individual strings and sequences. *Inform. and Comput.*, 187(1):49–79, 2003.
11. E. Mayordomo. A Kolmogorov complexity characterization of constructive Hausdorff dimension. *Inform. Process. Lett.*, 84(1):1–3, 2002.
12. J. S. Miller and A. Nies. Randomness and computability: Open questions. Submitted for publication, 2005.
13. A. Nies, *Computability and Randomness*, to appear.
14. P. Odifreddi. *Classical recursion theory*, volume 125 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1989.
15. J. Reimann. Computability and fractal dimension. Doctoral dissertation, Universität Heidelberg, 2004.
16. J. Reimann and T. A. Slaman. Randomness, entropy, and reducibility. Manuscript, in preparation, 2005.
17. B. Y. Ryabko. Noise-free coding of combinatorial sources, Hausdorff dimension and Kolmogorov complexity. *Problemy Peredachi Informatsii*, 22(3):16–26, 1986.
18. G. E. Sacks. *Degrees of unsolvability*. Princeton University Press, Princeton, N.J., 1963.

19. R. I. Soare. *Recursively enumerable sets and degrees*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1987.
20. L. Staiger. Kolmogorov complexity and Hausdorff dimension. *Inform. and Comput.*, 103(2):159–194, 1993.
21. L. Staiger. Constructive dimension equals Kolmogorov complexity. *Information Process. Lett.*, 93:149–153, 2005.
22. F. Stephan. Hausdorff dimension and weak truth-trable reducibility. Submitted for publication, 2005.
23. K. Tadaki. A generalization of Chaitin's halting probability Ω and halting self-similar sets. *Hokkaido Math. J.*, 31(1):219–253, 2002.
24. S. A. Terwijn. Complexity and randomness. Course Notes, 2003.
25. A. K. Zvonkin and L. A. Levin. The complexity of finite objects and the basing of the concepts of information and randomness on the theory of algorithms. *Uspehi Mat. Nauk*, 25(6(156)):85–127, 1970.