Interpreting $\mathbb{N}$ in the computably enumerable weak truth table degrees

André Nies*
The University of Chicago

Abstract

We give a first-order coding without parameters of a copy of $(\mathbb{N}, +, \times)$ in the computably enumerable weak truth table degrees. As a tool,we develop a theory of parameter definable subsets.

Given a degree structure from computability theory, once the undecidability of its theory is known, an important further problem is the question of the actual complexity of the theory. If the structure is arithmetical, then its theory can be interpreted in *true arithmetic*, i.e. $\mathrm{Th}(\mathbb{N}, +, \times)$. Thus an upper bound is $\emptyset^{(\omega)}$, the complexity of $\mathrm{Th}(\mathbb{N}, +, \times)$. Here an interpretation of theories is a many–one reduction based on a computable map defined on sentences in some natural way. An example of an arithmetical structure is $\boldsymbol{D}_T(\leq \emptyset')$, the Turing–degrees of Δ^0_2–sets. Shore [16] proved that true arithmetic can be interpreted in $\mathrm{Th}(\boldsymbol{D}_T(\leq \emptyset'))$. A stronger result is interpretability without parameters of a copy of $(\mathbb{N}, +, \times)$ in the structure (interpretability of structures is defined in [8], Ch. 5). The main purpose of this paper is to prove such a result for the structure $\mathcal{R}_{wtt}$ of computably enumerable weak truth table degrees. So far the undecidability of $\mathrm{Th}(\mathcal{R}_{wtt})$ is known [3]. This result brings a program closer to its completion which has been carried out by various researchers over the past years: to determine the complexity of the theory for structures from computability theory. We discuss some results. For the c.e. many–one and Turing degrees, it has been proved that a copy of $(\mathbb{N}, +, \times)$ can be interpreted without parameters ([13] and [14], respectively). For the c.e. truth–table degrees and the lattice $\mathcal{E}$ of c.e. sets under inclusion, interpretations of $\mathrm{Th}(\mathbb{N}, +, \times)$ in the theory have been given (for the first, see [15]; the second result is due to Harrington, see [7]). In $\mathcal{E}$ one cannot interpret a copy of $(\mathbb{N}, +, \times)$ [7], which shows that the stronger, model theoretic result is not always implied by the mere interpretability of the theory of $(\mathbb{N}, +, \times)$. For the structures $\mathcal{R}_m$ and $\mathcal{R}_T$ of c.e. many–one and c.e. Turing degrees as well as for $\mathcal{E}$, the methods employed (usually auxiliary codings of copies of $(\mathbb{N}, +, \times)$ *with* parameters and uniform definability results) have been used to obtain further results of a model theoretic

*Partially supported by NSF grant DMS–9803482

flavor about the structure (see the same references). For $\mathcal{R}_T$ one obtained $\emptyset$–definability of all jumps classes except Low_1. For both $\mathcal{R}_m$ and $\mathcal{R}_T$, restrictions on automorphisms were derived: each automorphism of $\mathcal{R}_m$ is arithmetical on any proper initial interval, and, dually each automorphism of $\mathcal{R}_T$ is arithmetical on any proper end interval (see [11] for the latter result). Finally for $\mathcal{E}$ one obtained elementary nonequivalence of relativizations. We hope that the new coding methods eventually lead to such results for $\mathcal{R}_{wtt}$.
Degree structures where so far only undecidability of the theory is known include the c.e. Q– and *btt*–degrees ([6] and [12]), as well as the enumeration degrees of Σ^0_2–sets ([17]).
Among the degree structures induced on c.e. sets, only $\mathcal{R}_{wtt}$ and $\mathcal{R}_m$ are *distributive* as an upper semilattice, namely they satisfy

$$\forall x \forall a \forall b [x \leq a \vee b \ \Rightarrow \ \exists a_0 \leq a \exists b_0 \leq b \ x = a_0 \vee b_0]. \tag{1}$$

See Lachlan [9] for a proof in the nontrivial case of $\mathcal{R}_{wtt}$. As a tool for proving our main result we develop a theory of two sorts of parameter definable subsets, using the distributivity of $\mathcal{R}_{wtt}$ in an essential way. One of them is the class of EN-sets. EN-sets are relatively definable without parameters in an *end segment*, i.e. an upward closed subset E of $\mathcal{R}_{wtt}$, while E is definable from two parameters $\boldsymbol{c}, \boldsymbol{d}$. The number $n \in \mathbb{N}$ will be represented by (parameters defining) any EN-set of size n, but there may also be infinite EN-sets. Using the combinatorics of EN-sets, we give first-order definitions in terms of parameters for whether two EN-sets have the same size, and of ternary relations corresponding to the operations $+$ and $\times$ which behave properly on the finite EN–sets. For instance, for $+$, we express that an EN-set is the disjoint union of two others.
The second type of uniformly definable set, called ID-set ("ID" stands for ideal) is needed to single out the finite EN-sets in a first-order way. The concept of ID-sets is dual to that of EN-sets. Thus ID-sets are relatively definable without parameters in an *initial segment*, actually in an ideal I, while I is definable from two parameters $\boldsymbol{c}, \boldsymbol{d}$. To single out finite EN–sets, we will compare EN-sets to ID-sets, using uniformly definable 1-1 maps between the first and the second.
The idea of representing a number n in a natural way by the class of EN-sets of that size sets our proof apart from the ones used for the other structures discussed above which make use of auxiliary codings of copies of $(\mathbb{N}, +, \times)$ *with* parameters. However, this idea was first used in [12] for the upper semilattice of c.e. equivalence relations modulo finite differences.
An important fact is that there is an easy way to produce finite EN-sets: each finite set of low degrees which pairwise join up to greatest c.e. *wtt*–degree is an EN-set (this uses an idea of Ambos–Spies). We first use this fact to give a quite elementary new proof of undecidability for $\mathrm{Th}(\mathcal{R}_{wtt})$. Slightly refining the proof yields the undecidability of $\Pi_5 - \mathrm{Th}(\mathcal{R}_{wtt})$ as a partial order. In Lempp and Nies [10] a coding of finite bipartite graphs based on ID-sets is developed, which even yields the undecidability of $\Pi_4 - \mathrm{Th}(\mathcal{R}_{wtt})$. The Π_2-theory of $\mathcal{R}_{wtt}$ as a partial order is decidable ([2]). Thus the following problem remains open.

Question 0.1 *Is the Π_3–theory of $\mathcal{R}_{wtt}$ as a p.o., or at least as an upper semilattice, undecidable ?*

1 Some terminology

We recall some definitions. See [14] for more details. For a first–order language L, a *scheme* for coding in an L-structure $\boldsymbol{A}$ is given by a list of L-formulas

$$\varphi_1, \dots, \varphi_n$$

with a shared parameter list $\overline{p}$, together with a correctness condition $\alpha(\overline{p})$.

Convention 1.1 *If a scheme S_X is given, $X, X_0, X_1, \dots$ denote objects coded via S_X by a list of parameters satisfying the correctness condition.*

Example 1.2 *A scheme S_g for defining a function g is given by a formula $\varphi_1(x, y; \overline{p})$ defining the relation between arguments and values, and a correctness condition $\alpha(x, y; \overline{p})$ which says (at least) that a function is defined: $\forall x \exists^{\leq 1} y \varphi_1(x, y; \overline{p})$*

Example 1.3 *A scheme for defining n-ary relations on $\boldsymbol{A}$ is given by a formula $\varphi(x_1, \dots, x_n; \overline{p})$ and a correctness condition $\alpha(\overline{p})$.*

Definition 1.4 *A class $\mathcal{C}$ of n-ary relations on $\boldsymbol{A}$ is* uniformly definable *in $\boldsymbol{A}$ if, for some scheme S for coding relations, $\mathcal{C}$ is the class of relations coded via S as the parameters range over tuples in $\boldsymbol{A}$ which satisfy the correctness condition.*

For instance, if $\boldsymbol{A}$ is a linear order and $\mathcal{C}$ is the set of closed intervals, then $\mathcal{C}$ is uniformly definable via the scheme consisting of $\varphi_1(x; a, b) \Leftrightarrow a \leq x \leq b$ and the correctness condition $\alpha(a, b) \Leftrightarrow a \leq b$.

Notation 1.5 As in Soare [18, p. 49], we assume that the use of the computation $\{e\}_s^A(x)$, $u(A; e, x, s) \leq s$. For $e = \langle e_0, e_1 \rangle$ let

$$[e](x) \simeq \max_{y \leq x} \varphi_{e_1}(y).$$

Let $[e]^A(x)$ be $\{e_0\}^A(x)$ if $[e](x)$ and $\{e_0\}^A(x)$ are defined, and the computation has use $\leq [e](x)$. Otherwise $[e]^A(x)$ is undefined . In a similar way define the approximations at stage s, namely $[e]_s(x)$ and $[e]_s^A(x)$.

Note that $A \leq_{wtt} B \ \Leftrightarrow \ A = [e]^B$ for some e. This implies that

$$\{\langle e, i \rangle : W_e \leq_{wtt} W_i\} \text{ is } \Sigma_3^0. \tag{2}$$

2 Uniformly definable classes in $\mathcal{R}_{wtt}$

We develop a coding without parameters of a copy of $(\mathbb{N}, +, \times)$ in $\mathcal{R}_{wtt}$. Recall that we plan to use two types of uniformly definable subsets.
We first prove some facts which lead to the concepts of EN- and ID-sets. Most of the facts are algebraic. We expose the duality between the two concepts, as far as the nonsymmetric framework of an upper semilattice which may not be a lattice allows this. In the following let $(D; \leq, \vee, 0, 1)$ be a distributive upper semilattice with least and greatest elements $0, 1$.

Lemma 2.1 *Suppose that $b, y_0, \ldots, y_n \in D$.*

1. *If $b \wedge y_i = 0$ for each $i \leq n$, then $b \wedge sup_i y_i = 0$*
2. *If $b \vee y_i = 1$ for each i, then there is $t \in D$ such that $b \vee t = 1$ and $t \leq y_i$ for each i.*

Proof. (i) If $0 < x \leq b, \sup_{i \leq n} y_i$, then by distributivity, there is an i and $r \in D$ such that $0 < r \leq x, y_i$. But then $r \leq b, y_i$, contrary to $b \wedge y_i = 0$.
(ii) If $n = 0$ let $t = y_0$. Else, since $y_1 \leq b \vee y_0$, we can choose a $t_1 \leq y_0$ and $b_1 \leq b$ such that $y_1 = b_1 \vee t_1$. Then, $1 = b \vee b_1 \vee t_1 = b \vee t_1$, so if $n \neq 1$, $y_2 \leq b \vee t_1$ implies that we can pick $t_2 \leq t_1$ and $b_2 \leq b$ such that $y_2 = b_2 \vee t_2$ and thus $1 = b \vee t_2$. Continuing in this way we obtain $t = t_n \leq y_0, \ldots y_n$ such that $b \vee t = 1$. ♢

For $d_0, \ldots, d_n \in D$, let

$$E(d_0, \ldots, d_n) \;=\; \{x \in D : \forall y[\forall i \leq n(y \leq d_i) \;\Rightarrow\; y \leq x]\}. \tag{3}$$

Thus $E(d_0, \ldots, d_n)$ is the set of upper bounds of the ideal $[0, d_0] \cap \ldots \cap [0, d_n]$. Note that $d_i \in E(d_0, \ldots, d_n)$ for each i. Finite EN-sets $\{p_0, \ldots, p_n\}$ will be sets which are relatively definable in $E(p_0, \ldots, p_n)$. First we need a characterization of the elements in such an end segment.

Lemma 2.2 *For $d_0, \ldots, d_n \in (D; \leq, \vee, 0, 1)$,*

$$x \in E(d_0, \ldots, d_n) \;\Leftrightarrow\; x = \inf_{i \leq n}(x \vee d_i).$$

Proof. For the direction from right to left, clearly $x \vee d_i \in E(d_0, \ldots, d_n)$ for each i. Hence, if the infimum exists, it is also an upper bound for the ideal $[0, d_0] \cap \ldots \cap [0, d_n]$.
For the direction from left to right, the argument is similar to the one used in the proof of Lemma 2.1 (ii). If $y \leq x \vee d_i$ for each i, then by distributivity we can choose $x_0 \leq x$ and $q_0 \leq d_0$ such that $y = x_0 \vee q_0$. If $n \geq 1$, choose $x_1 \leq x, q_1 \leq d_1$ such that $q_0 = x_1 \vee q_1$. Continuing in this way we obtain $q_n \leq d_n$ such that $q_{n-1} = x_n \vee q_n$. Moreover $q_n \leq d_0, \ldots, d_n$, so $q_n \leq x$. Hence $q_{n-1} \leq x, \ldots, q_0 \leq x$ and finally $y \leq x$. ♢

Definition 2.3 *For $x, y \in D$, we write $nd[x, y]$ if $x < y$ and the interval $[x, y]$ does not embed the 4-element boolean algebra preserving least and greatest element.*

Clearly $nd[x, y]$ can be expressed in the language of p.o. In the next lemma, (i) leads to the definition of EN-sets, and (ii) to the definition of ID-sets.

Lemma 2.4 *(i) Let $p_0, \dots, p_n$ be a finite sequence of elements of D such that for each i, $nd[p_i, 1]$ (in particular, $p_i < 1$) and for $i \neq j$, $p_i \vee p_j = 1$. Then $\{p_i : i \leq n\}$ is the set of minimal elements x in $E = E(p_0, \dots, p_n)$ such that $nd[x, 1]$.*

(ii) Let (a_i) be a finite or infinite sequence of elements of D such that for each i $nd[0, a_i]$ and for $i \neq j$, $a_i \wedge a_j = 0$. Then $\{a_i\}$ is the set of maximal elements x in I such that $nd[0, x]$, where I is the ideal of D generated by $\{a_i\}$.

Proof. (i) It is sufficient to prove that

$$x \in E \,\&\, nd[x, 1] \;\Rightarrow\; \exists j \; p_j \leq x.$$

Since $x < 1$, by Lemma 2.2 there is j such that $x \vee p_j < 1$. Moreover, by Lemma 2.1 there is t such that, for all $i \neq j$, $t \leq x \vee p_i$ and $x \vee p_j \vee t = 1$. We can suppose that $x \leq t$. By Lemma 2.2, $x = \inf_{k \leq n} x \vee p_k$, so $(x \vee p_j) \wedge t = x$. By $nd[x, 1]$, this implies $t = 1$, so $x \vee p_i = 1$ for $i \neq j$ and $x = \inf_{k \leq n} x \vee p_k = x \vee p_j$.
(ii) It is sufficient to prove that

$$x \in I \,\&\, nd[0, x] \;\Rightarrow\; \exists j \; x \leq a_j.$$

Since $x \in I$, $x \leq \sup_{i \leq n} a_i$ for some n. By distributivity, $x = \sup_{i \leq n} \widetilde{a}_i$ for some $\widetilde{a}_i \leq a_i$ $(i \leq n)$. Since $0 < x$, some $\widetilde{a}_j$ does not equal 0. By Lemma 2.1, $\widetilde{a}_j \wedge \sup_{i \leq n, i \neq j} \widetilde{a}_i = 0$, so $nd[0, x]$ implies that $\widetilde{a}_i = 0$ for $i \leq n, i \neq j$, hence $x = \widetilde{a}_j \leq a_j$. ♢

In the context of $\mathcal{R}_{wtt}$, we are able to give first-order definitions with parameters of the set E in (i) of the preceding Lemma, and also of I in (ii) if $(\boldsymbol{a}_i)$ is a finite or an infinite u.c.e. sequence. We use the following theorem of Ambos-Spies, Nies and Shore.

Theorem 2.5 ([3]) *Let I be a Σ^0_3-ideal of $\mathcal{R}_{wtt}$. Then there exists $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{R}_{wtt}$ such that $I = [\mathbf{o}, \boldsymbol{a}] \cap [\mathbf{o}, \boldsymbol{b}]$.* ♢

Degrees $\boldsymbol{a}, \boldsymbol{b}$ as above are called an *exact pair* for I. Note that, conversely, each ideal which has an exact pair is Σ^0_3, so that the theorem constitutes a uniform definability result for the class of Σ^0_3-ideals.

Lemma 2.6 *(i) Suppose $\{\boldsymbol{p}_0, \dots, \boldsymbol{p}_n\}$ is a subset of $\mathcal{R}_{wtt}$ such that $nd[\boldsymbol{p}_i, \mathbf{1}]$ for each i and $\boldsymbol{p}_i \vee \boldsymbol{p}_j = \mathbf{1}$ for $i \neq j$. Then $\{\boldsymbol{p}_0, \dots, \boldsymbol{p}_n\}$ is definable from two parameters $\mathbf{c}, \boldsymbol{d}$ via a formula $\varphi_P(x; c, d)$.*

(ii) *Suppose* $(\boldsymbol{a}_i)$ *is a finite or infinite u.c.e. sequence in* $\mathcal{R}_{wtt}$ *such that* $nd[\mathbf{o}, \boldsymbol{a}_i]$ *for each* i *and* $\boldsymbol{a}_i \vee \boldsymbol{a}_j = \mathbf{o}$ *for* $i \neq j$. *Then* $\{\boldsymbol{a}_i\}$ *is definable from two parameters* $\boldsymbol{c}, \boldsymbol{d}$ *via a formula* $\varphi_Z(x; c, d)$.

Proof. (i) Observe that $I = [\mathbf{o}, \boldsymbol{p}_0] \cap \ldots \cap [\mathbf{o}, \boldsymbol{p}_n]$ is a Σ^0_3-ideal by (2), so $I = [\mathbf{o}, \boldsymbol{c}] \cap [\mathbf{o}, \boldsymbol{d}]$ for some $\boldsymbol{c}, \boldsymbol{d}$. Thus $E(\boldsymbol{p}_0, \ldots, \boldsymbol{p}_n) = E(\boldsymbol{c}, \boldsymbol{d})$ is definable from $\boldsymbol{c}, \boldsymbol{d}$ via the formula $\psi(x; c, d) = \forall y[y \leq c, d \Rightarrow y \leq x]$. Let $\varphi_P(x; c, d)$ be the formula expressing that x is a minimal element in $\{z : \psi(z; c, d)\}$ such that $nd[x, 1]$.
(ii) Let I be the ideal generated by $\{\boldsymbol{a}_i\}$. It follows from (2) that I is Σ^0_3. So, once again, $I = [\mathbf{o}, \boldsymbol{c}] \cap [\mathbf{o}, \boldsymbol{d}]$ for some $\boldsymbol{c}, \boldsymbol{d}$. Let $\varphi_Z(x; c, d)$ be the formula expressing that x is a maximal element $\leq c, d$ such that $nd[0, x]$. ◊

We are now ready to specify the notions of EN-sets and ID-sets by appropriate schemes of the same type as in Example 1.3.

Definition 2.7 (i) *Let* S_P *the the scheme given by the formula* $\varphi_P(z; c, d)$ *and the correctness condition* $\alpha(c, d)$ *expressing that whenever* x, y *satisfy the formula and* $x \neq y$, *then* $x \vee y = 1$. *Subsets of* $\mathcal{R}_{wtt}$ *coded via* S_P *are called* EN-sets.

(ii) *Let* S_Z *the scheme given by the formula* $\varphi_Z(z; c, d)$ *and the correctness condition* $\beta(c, d)$ *expressing that whenever* x, y *satisfy the formula and* $x \neq y$, *then* $x \wedge y = 0$. *Subsets of* $\mathcal{R}_{wtt}$ *coded via* S_Z *are called* ID-sets.

Notice that subsets of finite EN-sets are EN-sets themselves. In particular, $\emptyset$ is the EN-set coded by $\boldsymbol{c} = \boldsymbol{d} = \mathbf{1}$. Also $\emptyset$ is the ID-set coded by $\boldsymbol{c} = \boldsymbol{d} = \mathbf{o}$.

3 Undecidability of $\mathrm{Th}(\mathcal{R}_{wtt})$

We develop a scheme S_C to code arbitrary relations between finite EN-sets. The coding methods of this section will also be used to obtain a coding of a copy of $(\mathbb{N}, +, \times)$.
The abundance of EN-sets stems from the fact that each low $\boldsymbol{p} \in \mathcal{R}_{wtt}$ satisfies $nd[\boldsymbol{p}, \mathbf{1}]$[1]. Thus, whenever $\boldsymbol{p}_0, \ldots, \boldsymbol{p}_n$ are low and $\boldsymbol{p}_i \vee \boldsymbol{p}_j = \mathbf{1}$ for $i \neq j$, then $\{\boldsymbol{p}_0, \ldots, \boldsymbol{p}_n\}$ is an EN-set. For each n, such *wtt*-degrees $\boldsymbol{p}_0, \ldots, \boldsymbol{p}_n$ can be easily obtained by the method of the Sacks splitting theorem (see Soare [18]). In view of later applications, we will prove a more general version of this in Proposition 3.2 below.

Theorem 3.1 *If* $\boldsymbol{p} \in \mathcal{R}_{wtt}$ *is low, then* $nd[\boldsymbol{p}, \mathbf{1}]$.

Proof. We slightly modify the proof of an extension of the Lachlan Non- Diamond Theorem in Ambos-Spies [1]. He proves that, if $\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{b}_0, \boldsymbol{b}_1$ are c.e. Turing degrees such that $\boldsymbol{a}_0 \vee \boldsymbol{a}_1 = \deg_T(\emptyset')$ and $\boldsymbol{b}_0 \vee \boldsymbol{b}_1$ is low, then, for some

[1]The author would like to thank Klaus Ambos-Spies for suggesting this.

$i \leq 1$, $\boldsymbol{a}_i$ is not $\boldsymbol{b}_i$-cappable. Here $\boldsymbol{a}$ is called *$\boldsymbol{b}$–cappable* if there is a $\boldsymbol{c} \not\leq \boldsymbol{b}$ such that $\boldsymbol{b} = \boldsymbol{a} \wedge \boldsymbol{c}$. An inspection of the proof reveals that it can be adapted to *wtt*-reducibility. (The T-reductions built during the construction have recursively bounded use anyway, and the proof of Lemma 6 [Lemma 9] goes through. In particular, if the reduction procedures occurring in requirement R_e are now *wtt*-reductions $[e_1]^{B_0}$ and $[e_2]^{B_1}$, then the step counting functions g in the proofs of those lemmas can be computed from B_0 $[B_1]$ with recursively bounded use. So the weaker hypothesis $C_0 \not\leq_{wtt} B_0$ $[C_1 \not\leq_{wtt} B_1]$ suffices.)
Here we use only the special case of the Theorem that $\boldsymbol{b}_0 = \boldsymbol{b}_1 = \boldsymbol{p}$. If $nd[\boldsymbol{p}, \mathbf{1}]$ fails, then there are $\boldsymbol{a}_0, \boldsymbol{a}_1 < \mathbf{1}$ such that $\boldsymbol{a}_0 \vee \boldsymbol{a}_1 = \mathbf{1}$ and $\boldsymbol{a}_0 \wedge \boldsymbol{a}_1 = \boldsymbol{p}$. So for both $i = 0$ and $i = 1$, $\boldsymbol{a}_i$ is $\boldsymbol{p}$–cappable via $\boldsymbol{c}_i = \boldsymbol{a}_{1-i}$. ◊

We now prove the existence of EN-sets relating in a certain way to given degrees.

Proposition 3.2 *Suppose that $\boldsymbol{u}_0, \ldots, \boldsymbol{u}_m < \mathbf{1}$. Then for each $n \geq 0$ there exist low $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_n \in \mathcal{R}_{wtt}$ such that $\{\boldsymbol{v}_0, \ldots, \boldsymbol{v}_n\}$ is an EN-set and $\boldsymbol{u}_i \vee \boldsymbol{v}_j < \mathbf{1}$ for each $i \leq m, j \leq n$.*

Proof. Choose c.e. sets $U_i \in \boldsymbol{u}_i$. We construct c.e. sets V_j such that the statement of the theorem holds with $\boldsymbol{v}_j = \deg_{wtt}(V_j)$.
To achieve $\boldsymbol{v}_j \vee \boldsymbol{v}_{j'} = \mathbf{1}$ $(j' \neq j)$ we ensure that $K = V_j \cup V_{j'}$ (where K is some creative set). For $nd[\boldsymbol{v}_j, \mathbf{1}]$, we make each V_j low and apply Theorem 3.1. We meet the standard lowness requirements

$$L_{e,j} : \exists^\infty s \ \{e\}^{V_j}(e)[s] \text{ is defined } \Rightarrow \{e\}^{V_j}(e) \text{ converges} .$$

Finally, for $\boldsymbol{u}_i \vee \boldsymbol{v}_j < \mathbf{1}$ $(0 \leq i \leq m, 0 \leq j \leq n)$ we meet the requirements

$$N_{e,i,j} : K \neq [e]^{U_i \oplus V_j},$$

by refraining from changing V_j till a permanent disagreement occurs (or $[e]^{U_i \oplus V_j}$ is partial). Let (R_k) be some priority listing of the L-type and N-type requirements. If R_k is $N_{e,i,j}$ let

$$\text{length}(k, s) = \min\{x : \forall y < x \ K(y) = [e]^{U_i \oplus V_j}(y)[s],$$

and let $r(k, s) = \max\{[e]_s(y) : y < \text{length}(k, s)$.
If R_k is a lowness requirement $L_{e,j}$, the restraint associated with R_k is

$$r(k, s) = u(V_{j,s}; e, e, s).$$

Construction. At stage $s+1$, if $K_s = K_{s+1}$ do nothing. Else, say y is the unique element in $K_{s+1} - K_s$. Determine the minimal k such that $y < r(k, s)$. If k fails to exist enumerate y into all sets V_j. Else let j be the number such that $R_k = L_{e,j}$ or $R_k = N_{e,i,j}$ for some e, i. Then V_j is the set such that enumerating y into V_j would violate $r(k, s)$. So enumerate y into $V_{j'}$, for each $j' \neq j$. This completes the description of the construction.

Clearly $K = V_j \cup V_{j'}$ for $j \neq j'$. By induction on k we prove:

Lemma 3.3 *Let $k \geq 0$.*

(i) The requirement R_k is met.

(ii) $r(k) = lim_s r(k, s)$ exists and is finite.

Proof. Assume the Lemma holds for all $h < k$. Choose a stage s_0 such that for all $h < k$, $r(h, s_0)$ has reached the limit, and K does not change below $\max_{h<k} r(h)$ at any stage $s \geq s_0$. Then at no stage $s \geq s_0$ can any number $y < r(k, s)$ enter V_j, where j is determined from k as in the construction: j is the number such that $R_k = L_{e,j}$ or $R_k = N_{e,i,j}$ for some e, i.
If $R_k = L_{e,j}$, then R_k is met, because if ever $\{e\}^{V_j}[s]$ converges for $s \geq s_0$, then this computation is preserved. Hence also $r(k, s)$ reaches its limit. Now suppose that $R_k = N_{e,i,j}$.
For (i), assume for a contradiction that $K = [e]^{U_i \oplus V_j}$. Then

$$\limsup \text{length}(k, s) = \infty.$$

We obtain a *wtt*-reduction of K to U_j as follows: given an input y, compute $s \geq s_0$ such that $\text{length}(k, s) > y$ and $U_i \lceil [e](y) = U_{i,s} \lceil [e](y)$. Then $r(k, t) \geq [e](y)$ for all $t \geq s$, so (by he monotonicity of the function $[e]$) $[e]^{U_i \oplus V_j} \lceil y + 1$ is protected from changing at stages $\geq s$. So $K(y) = [e]^{U_i \oplus V_j}(y)[s]$. Since $\boldsymbol{u}_i < \boldsymbol{1}$, we conclude that $N_{e,i,j}$ is met.
For (ii), let x be least such that $K(x) \neq [e]^{U_i \oplus V_j}(x)$. Let $s_1 \geq s_0$ be least such that, $[e](x)$ is defined, then $K(x)$ and $U_i \oplus V_j \lceil [e](x)$ have reached their final values at s_1. Then $\text{length}(k, s) \leq x$ from s_1 on, hence $r(k, s)$ reaches it limit. $\diamondsuit$

Our next goal is to code relations between arbitrary finite EN-sets.

Proposition 3.4 *There is a scheme S_C for coding objects of the form (P_0, P_1, R) in $\mathcal{R}_{wtt}$, where P_0, P_1 are EN-sets, which has the following property: if P_0, P_1 are finite, then for any $R \subseteq P_0 \times P_1$, (P_0, P_1, R) can be coded.*

Proof. S_C contains parameters $\boldsymbol{c}_0, \boldsymbol{d}_0, \boldsymbol{c}_1, \boldsymbol{d}_1$ coding P_0, P_1 and further parameters for the relation R. Suppose that $P_0 = \{\boldsymbol{p}_0, \ldots, \boldsymbol{p}_n\}$ and $P_1 = \{\boldsymbol{q}_0, \ldots, \boldsymbol{q}_m\}$. First we assume that, in addition,

$$\boldsymbol{p}_i \vee \boldsymbol{q}_j < \boldsymbol{1} \text{ for all } i, j. \tag{4}$$

We will reduce the general case to this.
As in the proof of Lemma 2.6(ii) there are $\boldsymbol{g}, \boldsymbol{h}$ such that

$$E(\boldsymbol{g}, \boldsymbol{h}) = E(\{\boldsymbol{p}_i \vee \boldsymbol{q}_j : R\boldsymbol{p}_i\boldsymbol{q}_j\}).$$

We claim that

$$R\boldsymbol{p}_i\boldsymbol{q}_j \;\Leftrightarrow\; \exists \boldsymbol{z} \in E(\boldsymbol{g}, \boldsymbol{h}) - \{\boldsymbol{1}\}[\boldsymbol{p}_i \vee \boldsymbol{q}_j \leq \boldsymbol{z}].$$

For the direction from left to right, simply let $\boldsymbol{z} = \boldsymbol{p}_i \vee \boldsymbol{q}_j$. For the other direction, suppose that the right hand side holds via $\boldsymbol{z} < \mathbf{1}$. By Lemma 2.2, $\boldsymbol{z} = \inf\{\boldsymbol{z} \vee \boldsymbol{p}_r \vee \boldsymbol{q}_s : R\boldsymbol{p}_r\boldsymbol{q}_s\}$. But, if not $R\boldsymbol{p}_i\boldsymbol{q}_j$, then $\boldsymbol{z} \vee \boldsymbol{p}_r \vee \boldsymbol{q}_s = \mathbf{1}$ for each pair $\boldsymbol{p}_r, \boldsymbol{q}_s$ in R, since $(i,j) \neq (r,s)$ and therefore $\boldsymbol{p}_i \vee \boldsymbol{q}_r = \mathbf{1}$ or $\boldsymbol{p}_j \vee \boldsymbol{q}_s = \mathbf{1}$. This contradicts $\boldsymbol{z} < \mathbf{1}$.
Now let

$$\begin{aligned}\widetilde{\varphi}_{rel}(x,y;c_0,d_0,c_1,d_1,g,h) \quad &\Leftrightarrow \quad \varphi_P(x;c_0,d_0) \;\&\; \varphi_P(y;c_1,d_1) \;\& \\ &\qquad \exists z < 1[x,y \leq z \;\&\; z \in E(g,h)].\end{aligned}$$

Then in this special case each $R \subseteq P_0 \times P_1$ can be coded via $\widetilde{\varphi}_{rel}$.
To remove the restriction (4) we imposed, we interpolate with a third EN- set. By Proposition 3.2, there is an EN-set $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_n$ such that, for all $k \leq n$, $\boldsymbol{p}_i \vee \boldsymbol{v}_k < \mathbf{1}$ and $\boldsymbol{q}_j \vee \boldsymbol{v}_k < \mathbf{1}$ $(i \leq n, j \leq m)$. Let $F : P_0 \mapsto \{\boldsymbol{v}_0, \ldots, \boldsymbol{v}_n\}$ be a bijection. Consider the relation $\widetilde{R}$ given by $\widetilde{R}\boldsymbol{v}_k\boldsymbol{q}_j \;\Leftrightarrow\; R[F^{-1}(\boldsymbol{v}_k)\boldsymbol{q}_j]$. Both $F \subseteq P_0 \times \{\boldsymbol{v}_0, \ldots, \boldsymbol{v}_n\}$ and $\widetilde{R} \subseteq \{\boldsymbol{v}_0, \ldots, \boldsymbol{v}_n\} \times P_1$ can be coded by parameters via $\widetilde{\varphi}_{rel}$. Then $R = F\widetilde{R}$ can be coded via the following formula (think of z as $F(x)$):

$$\begin{aligned}\varphi_{rel}(x,y;\overline{p}) \quad \Leftrightarrow \exists z \quad &[\widetilde{\varphi}_{rel}(x,z;c_0,d_0,c_2,d_2,g_0,h_0) \;\& \\ &\widetilde{\varphi}_{rel}(z,y;c_2,d_2,c_1,d_1,g_1,h_1)],\end{aligned}$$

where c_2, d_2 are parameters coding the auxiliary EN-set and $\overline{p}$ consists of all 10 parameters. ♢

The following proof is somewhat more elementary than the previously known ones, because it only uses the exact pair theorem 2.5, the technique of the Sacks splitting theorem and Theorem 3.1 (i.e. the technique of the Lachlan non-diamond theorem) as the recursion theoretic ingredients.

Theorem 3.5 ([3, 10]) Th($\mathcal{R}_{wtt}$) *is undecidable.*

Proof. We use the usual general framework to prove undecidability of theories indirectly, see e.g. [5]. The class $\mathcal{C}$ of finite directed graphs has a hereditarily undecidable theory. Using Proposition 3.4, $\mathcal{C}$ can be uniformly coded in $\mathcal{R}_{wtt}$. Hence Th($\mathcal{R}_{wtt}$) is undecidable. ♢

4 Interpreting a copy of $(\mathbb{N}, +, \times)$

In the following, it is vital to keep in mind the convention 1.1.

Theorem 4.1 *A copy of* $(\mathbb{N}, +, \times)$ *can be interpreted in* $\mathcal{R}_{wtt}$ *without parameters.*

We will use finite EN-sets to represent numbers. The scheme S_C from Proposition 3.4 enables us to express by a first-order condition on parameters that EN-sets have the same cardinality, and also the arithmetical operations. In the end we face the harder problem to single out finite EN-sets. (Note that, even if our examples were all finite, there is no reason to believe that all the defined via the scheme for EN-sets in Definition 2.7 are finite.)
We introduce the formulas without parameters to code $(\mathbb{N}, +, \times)$. We use formulas $\varphi_{num}(\overline{x}), \varphi_{=}(\overline{x}, \widetilde{y}), \varphi_{+}(\overline{x}, \overline{y}, \overline{z})$ and $\varphi_{\times}(\overline{x}, \overline{y}, \overline{z})$, where $\overline{w}$ stands for a pair of variables w_0, w_1 which represent an exact pair needed to code an EN-set. The formula $\varphi_{num}(\overline{x})$ will be considered last, but of course it will imply the correctness condition for S_P, since $\overline{x}$ is thought of as coding an EN-set.

1. Equality

Let $\varphi_{\equiv}(\overline{x}, \overline{y})$ be a formula expressing

$$\exists C[C \text{ is bijection } P_{\overline{x}} \mapsto P_{\overline{y}}],$$

using the scheme S_C from Proposition 3.4. By that proposition, if $P_{\overline{\boldsymbol{a}}}$ and $P_{\overline{\boldsymbol{e}}}$ are finite, then

$$|P_{\overline{\boldsymbol{a}}}| = |P_{\overline{\boldsymbol{e}}}| \Leftrightarrow \mathcal{R}_{wtt} \models \varphi_{\equiv}(\overline{\boldsymbol{a}}, \overline{\boldsymbol{e}}).$$

2. The arithmetical operations

Let $\varphi_{+}(\overline{x}, \overline{y}, \overline{z})$ be a formula expressing that $P_{\overline{z}}$ can be partitioned into two sets of the same size as $P_{\overline{x}}$ and $P_{\overline{y}}$, respectively:

$$\exists \overline{u} \exists \overline{v} [\varphi_{\equiv}(\overline{x}, \overline{u}) \ \& \ \varphi_{\equiv}(\overline{y}, \overline{v}) \ \& \ P_{\overline{z}} = P_{\overline{u}} \cup P_{\overline{v}} \ \& \ P_{\overline{u}} \cap P_{\overline{v}} = \emptyset].$$

It can easily be checked that, for finite $P_{\overline{\boldsymbol{a}}}, P_{\overline{\boldsymbol{e}}}, P_{\overline{\boldsymbol{c}}}$

$$|P_{\overline{\boldsymbol{a}}}| + |P_{\overline{\boldsymbol{e}}}| = |P_{\overline{\boldsymbol{c}}}| \Leftrightarrow \mathcal{R}_{wtt} \models \varphi_{+}(\overline{\boldsymbol{a}}, \overline{\boldsymbol{e}}, \overline{\boldsymbol{c}}).$$

For the direction from left to right one uses that subsets of $P_{\overline{\boldsymbol{c}}}$ are again EN-sets. For $\varphi_{\times}(\overline{x}, \overline{y}, \overline{z})$ we say in terms of definable projection maps that $P_{\overline{z}}$ has the same size as the cartesian product $P_{\overline{x}} \times P_{\overline{y}}$. Thus $\varphi_{\times}(\overline{x}, \overline{y}, \overline{z})$ expresses

$$\begin{array}{ll} \exists C_1 \exists C_2 & C_1 : P_{\overline{z}} \mapsto P_{\overline{x}} \text{ onto } \ \& \ C_2 : P_{\overline{z}} \mapsto P_{\overline{y}} \text{ onto } \ \& \\ & \forall a \in P_{\overline{x}} \, \forall b \in P_{\overline{y}} \, \exists! q \in P_{\overline{z}} [C_1(q) = a \ \& \ C_2(q) = b]. \end{array}$$

Then, for finite $P_{\overline{\boldsymbol{a}}}, P_{\overline{\boldsymbol{e}}}, P_{\overline{\boldsymbol{c}}}$

$$|P_{\overline{\boldsymbol{a}}}||P_{\overline{\boldsymbol{e}}}| = |P_{\overline{\boldsymbol{c}}}| \Leftrightarrow \mathcal{R}_{wtt} \models \varphi_{\times}(\overline{\boldsymbol{a}}, \overline{\boldsymbol{e}}, \overline{\boldsymbol{c}}).$$

Recognizing finiteness

To recognize in a first-order way that an EN-set coded by two parameters is finite, the idea is to compare EN-sets to fragments of a uniformly definable subclass of the ID-sets. ID-sets are not as easy to construct as EN-sets, but a more involved construction actually yields a u.c.e. *infinite* ID-set

$$Z^* = \{\boldsymbol{a}_i : i \in \mathbb{N}\}.$$

To specify the uniformly definable subclass of the class of ID-sets we will impose conditions on parameters $\boldsymbol{c}, \boldsymbol{d}$ coding $Z = Z_{\boldsymbol{c},\boldsymbol{d}}$ which are satisfied by Z^* and *imply* that

1. when $\boldsymbol{x}$ ranges through degrees $\leq \boldsymbol{c}, \boldsymbol{d}$, then $|Z \cap [\mathbf{o}, \boldsymbol{x}]|$ assumes all finite cardinalities
2. if $|P| = |Z \cap [\mathbf{o}, \boldsymbol{x}]|$, $\boldsymbol{x} \leq \boldsymbol{c}, \boldsymbol{d}$, then a bijection between the two sets can be uniformly defined.

ID-sets Z satisfying the conditions to be formulated will be called *good*. For the special good ID-set $Z^* = Z^*_{\boldsymbol{c},\boldsymbol{d}}$, $Z^* \cap [\mathbf{o}, \boldsymbol{x}]$ is finite for $\boldsymbol{x} \leq \boldsymbol{c}, \boldsymbol{d}$. The formula φ_{num} implies about P that for each good $Z_{\boldsymbol{c},\boldsymbol{d}}$, a bijection between P and some set $Z \cap [\mathbf{o}, \boldsymbol{x}]$, $\boldsymbol{x} \leq \boldsymbol{c}, \boldsymbol{d}$, exist.
The set Z^* is obtained applying a rather hard theorem of Ambos-Spies and Soare [4]. To ensure property (2.) above, one has to make all the degrees $\boldsymbol{a}_i$ low. An easier result in Lempp and Nies [10] could also be used, but this has the disadvantage that the actual construction needs to be extended by adding new requirements to make the degrees $\boldsymbol{a}_i$ low.

Main Lemma 4.2 ([4]) *There exists a u.c.e. sequence $(A_i)_{i \in \mathbb{N}}$ such that each A_i is low, A_i, A_j form a T- minimal pair for $i \neq j$ and, where $\boldsymbol{a}_i = deg_{wtt}(A_i)$, $nd[\mathbf{o}, \boldsymbol{a}_i]$ for each i. Thus $Z^* = \{\boldsymbol{a}_i\}$ is an ID-set.*

Proof. Recall that noncomputable c.e. set C is *non-bounding* if there is no minimal pair A, B such that $A, B \leq_T C$. This definition makes sense also for wtt-reducibility. Clearly, C is wtt-non-bounding iff $nd[\mathbf{o}, \boldsymbol{d}]$ for each nonzero $\boldsymbol{d} \leq \boldsymbol{c} = \deg_{wtt}(C)$.
In Ambos-Spies et al. [3], Lemma 6, it is proved that each non-bounding C is also wtt-non-bounding. From Ambos-Spies and Soare [4] one obtains a u.c.e. sequence (A_i) such that each A_i is T-non-bounding and A_i, A_j form a T-minimal pair for $i \neq j$. Since there is a uniform construction to produce from a given c.e. set A a low set $\widetilde{A}$ such that $\widetilde{A}$ is non-computable if A is [18], we can assume that each set A_i is low. $\diamondsuit$

Definition 4.3 *An ID-set Z defined from parameters $\boldsymbol{c}, \boldsymbol{d}$ is* good *if*

(i) $\forall \boldsymbol{x} \leq \boldsymbol{c}, \boldsymbol{d} (Z \not\subseteq [\mathbf{o}, \boldsymbol{x}])$

(ii) $\forall \boldsymbol{x} \leq \boldsymbol{c}, \boldsymbol{d}\, \exists \widetilde{P}$

$$\{\langle \boldsymbol{u}, \boldsymbol{v} \rangle : \boldsymbol{u} \leq \boldsymbol{v} \ \& \ \boldsymbol{u} \in Z \cap [\mathbf{o}, \boldsymbol{x}] \ \& \ \boldsymbol{v} \in \widetilde{P}\} \tag{5}$$

is a bijection between $Z \cap [\mathbf{o}, \boldsymbol{x}]$ and $\widetilde{P}$.

Being good can be expressed by a first-order condition on $\boldsymbol{c}, \boldsymbol{d}$. Moreover, (i) implies that Z is infinite: else $\boldsymbol{x} = \sup Z$ is below $\boldsymbol{c}, \boldsymbol{d}$, contrary to (i).
We will prove that any u.c.e ID-set Z of low *wtt*-degrees is good, when defined from an exact pair for the Σ_3^0-ideal generated by Z. In particular the set $Z^* = \{\boldsymbol{a}_i\}$ from the Main Lemma 4.2 is good. Assuming this fact, we now give a first order condition on parameters expressing finiteness of an EN-set P.

Lemma 4.4 *P is finite* $\Leftrightarrow$ $\forall \boldsymbol{a}, \boldsymbol{b}[Z_{\boldsymbol{a},\boldsymbol{b}}$ *good* $\Rightarrow \exists \boldsymbol{x} \leq \boldsymbol{a}, \boldsymbol{b}$

$$\exists \widetilde{P}[\textit{the map}(5) \textit{ is a bijection } \ \& \ \exists C \ C \textit{ is bijection } P \leftrightarrow \widetilde{P}]].$$

Proof. For the direction from left to right, assume that P is finite. Because good *ID*-sets are infinite, we can choose $F \subseteq Z$ such that $|F| = |P|$. If $\boldsymbol{x} = \sup F$ (with the convention $\sup \emptyset = \mathbf{o}$), then by (ii) of Lemma 2.1 $F = [\mathbf{o}, \boldsymbol{x}] \cap Z$. Choose $\widetilde{P}$ satisfying (5). By Proposition 3.4, a bijection $P \leftrightarrow \widetilde{P}$ can be coded via S_C.
For the other direction, let $\boldsymbol{a}, \boldsymbol{b}$ be an exact pair coding the set Z^* obtained from the Main Lemma 4.2. If $\boldsymbol{x} \leq \boldsymbol{a}, \boldsymbol{b}$, then $\boldsymbol{x} \leq \boldsymbol{a}_0, \ldots, \boldsymbol{a}_n$ for some n. By Lemma 2.1, $\boldsymbol{a}_k \wedge \boldsymbol{x} = \mathbf{o}$ for all $k > n$, so $Z \cap [\mathbf{o}, \boldsymbol{x}]$ is finite. Thus P is finite. $\diamondsuit$

Finally, we supply the proof that any infinite u.c.e. ID-set Z of low *wtt*-degrees is good. Let Z be such a set, coded by an exact pair $\boldsymbol{a}, \boldsymbol{b}$. By a similar argument as above, $Z \cap [\mathbf{o}, \boldsymbol{x}]$ is finite for any $\boldsymbol{x} \leq \boldsymbol{a}, \boldsymbol{b}$. Since all degrees in Z are low and the finite EN–sets are closed downwards, it is now sufficient to prove the following.

Lemma 4.5 *Suppose that $\boldsymbol{a}_0, \ldots, \boldsymbol{a}_n$ are low pairwise incomparable degrees in $\mathcal{R}_{wtt}$. Then there is an EN-set $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_n$ such that*

$$\boldsymbol{a}_i \leq \boldsymbol{v}_j \ \Leftrightarrow \ i = j.$$

Proof. Choose c.e. sets $A_i \in \boldsymbol{a}_i$. We construct c.e. sets V_j such that the statement of the theorem holds with $\boldsymbol{v}_j = \deg_{wtt}(A_j \oplus V_j)$. Clearly $\boldsymbol{a}_i \leq \boldsymbol{v}_i$. To ensure $\boldsymbol{a}_i \not\leq \boldsymbol{v}_j$ for $i \neq j$, we meet the requirements

$$N_{e,i,j}: \ A_i \neq [e]^{A_j \oplus V_j} \ (i \neq j),$$

by the same strategy as in the proof of Proposition 3.2: refrain from changing V_j till a permanent disagreement occurs. We will define some priority listing $(R_k)_{k \in \mathbb{N}}$ of all the requirements. If R_k is $N_{e,i,j}$ let

$$\text{length}(k, s) = \min\{x : \forall y < x \ A_i(y) = [e]^{A_j \oplus V_j}(y)[s],$$

and let $r(k, s) = \max\{[e]_s(y) : y < \text{length}(k, s)\}$.
To achieve $\boldsymbol{v}_j \vee \boldsymbol{v}_{j'} = \mathbf{1}$ ($j' \neq j$) as in Proposition 3.2 we ensure that $K = V_j \cup V_{j'}$. For $nd[\boldsymbol{v}_j, \mathbf{1}]$, we make each $A_j \oplus V_j$ low and apply Theorem 3.1. Lowness is achieved by the side effects of the "pseudolowness requirements"

$$L_{e,j} : \exists^\infty s \ \{e\}^{A_j \oplus V_j}(e)[s] \ \text{ is defined } \ \Rightarrow \ \{e\}^{A_j \oplus V_j}(e) \ \text{ converges} .$$

While $L_{e,j}$ may fail to be met, it will produce enough restraint to ensure $(A_j \oplus V_j)' \equiv_T \emptyset'$. We use a standard technique introduced by Robinson. By the recursion theorem, we can assume that the sets $V_0, \ldots, V_n$ with specific enumerations are *given* (see comment at the end). Since each set A_i $(i \leq n)$ is low, the following property of e, j and a stage number $\widetilde{s}$ can be checked with an oracle $\emptyset'$:

$$\exists s \geq \widetilde{s} \ [\{e\}^{A_j \oplus V_j}(e)[s] \text{ is defined via an } A_j\text{-correct computation}]. \qquad (6)$$

By the Limit Lemma ([18]) we choose a computable function $g(\widetilde{s}, e, j, t)$ such that $\lim_t g(\widetilde{s}, e, j, t)$ exists, has value 0 or 1, and the limit equals 1 iff (6) holds. Let (R_k) be some priority listing of all the requirements.

Construction. At Stage 0 initialize all the lowness requirements.
Stage $s+1$. First determine the restraint $r(k,s)$ for all $k < s$ such that R_k is a lowness requirement $L_{e,j}$. Let $\widetilde{s} < s$ be greatest such that R_k was initialized at $\widetilde{s}$. If $\{e\}^{A_j \oplus V_j}(e)[s]$ is undefined, let $r(k,s) = 0$. Else let u be the use of this computation and find the least $t \geq s$ such that either

a) $A_{j,t+1}|u \neq A_{j,t}|u$, or

b) $g(\widetilde{s}, e, i, t) = 1$.

Since (6) is equivalent to $\lim_t g(\widetilde{s}, e, j, t) = 1$ and the computation at s seems to provide a witness for (6), one of the two cases has to apply. In Case a) let $r(k,s) = 0$, while in Case b) $r(k,s) = u$.

Now, if $K_s = K_{s+1}$ terminate stage $s+1$ here. Else, say y is the unique element in $K_{s+1} - K_s$. Determine the minimal k such that $y < r(k,s)$. If k fails to exist enumerate y into all the sets V_j. Else let j be the number such that $R_k = L_{e,j}$ or $R_k = N_{e,i,j}$ for some e, i. Enumerate y into $V_{j'}$, for each $j' \neq j$. Initialize all the lowness requirements R'_k, $k' > k$. This completes the description of the construction.

Lemma 4.6 *Let* $k \geq 0$.

(i) If R_k *is* $N_{e,i,j}$*, then the requirement* R_k *is met.*

(ii) $r(k) = lim_s r(k,s)$ *exists and is finite.*

Proof. Assume the Lemma holds for all $h < k$. Choose a stage s_0 such that for all $h < k$, $r(h, s_0)$ has reached the limit, and K does not change below $\max_{h<k} r(h)$ at any stage $s \geq s_0$.
If R_k is $N_{e,i,j}$, we can prove (i) and (ii) as in Proposition 3.2. In particular, if $A_i = [e]^{A_j \oplus V_j}$, then one can obtain a *wtt* reduction procedure of A_i to A_j, contrary to the assumption that $\boldsymbol{a}_i, \boldsymbol{a}_j$ are incomparable.

Now suppose that R_k is $L_{e,j}$. We have to show that $\lim_s r(k,s)$ is finite. Let $\widetilde{s}$ be the greatest stage where R_k is initialized (necessarily $\widetilde{s} \leq s_0$), and pick $s \geq s_0$ where $g(\widetilde{s}, e, j, s)$ has reached its limit. If the limit is 0, then $r(k,t) = 0$ for all $t \geq s$. Else, by (6) and the definition of g there is a least stage $t \geq \widetilde{s}$ such that $\{e\}^{A_j \oplus V_j}(e)[t]$ is defined via an A_j -correct computation with use u . Then at stage t we define $r(k,t) = u$. Since R_k is not initialized at stages $> \widetilde{s}$, the computation $\{e\}^{A_j \oplus V_j}(e)[t]$ is preserved. So $r(k,s) = u$ for all $s \geq t$. ♢

Lemma 4.7 *$A_j \oplus V_j$ is low for each $j \leq n$.*

Proof. Given e, we have to determine with a $\emptyset'$-oracle whether $\{e\}^{A_j \oplus V_j}(e)$ converges. Let k be such that R_k is $L_{e,j}$. Note that, in the proof of the preceding lemma, we can determine $\widetilde{s}$ using a $\emptyset'$-oracle. Then, by (6),

$$\lim_t g(\widetilde{s}, e, j, t) = 0 \ \Rightarrow\ \{e\}^{A_j \oplus V_j}(e) \text{ diverges},$$

and by the argument above,

$$\lim_t g(\widetilde{s}, e, j, t) = 1 \ \Rightarrow\ \{e\}^{A_j \oplus V_j}(e) \text{ converges}.$$

The use of the recursion theorem deserves a comment: We are given some c.e. sets $V_0, \ldots, V_n$ via a partial recursive enumeration function ψ which maps s to a strong index for $V_0 \oplus \ldots \oplus V_n[s]$. From this the construction produces a similar enumeration $\widetilde{\psi}$ for sets $\widetilde{V}_0, \ldots, \widetilde{V}_n$. By the recursion theorem, there must be ψ such that $\widetilde{\psi} = \psi$, and in particular $V_j = \widetilde{V}_j$ for $j \leq n$. The function g actually contains an extra argument, namely an index for ψ, and in the discussion above we assume that this extra argument is an index such that $\widetilde{\psi} = \psi$. ♢

References

[1] K. Ambos-Spies. An extension of the nondiamond thoerem in classical and α-recursion theory. *J. Symbolic Logic*, 49:586–607, 1984.

[2] K. Ambos-Spies, P. A. Fejer, S. Lempp, and M. Lerman. Decidability of the two-quantifier theory of the recursively enumerable weak truth-table degrees and other distributive upper semi-lattices. *Journal of Symbolic Logic*, 61(3):880–905, 1996.

[3] Klaus Ambos-Spies, Andre Nies, and Richard A. Shore. The theory of the recursively enumerable weak truth-table degrees is undecidable. *J. Symbolic Logic*, 57:864–874, 1992.

[4] R.I. Ambos-Spies, K. Soare. The recursively enumerable degrees have infinitely many one types. *Ann. Pure Appl. Logic*, 44:1–23, 1989.

[5] S. Burris and H. P. Sankappanavar. A course in universal algebra. Springer–Verlag, Berlin, 1981.

[6] R. Dowey, A. Nies, and G. LaForte. Enumerable sets and quasireducibility. *Ann. Pure Appl. Logic*, 95:1–35, 1998.

[7] L. A. Harrington and A. Nies. Coding in the lattice of enumerable sets. *Adv. in Math.*, 133:133–162, 1998.

[8] Wilfrid Hodges. Model Theory. Enzyklopedia of Mathematics. Cambridge University Press, Bangkok, 1993.

[9] Alistair H. Lachlan. Embedding nondistributive lattices in the recursively enumerable degress. In W. Hodges, editor, *Conference in Mathematical Logic, London, 1970*, volume 255 of *Lecture Notes in Mathematics*, pages 149–177, Heidelberg, 1972. Springer–Verlag.

[10] S. Lempp and A Nies. Undecidability of the 4-quantifier theory for the recursively enumerable turing and wtt degrees. *J. Symbolic Logic*, 60:1118–35, 1995.

[11] A. Nies. Parameter definable subsets of the computably enumerable degrees. To appear.

[12] A. Nies. Definability and undecidability in recursion theoretic semilattices. PhD thesis, Universität Heidelberg, 1992.

[13] A Nies. The last question on recursively enumerable many-one degrees. *Algebra i Logika*, 33(5):550–563, 1995. English Translation, Consultants Bureau, NY, July 1995.

[14] A. Nies, R. Shore, and T. Slaman. Interpretability and definability in the recursively enumerable turing-degrees. *Proc. Lond. Math. Soc.*, 3(77):241–291, 1998.

[15] A. Nies and R. A. Shore. Interpreting true arithmetic in the theory of the r. e. truth table degrees. *Ann. Pure Appl. Logic*, 75:269–311, 1995.

[16] Richard A. Shore. The theory of the degrees below $0'$. *J. London Math. Soc.*, 24:1–14, 1981.

[17] Theodore A. Slaman and W. Hugh Woodin. Definability in the enumeration degrees. *Arch. Math. Logic*, 36:255–267, 1997.

[18] Robert I. Soare. Recursively Enumerable Sets and Degrees. Perspectives in Mathematical Logic, Omega Series. Springer–Verlag, Heidelberg, 1987.

Department of Mathematics, The University of Chicago, 5734 S. University Av., Chicago IL 60637, USA, e-mail nies@math.uchicago.edu
Website http://www.math.uchicago.edu/~nies