Randomness in effective descriptive set theory

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ABSTRACT. An analog of ML-randomness in the effective descriptive set theory setting is studied, where the r.e. objects are replaced by their Π^1_1 counterparts. We prove the analogs of the Kraft-Chaitin Theorem and Schnorr's Theorem. In the new setting, while K-trivial sets exist that are not hyper-arithmetical, each low for random set is. Finally we study a very strong yet effective randomness notion: Z is strongly random if Z is in no null Π^1_1 set of reals. We show that there is a greatest Π^1_1 null set, that is, a universal test for this notion.

1. Introduction

A reasonable intuitive view is that an infinite sequence of 0's and 1's is random if it does not satisfy any properties of probability zero. However, one has to restrict the type of properties considered to obtain a sound formal definition of randomness, for instance since being equal to that sequence also is a null property. To do so, usually one uses algorithmic notions. A commonly accepted formalization is the one given by Martin-Löf [**6**], based on uniformly r.e. open sets. He defined a sequence to be random if it does not have any property of effective Σ^0_1 measure zero. A MARTIN-LÖF TEST (ML-test) is a uniformly r.e. sequence $\{U_i\}_{i\in\omega}$ of Σ^0_1-classes such that $\mu(U_i) \leq 2^{-i}$. A set $\mathcal{A} \subseteq 2^\omega$ is MARTIN-LÖF NULL if there is a ML-test $\{U_i\}_{i\in\omega}$ such that $\mathcal{A} \subseteq \bigcap_i U_i$. A set A is MARTIN-LÖF RANDOM if $\{A\}$ is not ML-null. There is an extensive theory of ML-randomness. For instance, Schnorr's Theorem states that Z is ML-random iff there exists b such that $K_{r.e.}(Z|n) > n - b$ at every n, where $K_{r.e.}$ is the prefix free complexity defined in terms of the universal recursively enumerable prefix free machine.

Effective descriptive set theory provides the Π^1_1-sets of natural numbers as a high level analog of the r.e. sets. Such a set can be thought of as being enumerated during stages formed by the recursive ordinals. It certainly makes sense to restrict the allowed properties using tools from effective descriptive set theory, instead of (classical) computability theory. Thus we replace the r.e. test and machine concepts mentioned above by their Π^1_1 analogs. We show that Schnorr's Theorem and a further major tool, the Kraft-Chaitin Theorem, persist in the new setting. In this

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context there are considerable new technical problems arising from the presence of limit stages.

A lot of recent research is centered on K-trivial sets, a notion opposite to ML-randomness. A is K-TRIVIAL if there is a constant b such that $K_{r.e.}(A \upharpoonright n) \leq K_{r.e.}(n) + b$ for each n (here the number n is identified with a string corresponding to its binary representation). There are r.e. non-computable K-trivial sets, but all are Δ^0_2 (see [**1**]). A is K-trivial if and only if A is low for ML-random, namely each ML-random set is already random relative to A [**10**]. In particular, K-triviality is closed downward under Turing reducibility. This coincidence has been extended to a further class introduced by Kučera [**5**]: A is a BASE FOR ML-RANDOMNESS (or base, in brief) if $A \leq_T Z$ for some Z which is ML-random relative to A. Each low for ML-random set is such a base. In [**2**] it is shown that each base is K-trivial. Thus all the three notions coincide, being K-trivial, low for ML-random and a base for ML-randomness.

Surprisingly, these coincidences are limited to the r.e. case. We show that in the Π^1_1 case, while a K-trivial Π^1_1 set exists which is not hyper-arithmetical, the only low for ML-random sets (and in fact, the only bases) are the hyper-arithmetical sets.

Finally we consider the even stronger randomness notion where the null properties to be avoided are the Π^1_1-sets of reals. We prove that there is a largest such set, that is, a universal test for this randomness notion. Therefore this notion, first mentioned in Sacks [**11**, Exercise 2.5.IV], is a natural one deserving further exploration.

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2. Basics

We identify a string σ in $2^{<\omega}$ with the natural number n such that the binary representation of $n+1$ is 1σ. Sets are subsets of ω unless otherwise stated. They are identified with infinite strings over $\{0,1\}$. $Z \upharpoonright n$ denotes the string $Z(0)\dots Z(n-1)$. A set Z is LEFT-R.E. if $\{\sigma : \sigma <_L Z\}$ is r.e. ($<_L$ is the usual lexicographical ordering on $2^{<\omega}$). Similarly we define left-Π^1_1 sets. Topological notions refer to the space 2^ω with the product topology. For σ a finite binary string, we let $[\sigma]$ be the set of all $Z \in 2^\omega$ which extend σ; in other words, $[\sigma]$ is the basic clopen set canonically described by σ. A clopen set is a finite union of basic clopen sets. For $D \subset 2^{<\omega}$ we let $[D]^{\preceq} = \bigcup\{[\sigma] : \sigma \in D\}$.

We generally refer to Sacks [**11**] for effective descriptive set theory. In particular, $\mathcal{O}$ is the set of ordinal notations, a Π^1_1 complete set, ω_1^{ck} is the least non-recursive ordinal, and ω_1^A is the least ordinal not recursive in the set A.

Given a Π^1_1 set $S \subseteq \omega$, one can effectively obtain a u.r.e. sequence $(R_e)_{e\in\omega}$ of linear orders on initial segments of ω such that, for each y, $y \in S \Leftrightarrow R_y$ is well-ordered. See [**11**, 5.3.I] and Section 5 for more details, or [**3**, Thms 25.3, 25.12].

For $y \in S$, we view the order type $\alpha = |R_y|$ as the stage when y is enumerated into S, in an enumeration through stages which are recursive ordinals. We replace R_y by $\omega R_y + y + 1$, so that we may assume that at each stage, at most one element is enumerated, and none at a limit stage. In the following, each Π^1_1 set S comes with such an enumeration. For each ordinal $\alpha \leq \omega_1^{\mathrm{ck}}$, we let $S_\alpha = \{y : |R_y| < \alpha\}$ (so that $S_{\omega_1^{\mathrm{ck}}}$ is the whole set).

We also make use of a set-theoretic representation of Π^1_1-sets. Here and below Σ_1 refers to the Levy hierarchy: Thus a Σ_1 formula is a formula in the language of set

theory which has the form $\exists x_1 \exists x_2 ... \exists x_n \, \varphi_0$, where φ_0 uses only bounded quantifiers, namely quantifiers of the form $\exists z \in y$ and $\forall z \in y$.
We frequently use the following.

THEOREM 2.1 (Spector-Gandy). *$S \subseteq \omega$ is Π^1_1 iff there is a Σ_1-fmla $\varphi(y)$ such that $S = \{y \in \omega : L(\omega_1^{\mathrm{ck}}) \models \varphi(y)\}$.*

It easy to see that each Π^1_1 set is of this form: $\varphi(y)$ expresses that R_y is isomorphic to an ordinal, namely, $\exists \alpha \exists g [g : (\omega, R_y) \cong (\alpha, \in)$. For the converse, see [**11**, 1.3.VII].

This important theorem enables us to apply the techniques of recursion theory to effective descriptive set theory. Instead of enumeration over the natural numbers, we enumerate over $L(\omega_1^{\mathrm{ck}})$. Π^1_1 sets in particular play a role analogous to recursively enumerable sets. It should be mentioned already at this stage of exposition that the *limit ordinals* less than ω_1^{ck} play a role in effective descriptive set theory that has no counterpart in recursion theory.
Our use of the Spector-Gandy Theorem to build Π^1_1 sets S can be made more explicit as follows. An ENUMERATION of S is a Σ_1 (over $L(\omega_1^{\mathrm{ck}})$) function $\omega_1^{\mathrm{ck}} \mapsto \omega \cup \{\mathrm{nil}\}$ (where nil is a further element, say ω). A CONSTRUCTION C of S is given by a Σ_1 function over $L(\omega_1^{\mathrm{ck}})$ which tells us what to enumerate at stage α, given the enumeration up to α. Formally C is a Σ_1 function over $L(\omega_1^{\mathrm{ck}})$ mapping $\langle \alpha, f \upharpoonright \alpha \rangle$ to the number to be enumerated at α, or to nil if no number is enumerated. By transfinite recursion in $L(\omega_1^{\mathrm{ck}})$, a unique f exists for each C (see [**11**, pg. 155]). However, we will not be that formal below.

Prefix free machines and prefix free complexity.

DEFINITION 2.2. *A* PREFIX FREE MACHINE *is a possibly partial function $M : 2^{<\omega} \mapsto 2^{<\omega}$ with Π^1_1 graph such that dom(M) is an antichain under the prefix relation of strings $\preceq$.*

PROPOSITION 2.3. *There is an effective listing $(M_e)_{e \in \omega - \{0\}}$ of all prefix free machines.*

Proof. Let $(S_e)_{e \in \omega - \{0\}}$ be an effective listing of the Π^1_1-sets $\subseteq 2^{<\omega} \times 2^{<\omega}$. Thus $\langle \sigma, y \rangle \in S_e \Leftrightarrow R^e_{\sigma,y}$ is well-ordered, where $(R^e_{\sigma,y})$ is a u.r.e sequence of linear orders as above. Now let $\langle \sigma, y \rangle \in M_e \Leftrightarrow R^e_{\sigma,y}$ is well-ordered, and
$\forall \langle \rho, z \rangle \forall g \, [(\rho \prec \sigma \; \vee \; (\rho = \sigma \; \& \; z \neq y)) \; \Rightarrow$
g is not an order preserving embedding of $R^e_{\rho,z}$ into $R^e_{\sigma,y}$].
(Informally, no substring ρ of σ and no other value for σ has been enumerated before.) Clearly this is a Π^1_1 condition uniformly in e. If S_e is a prefix free machine, then $M_e = S_e$. □

As a consequence, there is a UNIVERSAL PREFIX FREE MACHINE $\mathbf{U}$, given by
$$\mathbf{U}(0^{d-1}1\sigma) = M_d(\sigma).$$
If $\mathbf{U}(\sigma) = y$, we say that σ is a *U-description* of y.
Let
$$K(y) = \min\{|\sigma| : U(\sigma) = y\}.$$
For any $\alpha \leq \omega_1^{\mathrm{ck}}$, we let $\mathbf{U}_\alpha(\sigma) = y$ if $\langle \sigma, y \rangle \in \mathbf{U}_\alpha$, and
$$K_\alpha(y) = \min\{|\sigma| : U_\alpha(\sigma) = y\}.$$

Note that for $\alpha < \omega_1^{\mathrm{ck}}$, "$K_\alpha(y) = u$" is a Δ_1 relation over $L(\omega_1^{\mathrm{ck}})$, and "$K(y) \le u$" is Σ_1 over $L(\omega_1^{\mathrm{ck}})$, , and hence Π^1_1, being equivalent to "$\exists \alpha\, \exists y\, (|y| \le u\ \&\ \mathbf{U}_\alpha(y) = x)$". Recall that each Π^1_1 set is many-one reducible to Kleene's $\mathcal{O}$ [**11**, I.5.4]. As a consequence, $K \le_T \mathcal{O}$, since $\mathcal{O}$ can determine the value $K(x)$.

3. A high level analog of ML-randomness

We prove that the analogs of the Kraft-Chaitin theorem, Schnorr's Theorem and the Kučera-Gács Theorem are valid in the Π^1_1-case. We make use of some material from [**8**]. *Throughout, we use the terminology and notation of the r.e. case with the new interpretations.*

3.1. The Kraft-Chaitin Theorem.

Definition 3.1. *A Π^1_1 set $W \subseteq \omega \times 2^{<\omega}$ is a* Kraft-Chaitin set *(KC set) if* $\sum_{\langle r,y\rangle \in W} 2^{-r} \le 1$.

Theorem 3.2. *From a Kraft-Chaitin set W one can effectively obtain a prefix free machine M such that*

$$\forall \langle r,y\rangle \in W \exists w\ (|w| = r\ \&\ M(w) = y).$$

We say that M is a prefix free machine for *W.*

Proof. As remarked above, W comes with an enumeration of elements at certain successor stages α, at most one per stage. Here the elements are axioms, of the form $\langle r,y\rangle$. We turn this enumeration into a stage-by-stage construction of a prefix free machine M, as defined in 2.2.
Construction of M. At a successor stage $\alpha = \beta+1$, if an axiom $\langle r,y\rangle$ is enumerated into W we will find a string w of length r, and we set $M(w) = y$. We let $D_0 = \{\emptyset\}$. At each stage $\gamma \ge 0$ we have an antichain $D_\beta \in L(\omega_1^{\mathrm{ck}})$ of strings (the set of extensions of strings in D_γ is our reservoir of future w-values, and strings in this set are called *unused*). With each string x we associate the half-open interval $I(x) \subseteq [0,1)$ of real numbers whose binary representation (containing infinitely many 0's) extends x. Thus for instance $I(011) = [3/8, 1/2)$.
Let z be the longest string in D_β of length $\le r$. Choose w so that $I(w)$ is the leftmost subinterval of $I(z)$ of length 2^{-r}, i.e., let $w = z0^{r-|z|}$. To obtain D_α, first remove z from D_β. If $w \neq z$ then also add the strings $z0^i1$, $0 \le i < r - |z|$.
At limit stages η we let

$$D_\eta = \{x : \exists \gamma < \eta\, \forall \alpha\, [\gamma < \alpha < \eta \Rightarrow x \in D_\alpha\}.$$

This ends the construction. We will see that a string can appear in D_α at most once, so that actually $D_\eta = \lim_{\gamma \to \eta} D_\gamma$. In Claim 3.3 below we verify a number of properties in order to show that for each axiom $\langle r,y\rangle$, z as above exists, and therefore one can assign a string w of length r to the axiom. Let $E_\alpha = \bigcup\{I(x) : x \in D_\alpha\}$ be the set of reals corresponding to D_α. At a limit stage η, the measure of unused strings is $\lambda(G_\eta)$, where $G_\eta = \bigcap_{\alpha<\eta} E_\alpha$. To be able to get beyond this limit stage, we want to replace G_η by D_η. The main statement (i) below says that this substitution is legal, because $E_\eta \subseteq G_\eta$ and $\lambda(G_\eta - E_\eta) = 0$. We first illustrate the construction with an example showing that this null set may be non-empty. Suppose at stage $i < \omega$ the axiom $\langle 2i+1, y_i\rangle$ is enumerated. Then $G_\omega - E_\omega = \{1/3\}$. For $D_0 = \{\emptyset\}$, $z_0 = \emptyset$, $w_0 = 00$; $D_1 = \{01, 1\}$, $z_1 = 01$, $w_1 = 0100$; $D_2 = \{0101, 011, 1\}$,

$z_2 = 0101$, $w_2 = 010100$ etc. Then $D_\omega = \{(01)^i 1 : i \in \omega\}$. 1/3 has the binary representation $0.010101\ldots$, so that $1/3 \in E_i$ for each i, but $1/3 \notin E_\omega$.

CLAIM 3.3. (i) *For each stage α, $E_{\alpha+1} \subseteq E_\alpha$. If $\alpha = \eta$ is a limit ordinal, then $E_\eta \subseteq G_\eta := \bigcap_{\beta<\eta} E_\beta$. Moreover, $\lambda(G_\eta - E_\eta) = 0$.*

(ii) *If an axiom is enumerated at stage α, then one can at that stage choose z, and hence w.*

(iii) *The strings in D_α have different lengths and form an antichain. (In fact, for $x, y \in D_\alpha$, $|x| < |y| \Leftrightarrow x <_L y$, that is, the intervals $I(x)$ get longer as one moves to the right.)*

(iv) *$\{I(z) : z \in D_\alpha\} \cup \{I(w_\beta) : \beta \le \alpha$ & w_β defined$\}$ induces a partition of a conull subset P_α of $[0,1)$.*

Proof. Inductively assume (i)-(iv) hold for all $\gamma < \alpha$.
(i) Clearly $E_{\alpha+1} \subseteq E_\alpha$. If $\alpha = \eta$ is a limit ordinal, to show $E_\eta \subseteq G_\eta$, let $\beta < \eta$. If $r \in E_\eta$, then $r \in I(x)$ for some $x \in D_\eta$, so there is γ, $\beta < \gamma < \eta$, such that $x \in D_\gamma$. Inductively $E_\gamma \subseteq E_\beta$. Thus $r \in E_\beta$.
We verify $\lambda E_\eta \ge \lambda G_\eta$, by showing $\lambda E_\eta \ge \lambda G_\eta - 2^{-k+1}$ for any $k \in \omega$. Write λG_η in binary form, $\lambda(G_\eta) = \sum_{d \in A} 2^{-d}$, where $A \subseteq \omega$. Since $(\lambda E_\gamma)_{\gamma<\eta}$ is non-increasing and converges to λG_η, there is $\gamma < \eta$ such that $2^{-k+1} + \sum_{d \in A \cap k} 2^{-d} \ge \lambda E_\gamma$. Let $A \cap k = \{d_1, d_2, ..., d_N\}$. For each α, $\gamma < \alpha < \delta$, let z_i^α $(1 \le i \le N)$ be the elements of D_α such that $|z_i^\alpha| = d_i$. Such strings exist by inductive hypothesis (iii) for α. If $z \in D_\beta - D_{\beta+1}$ for some $\beta < \eta$, then $z \preceq w_\beta$, so $z \notin D_\delta$ for any δ, $\beta < \delta < \eta$ by inductive hypothesis (iv) for δ (in brief, z cannot reappear after disappearing). Since there are only 2^{d_i} possibilities for z_i^α, we eventually settle on some strings z_i, hence $z_i \in D_\eta$. Thus

$$\lambda(E_\eta) \ge \sum_{1 \le i \le N} 2^{-|z_i|} \ge \lambda(E_\gamma) - 2^{-k+1} \ge \lambda(G_\eta) - 2^{-k+1}$$

as required.
(ii) Suppose the axiom $\langle r, y\rangle$ is enumerated at stage $\alpha = \beta + 1$. If z_α fails to exist, then r is less than the length of each string in D_β. By (iii) for β, $\lambda E_\beta = \sum\{2^{-|z|} : z \in D_\beta\}$, so by (iv) for β,

$$2^{-r} + \sum\{2^{-m} : \text{ an axiom } \langle m, z\rangle \text{ is enumerated at a stage } \le \beta\} > 1,$$

contrary to the assumption that W is a KC-set.
(iii) This is clear for successor stages α, because the intervals $I(w_\gamma)$, $\gamma \le \alpha$ and w_γ defined, are disjoint. Then the property persists to limit ordinals by the definition of D_η.
(iv) Again, this is clear for successor stages $\alpha = \beta + 1$, in which case we may define $P_\alpha = P_\beta$. If $\alpha = \eta$ is a limit ordinal, then let P_η be the intersection of of the sets P_γ, $\gamma \le \eta$ and the complements of the null sets $G_\gamma - E_\gamma$ from (ii). Then for each $\beta < \eta$, P_η is partitioned by E_β and $I(w_\gamma)$, $\gamma \le \beta$, w_γ defined. So P_η is partitioned into G_η and $I(w_\gamma), \gamma < \eta$. Since G_η is partitioned on P_η into the intervals $I(w)$, $w \in D_\eta$, we have shown (iv) for η. □

3.2. The Coding Theorem. For a prefix free machine D, the probability that D outputs x is

$$P_D(x) = \lambda\{\sigma : D(\sigma) = x\}.$$

Clearly, $2^{-K(x)} \leq P_{\mathbf{U}}(x)$. We show that, for some constant c, $\forall x\ 2^c 2^{-K(x)} \geq P_D(x)$. This also holds at certain ordinal stages. For $\alpha \leq \omega_1^{\mathrm{ck}}$, let $P_{D,\alpha}(x) = \lambda\{\sigma : D_\alpha(\sigma) = x\}$. For $g : \omega_1^{\mathrm{ck}} \mapsto \omega_1^{\mathrm{ck}}$, we say that a limit ordinal $\lambda \leq \omega_1^{\mathrm{ck}}$ is g-CLOSED if $\forall \alpha < \lambda\, [g(\alpha) < \lambda]$.

THEOREM 3.4 (Coding Theorem). *For each prefix free machine D, there is a Σ_1 over $L(\omega_1^{\mathrm{ck}})$ function $g_D = g : \omega_1^{\mathrm{ck}} \mapsto \omega_1^{\mathrm{ck}}$ and a constant c such that, for each g-closed $\lambda \leq \omega_1^{\mathrm{ck}}$*

$$\forall x\ 2^c 2^{-K_\lambda(x)} \geq P_{D,\lambda}(x).$$

Proof. One enumerates a KC set L, "accounting" the enumeration of axioms $\langle r, x\rangle$ against the open sets generated by the D-descriptions of x. Of course, for different outputs x, these open sets are disjoint.
Construction of L.

> *Stage s.* If x is a string, $r \in \omega$ is least such that $P_{D,s}(x) \geq 2^{-r+1}$, and the axiom $\langle r, x\rangle$ is not in L yet, then put $\langle r, x\rangle$ into L.

For a string x, let α_x be the greatest stage at which an axiom $\langle r, x\rangle$ is put into L. Then $P_{D,\alpha}(x) \geq 2^{-r+1}$. Hence all such axioms together contribute at most 1/2. The total weight of all axioms $\langle r', x\rangle$ enumerated at previous stages is $\leq 2^{-r}$ since $r' > r$ for such an axiom, and there is at most one for each length r'. Thus L is a KC set.
Let c_L be the coding constant for L given by Theorem 3.2. The function g is the delay it takes the universal machine to react to an enumeration of an axiom into L. Thus for $\alpha < \omega_1^{\mathrm{ck}}$,

$$g(\alpha) = \mu\beta \forall \langle r, x\rangle \in L_\alpha\, [K_\beta(x) \leq r + c_L].$$

If r is least such that $P_{D,\lambda}(x) > 2^{-r+1}$, then at the least stage $\alpha < \lambda$ where $P_{D,\alpha}(x) \geq 2^{-r+1}$, we enumerate $\langle r, x\rangle$ and cause $K_\lambda(x) \leq K_{g(\alpha)}(x) \leq r + c_L$, since λ is g-closed. By the minimality of r, $2^{-r+2} \geq P_{D,\lambda}(x)$, hence $2^{c_L+2} 2^{-K_\lambda(x)} \geq 2^{-r+2} \geq P_{D,\lambda}(x)$. Thus $c = c_L + 2$ is as required. □

3.3. Some properties of K. We apply the Coding Theorem in order to obtain an estimate of the number of strings with small K-complexity.

THEOREM 3.5. *There is a constant $\mathbf{c} \in \omega$ and a Σ_1 over $L(\omega_1^{\mathrm{ck}})$ function $g : \omega_1^{\mathrm{ck}} \mapsto \omega_1^{\mathrm{ck}}$ such that the following hold for each g-closed $\delta \leq \omega_1^{\mathrm{ck}}$.*
(i) $\forall d \forall n \quad |\{x : |x| = n\ \&\ K_\delta(x) \leq n + K_\delta(n) - d\}| \leq 2^{\mathbf{c}} 2^{n-d}$
(ii) $\forall b \forall n \quad |\{x : |x| = n\ \&\ K_\delta(x) \leq K_\delta(n) + b\}| \leq 2^{\mathbf{c}} 2^{b}$

Proof. Let D be the prefix free machine given by $D(\sigma) = |U(\sigma)|$, and let g be the function obtained in the coding theorem for D. Let $\mathbf{c}$ be the constant such that for each n, $2^{\mathbf{c}} 2^{-K_\delta(n)} \geq P_{D,\delta}(n)$, given by the Coding Theorem.
(i). If $|x| = n$ and $K_\delta(x) \leq n + K_\delta(n) - d$, then a shortest description of x contributes at least $2^{-n-K_\delta(n)+d}$ to $P_{D,\delta}(n)$. If there were more that 2^{n+c-d} many such x, then $P_{D,\delta}(n) > 2^{n+c-d} 2^{-n-K_\delta(n)+d} = 2^c 2^{-K_\delta(n)}$, a contradiction.
(ii). This follows from (i), by letting $d = n - b$. □

3.4. The Π^1_1 version of ML-randomness. In what follows we use λ to denote the product measure on 2^ω. Of course we are also using λ to denote Lebesgue measure on the unit interval, but under the identification provided by binary representation of a conull subset of the unit interval with a conull subset of 2^ω these two senses of λ coincide.
A ML-TEST is a sequence $(S_m)_{m\in\omega-\{0\}}$ of uniformly Σ_1 over $L(\omega_1^{\mathrm{ck}})$ open sets such that $\forall m \in \omega - \{0\}\ \lambda S_m \leq 2^{-m}$. Z is ML-RANDOM if Z passes each ML-test in the sense that $Z \notin \bigcap_m S_m$.
Let MLR denote the class of ML-random sets, and Non-MLR its complement in 2^ω.
For $b \in \omega^+$, let $\mathbf{R}_b = [\{x \in 2^{<\omega} :\ K(x) \leq |x| - b\}]$.

PROPOSITION 3.6. *$(\mathbf{R}_b)_{b\in\omega-\{0\}}$ is a ML-test.*

Proof. The condition "$K(x) \leq |x|-b$" is equivalent to $\exists\sigma, \alpha\ \mathbf{U}_\alpha(\sigma) = x\ \&\ |\sigma| \leq |x|-b$, which is a Σ_1-property of x and b. Hence the sequence of open sets $(\mathbf{R}_b)_{b\in\omega-\{0\}}$ is Σ_1. To show $\lambda\mathbf{R}_b \leq 2^{-b}$, let V_b be the set of strings in $\mathbf{R}_b$ which are minimal under the prefix ordering. For each $x \in V_b$, $K(x) \leq |x|-b$, so $2^{-|x^*|} \geq 2^b 2^{-|x|}$ (here x^* denotes a shortest $\mathbf{U}$-description of x). Because $\mathbf{U}$ is a prefix free machine,

$$1 \geq \sum\{2^{-|x^*|} : x \in V_b\} \geq 2^b \sum\{2^{-|x|} : x \in V_b\},$$

hence $\lambda\mathbf{R}_b \leq 2^{-b}$. □

We now begin on the analogue of Schnorr's theorem for the hyperarithmetical context. Recall that Schnorr's original theorem stated that Z is ML-random with respect to recursively enumerable tests if and only if for $K_{r.e.}$, the prefix free complexity defined in terms of the universal recursively enumerable prefix free machine, there exists b with $K_{r.e.}(Z|n) > n - b$ at every n.
Although the statement of this theorem carries across with only the obvious changes, the proof does not. The new obstacle arises at limit stages. We describe the measure theoretic lemmas which are necessary to meet this fresh obstacle, then we prove the hyperarithmetical version of Schnorr, and then finally we indicate why the original proof refuses a cut and paste adaption to the present context.
In the arguments below we think of 2^ω as coming equipped with an enumeration of the standard basis consisting exactly of all the clopen sets.

LEMMA 3.7. *Given an open $S \subseteq 2^\omega$ such that $S \in L(\omega_1^{\mathrm{ck}})$, a clopen subset U of 2^ω and a rational $\epsilon > 0$, we may in an effective (i.e., Δ_1 over $L(\omega_1^{\mathrm{ck}})$) manner obtain a clopen set C such that $C \supset U \setminus S$ and $\lambda(C) < \lambda(U \setminus S) + \epsilon$.*

Proof. From S one may effectively (in the above sense) obtain an $L(\omega_1^{\mathrm{ck}})$ sequence $(\sigma_n)_{n\in\omega}$ such that $S = \bigcup_n[\sigma_n]$. For each k consider the clopen set

$$C_k = \bigcup\{[\rho] : |\rho| = k\ \&\ \rho \subseteq U\ \&\ \forall n\, \sigma_n \not\preceq \rho.\}$$

Then $\bigcap_k C_k = U \setminus S$, since $[\sigma_n] \cap C_k = \emptyset$ whenever $k \geq |\sigma_n|$. So one may in an effective (over $L(\omega_1^{\mathrm{ck}})$ way determine k such that $\lambda(C) \leq \lambda(U - S) + \epsilon$. □

Next we cover an effective sequence of basic clopen sets by such a sequence which is almost disjoint in the sense that the sum of the measures is small.

PROPOSITION 3.8. *Let $\alpha \mapsto U_\alpha$ be a Σ_1 over $L(\omega_1^{\mathrm{ck}})$ function mapping ordinals to basic clopen sets in 2^ω. Then we may find, uniformly in the sequence $(U_\alpha)_{\alpha<\omega_1^{\mathrm{ck}}}$*

and rational $\epsilon > 0$, a Σ_1 over $L(\omega_1^{\mathrm{ck}})$ mapping $\alpha \mapsto C_\alpha$ of ordinals to clopen sets such that at each $\beta \le \omega_1^{\mathrm{ck}}$

$$\bigcup_{\alpha<\beta} U_\alpha \subset \bigcup_{\alpha<\beta} C_\alpha, \text{ and } \sum_{\alpha<\beta} \lambda(C_\alpha) \le \lambda(\bigcup_{\alpha<\beta} U_\alpha) + \epsilon.$$

Proof. Let $(\rho_n)_{n\in\omega}$ be a computable listing of $2^{<\omega}$. Let

$$X_\beta = \{m : [\rho_m] \subset U_\beta\} \setminus \{m : [\rho_m] \subset \bigcup_{\alpha<\beta} U_\alpha\}$$

(see the explanatory remark after the proof of Theorem 3.9.) As long as U_β is not included in the union of the earlier U_α's we will have $X_\beta \neq \emptyset$. Clearly, $\beta \mapsto X_\beta$ is Σ_1 over $L(\omega_1^{\mathrm{ck}})$. At each stage β, applying 3.7 for $S = \bigcup_{\alpha<\beta} U_\alpha$ and $U = U_\beta$, we choose a clopen set C_β such that

$$U_\beta \setminus (\bigcup_{\alpha<\beta} U_\alpha) \subset C_\alpha,$$

$$\lambda(C_\beta) < \lambda(U_\beta \setminus \bigcup_{\alpha<\beta} U_\alpha) + \sum_{m\in X_\beta} 2^{-m-2} \cdot \epsilon.$$

Then at any stage β we have

$$\sum_{\alpha<\beta} \lambda(C_\alpha) - \lambda(\bigcup_{\alpha<\beta} U_\alpha) \le$$

$$\sum_{\alpha<\beta} (\lambda(C_\alpha) - \lambda(U_\alpha \setminus \bigcup_{\gamma<\alpha} U_\gamma)) \le$$

$$\sum_{m\in\bigcup_{\alpha<\beta} X_\alpha} 2^{-m-2}\epsilon \le \frac{\epsilon}{2} < \epsilon.$$

□

This proposition allows itself to be further massaged. Given the sequence $(C_\beta)_\beta$ arising as above, we can break them up into basic clopen sets, and in this way find a new sequence $([x_\beta])_\beta$, each $x_\beta \in 2^\omega$,

$$\bigcup C_\beta = \bigcup [x_\beta],$$

$$\sum \lambda(C_\beta) = \sum \lambda[x_\beta],$$

and the assignment $\beta \mapsto x_\beta$ is still Σ_1 over $L(\omega_1^{\mathrm{ck}})$.

THEOREM 3.9. *The following are equivalent.*

(i) *Z is ML-random*
(ii) *$\exists b\ \forall n\ K(Z \upharpoonright n) > n - b$, that is, $\exists b\ Z \notin \mathbf{R}_b$.*

Proof. (i)⇒(ii) holds because $(\mathbf{R}_b)_{b\in\omega-\{0\}}$ is a ML-test. For (ii)⇒(i), suppose that (i) fails for Z, that is $Z \in \bigcap_m S_m$ for a ML-test $(S_m)_{m\in\omega-\{0\}}$. We may assume that $\lambda S_m \le 2^{-2m-1}$ and $S_m = \bigcup_{\beta<\omega_1^{\mathrm{ck}}} U_\beta^m$ where each U_β^m is basic clopen, and the associated map $(m, \beta) \mapsto U_\beta^m$ Σ_1 over $L(\omega_1^{\mathrm{ck}})$.

Following 3.8 we may find a Σ_1 over $L(\omega_1^{\mathrm{ck}})$ map $(m, \beta) \mapsto x_\beta^m$ such that at each m

$$S_m \subset \bigcup_\beta [x_\beta^m],$$

$$\lambda(\bigcup_\beta [x_\beta^m]) < 2^{-2m}.$$

In particular, at each m,

$$\sum_{\beta} 2^{m-|x^m_\beta|} < 2^m \sum \lambda([x^m_\beta]) < 2^m(2^{-2m}) = 2^{-m},$$

and hence $L = \{\langle |x^m_\beta| - m, x^m_\beta\rangle : m \in \omega, \beta < \omega_1^{\mathrm{ck}}\}$ is a KC set. Let M_d be the prefix-free machine for L given by the KC-Theorem 3.2. Given b, let $m = b+d+1$. Since $Z \in S_m$, $x^m_i \prec Z$ for some i. Because of the axiom enumerated for compressing $x = x^m_i$, $K(x) \leq |x| - m + d + 1 = |x| - b$. □

Certain steps were taken in the course of the proof above which did not need to be considered in Schnorr's original argument. There is a kind of continuing approximation, and giving ground, with the sets X_α from 3.8 serving as a kind of clock – letting us know how much to give, so that at the end of the process we did not give in too far.
The reason for this extra precaution can be illustrated by the following kind of example which could arise in 3.8 if we try to steadfastly insist that

$$\sum_{\alpha<\beta} \lambda(C_\alpha) = \lambda(\bigcup_{\alpha<\beta} U_\alpha).$$

We could be given an open set S with $\lambda(S) < 2^{-2}$, S enumerated as $(U_\alpha)_{\alpha\in\omega_1^{\mathrm{ck}}}$. In the naive attempt to copy Schnorr's earlier argument we try to effectively build a corresponding KC set, $\{\langle r_\alpha, y_\alpha\rangle : \alpha < \omega_1^{\mathrm{ck}}\}$ which has

$$\sum 2^{-r_\alpha} = \lambda(S),$$

and at each α we have some ordinal $\gamma(\alpha) < \omega_1^{\mathrm{ck}}$

$$\bigcup_{\beta<\alpha} U_\beta = \bigcup_{\beta<\gamma(\alpha)} [y_\beta],$$

$$\sum_{\beta<\gamma(\alpha)} 2^{-r_\beta} = \lambda(\bigcup_{\beta<\alpha} U_\beta).$$

It could then happen that at ω we already have that $\bigcup_{n<\omega} C_n$ contains the interval $[0, 1/4]$ with the exception of a Cantor set of positive measure. Eventually we are going to settle on some stage $\gamma(\omega)$ with $\bigcup_{\beta<\gamma(\omega)}[y_\beta]$ equal to that complement. But there is no way of doing this which will rule out the possibility of the unpleasant discovery at the next stage that $U_{\gamma(\omega)+1}$ includes some non-null piece of the Cantor set, at which there is no way of choosing the next $\langle r_\beta, y_\beta\rangle$ without overbiting.
Thus, Z is ML-random just if Z is in the complement of some open set $\mathbf{R}_b$, that is the set of paths through a Σ^1_1 subtree of $2^{<\omega}$. Recall the version of the Gandy low basis theorem for Σ^1_1-sets (folklore): A non-empty Σ^1_1 class always contains a member Z with $\mathcal{O}^Z \leq_T \mathcal{O}$. Thus:

Proposition 3.10. *There is a ML-random set Z such that $\mathcal{O}^Z \leq_T \mathcal{O}$.*

One can also consider the analog of Chaitin's halting probability, in order to obtain a ML-random set Z which is left-Π^1_1. Let

$$\Omega = \lambda(\mathrm{dom}\mathbf{U}) = \sum\{2^{-|\sigma|} : \mathbf{U}(\sigma)\downarrow\}.$$

Adapting Chaitin's proof one can show that Ω is ML-random.

3.5. An analog of the Kučera-Gács Theorem. Finite hyperarithmetical reducibility $\leq_{\mathrm{fin-h}}$ between sets $X, Y \subseteq \omega$ is a restriction of hyperarithmetical reducibility, where the use is finite for each input.

Definition 3.11.
(i) A $\mathrm{fin-h}$ reduction procedure is a partial function $\Phi : 2^{<\omega} \mapsto 2^{<\omega}$ with Π^1_1 graph (or, equivalently, Σ_1 over $L(\omega_1^{\mathrm{ck}})$ graph) such that the domain is closed under prefixes, and, if $\Phi(t)\downarrow$, then $s \preceq t \Rightarrow \Phi(s) \preceq \Phi(t)$.
(ii) $A = \Phi^Z$ if $\forall n \exists m\ \Phi(Z \upharpoonright m) \succeq A \upharpoonright n$. $A \leq_{\mathrm{fin-h}} Z$ if $A = \Phi^Z$ for some $\mathrm{fin-h}$ reduction.
(iii) $A \leq_{\mathrm{wtt-h}} Z$ if $A = \Phi^Z$ for some $\mathrm{fin-h}$ reduction such that the use is recursively bounded.

Notice that if A is hyperarithmetical, then $A \leq_{\mathrm{fin-h}} Z$ for any Z, because $\{\sigma : \sigma \preceq A\}$ is Π^1_1.

Theorem 3.12. *Let $b \in \omega - \{0\}$ and let Q be the (closed Σ^1_1) class of ML-random sets $2^\omega - \mathbf{R}_b = \{Z : \forall n\ K(Z \upharpoonright n) > n - b\}$. For each A, there is $Z \in Q$ such that $A \leq_{\mathrm{wtt-h}} Z$.*

Proof. For $S \subseteq 2^\omega$, $\lambda(S|z)$ denotes the local measure $2^{|z|}\lambda(S \cap [z])$. For each n, $\lambda(S)$ is the average, over all strings z of length n, of the local measures $\lambda(S|z)$.

Lemma 3.13. *Suppose $S \subseteq 2^\omega$ is measurable, $r \in \omega$ and $\lambda(S|x) \geq 2^{-r}$. Then there are $y_0, y_1 \succeq x$, $|y_i| = |x| + r + 1$, such that $\lambda(Q|y_i) \geq 2^{-(r+1)}$ for $i = 0, 1$.*

Proof. We may assume that $x = \emptyset$. Let y_0 be a string of length $r+1$ such that $\lambda(Q|y_0)$ is greatest among those strings, in particular $\lambda(Q|y_0) \geq 2^{-(r+1)}$ since the average is $\geq 2^{-(r+1)}$. Since $\lambda(Q \cap [y_0]) \leq 2^{-(r+1)}$,

$$\sum_{y \neq y_0\ \&\ |y|=r+1} \lambda(Q \cap [y]) \geq 2^{-(r+1)},$$

or $\sum_{y \neq y_0\ \&\ |y|=r+1} \lambda(Q|y) \geq 1$. Hence there is a further $y_1 \neq y_0$ of length $r+1$ such that $\lambda(Q|y_1) \geq 2^{-(r+1)}$. ◊

Let $f(r) = r(r+1)/2$ and consider the closed Π^1_1 class $\widehat{Q}$ given by the tree

$$\{y : \forall r. f(r) \leq |y|\ [\lambda(Q|(y \upharpoonright f(r))) \geq 2^{-r}]\}\ .$$

Define a tree $\mathcal{T}$ of strings $(x_\tau)_{\tau \in 2^{<\omega}}$, where $|x_\tau| = f(|\tau|)$. Let $x_\emptyset = \emptyset$. If x_τ has been defined, let $x_{\tau 0}$ be the leftmost y on $\widehat{Q}$ such that $x_\tau \prec y$ and $|y| = f(|\tau|+1)$. Let $x_{\tau 1}$ be the rightmost such y. By Lemma 3.13, $x_{\tau 0}$ and $x_{\tau 1}$ exists and are distinct. For each A, the ML-random set Z coding A is simply the path $\bigcup_{\tau \prec A} x_\tau$ determined by A.
We verify $A \leq_{\mathrm{wtt-h}} Z$, where f is the computable bound on the use. Given an input n, to determine $A(n)$, let $x = Z \upharpoonright f(n)$ and let $y = Z \upharpoonright f(n+1)$. Find α such that $Q_\alpha \cap \{v : x \preceq v\ \&\ |v| = |y|\ \&\ v <_L y\} = \emptyset$, or $Q_\alpha \cap \{v : x \preceq v\ \&\ |v| = |y|\ \&\ v >_L y\} = \emptyset$. In the first case, output 0, while in the second case, output 1. □

4. Lowness properties

4.1. *K*-triviality.

Definition 4.1. (i) *A is K-TRIVIAL if, for some $b \in \omega$,*

$$\forall n\ K(A \restriction n) \leq K(n) + b.$$

(ii) *Given a limit ordinal* $\eta \leq \omega_1^{\mathrm{ck}}$*,* A *is* K-TRIVIAL AT η *if for some* $b \in \omega$*,*

$$\forall n\, K_\eta(A \restriction n) \leq K_\eta(n) + b.$$

Thus K-trivial is the same as K-trivial at ω_1^{ck}.
Using the Π^1_1-version of the KC Theorem (Theorem 3.2 above), one can adapt the cost function construction from [**1**] (also see [**10**, Theorem 4.2] in order to show:

THEOREM 4.2. *There is a K trivial Π^1_1 set A which is not hyper-arithmetical.* □

Recall our convention that no element is enumerated into a Π^1_1 set at a limit stage. Then, A is K-trivial at η iff for some $b \in \omega$, $\forall n \forall \alpha < \eta\ \exists \beta < \eta\ K_\beta(A \restriction n) \leq K_\alpha(n) + b$.
Fix b and $\eta \leq \omega_1^{\mathrm{ck}}$. The subsets of ω which are K-trivial at η are the paths of the following tree:

$$T_{\eta,b} = \{s : \forall t \preceq s\, K_\eta(t) \leq K_\eta(|t|) + b\}.$$

If $\eta < \omega_1^{\mathrm{ck}}$ then $T_{\eta,b}$ is hyper-arithmetical, by Δ_1 comprehension in $L(\omega_1^{\mathrm{ck}})$ (see [**11**, p. 67]) $T_{\eta,b}$ is a subset of $2^{<\omega}$ which is Δ_1 (with $\eta \in L(\omega_1^{\mathrm{ck}})$ as a parameter).
Let g_D be the function obtained in Theorem 3.4, where $D(x) = |\mathbf{U}(x)|$. Recall that η a g_D-closed if $\forall \alpha < \eta\, [g_D(\alpha) < \eta]$. We show that for such $\eta < \omega_1^{\mathrm{ck}}$, if A is K-trivial at η, then A is hyper-arithmetical.

THEOREM 4.3. *Let $\eta < \omega_1^{\mathrm{ck}}$ be g_D-closed.*

(i) *There is $\mathbf{c} \in \omega$ such that the following holds: for each b there are at most $2^{\mathbf{c}+b}$ sets that are K-trivialat η with constant b.*
(ii) *If a set A is K-trivial at η for $\eta < \omega_1^{\mathrm{ck}}$ then A is hyper-arithmetical.*
(iii) *Each K-trivial set is computable in $\mathcal{O}$.*

Proof. By Theorem 3.5 (ii), there is a constant $\mathbf{c}$ such that the size of each level of $T_{\eta,b}$ is at most $2^{\mathbf{c}+b}$, which shows (i). Note that each path A of $T_{\eta,b}$ is isolated, hence recursive in $T_{\eta,b}$. For (ii), if $\eta < \omega_1^{\mathrm{ck}}$ this shows A is hyper-arithmetical. For (iii), note that since $K \leq_T \mathcal{O}$, the tree $T_{\omega_1^{\mathrm{ck}},b}$ is computable in $\mathcal{O}$. Now argue as in (ii). □

PROPOSITION 4.4. *If A is K-trivial via b and $\omega_1^A = \omega_1^{\mathrm{ck}}$, then A is hyper-arithmetical.*

Proof. We show that A is K-trivial at η via b, for some g_D-closed η. We define by recursion a function $h : \omega \to \omega_1^{\mathrm{ck}}$ which is Σ_1 over $L^{\omega_1^{\mathrm{ck}}}[A]$: let $h(0) = 0$, and

$$h(n+1) = \mu\beta > g_D(h(n))\, \forall m \leq n\, K_\beta(A \restriction m) \leq K_\beta(m) + b.$$

Since A is K-trivial, $h(n)$ is defined for each $n \in \omega$. Let $\eta = \sup(\mathrm{range}(h))$, then $\eta < \omega_1^A = \omega_1^{\mathrm{ck}}$, so η is as required. □

4.2. Lowness for ML-randomness. The notion of ML-randomness and the theorems in subsection 3.4 can be relativized to oracle sets A in the usual way. MLR^A denotes the class of sets which are ML-random relative to A. A set A is LOW FOR ML-RANDOM if $\mathsf{MLR}^A = \mathsf{MLR}$. A is a STRONG BASE FOR ML-RANDOMNESS if $A \leq_{\mathrm{fin-h}} Z$ for some $Z \in \mathsf{MLR}^A$ (see Definition 3.11). By Theorem 3.12, if A is low for ML-random then A is a strong base for ML-randomness. (We say *strong* base because the reduction is $\leq_{\mathrm{fin-h}}$ and not merely $\leq_h$. The theory for $\leq_h$ remains unexplored.)

THEOREM 4.5. *A is a strong base for ML-randomness iff A is hyper-arithmetical.*

Proof. If A is hyper-arithmetical, then $A \leq_{\mathrm{fin-h}} Z$ for each Z, so A is a strong base. Now suppose that A is a strong base, namely $A = \Phi^Z$ for some fin − h reduction Φ and $Z \in \mathsf{MLR}^A$. First we show that $\omega_1^A = \omega_1^{\mathrm{ck}}$. We may assume that A is not hyper-arithmetical, so that $\lambda\{Y : A = \Phi^Y\} = 0$ (see [**11**, 2.4.IV]). For each k, let

$$V_k = [\{\rho : A \restriction k \leq \Phi^\rho\}]^{\preceq} = [\{\rho : \exists \alpha < \omega_1^{\mathrm{ck}} \, A \restriction k \leq \Phi^\rho_\alpha\}]^{\preceq}$$

(recall that, for a set of strings G, $[G]^{\preceq}$ is the open set generated by G). If $\omega_1^A > \omega_1^{\mathrm{ck}}$, then V_k is uniformly hyperarithmetical relative to A, so the function $k \mapsto V_k$ is in $L(\omega_1^A)[A]$. Note that, since $\omega_1^A > \omega_1^{\mathrm{ck}}$, the binary statement "$\lambda W \leq q$", for open $W \in L(\omega_1^A)[A]$ and a rational q, is Σ_1 over $L(\omega_1^A)[A]$. So the function

$$h(n) = \mu k \, \lambda V_k \leq 2^{-n}$$

is also Σ_1 over $L(\omega_1^A)[A]$. Then $(V_{h(n)})_{n \in \omega - \{0\}}$ is a ML-test relative to A which succeeds on Z, contrary to the hypothesis that $Z \in \mathsf{MLR}^A$.
The principal part of the proof is to show that a strong base A is K-trivial. Then, by Proposition 4.4, A is hyper-arithmetical. To show that A is K-trivial, one proceeds exactly as in the proof of the corresponding theorem in the r.e. case, [**2**, Thm 2.1] (also see [**8**]), with mere notational changes. One restricts the enumeration into open sets $C^\tau_{d,\alpha}$ to successor stages, and for limit stages η, one defines $C^\tau_{d,\eta} = \bigcup_{\alpha<\eta} C^\tau_{d,\alpha}$. The verification works as before, making use of our Π^1_1 version of the Kraft-Chaitin theorem. □

COROLLARY 4.6. *Each low for ML-random set is hyperarithmetical.*

Proof. Immediate from Theorems 3.12 and 4.5. □

We first had a more technical but direct proof of this corollary, along the lines of the direct proof that in the r.e. case, each low for ML-random is Δ^0_2 (see [**9**]).

5. An even stronger effective notions of randomness

We consider the even stronger randomness notion where the null properties to be avoided are simply the Π^1_1 sets of reals (we will write "Set", capitalized, when we mean a set of reals).
Some preliminaries. According to [**11**, 5.2.I], a Π^1_1-Set (also called predicate) $S(Z)$ can be written in the normal form $\forall f \exists n R(\overline{f}(n), Z)$ where R is recursive and $\overline{f}(n)$ is defined to be the tuple $(f(0), \ldots, f(n-1))$. This gives an indexing of the Π^1_1-sets. Sacks also obtains a recursive functional Ψ^Z_R such that Ψ^Z_R is a set of codes for tuples in $\omega^{<\omega}$ (sequence numbers) and $S(Z) \Leftrightarrow \Psi^Z_R$ is well-founded (under the prefix relation on sequence numbers). Using the length-lexicographical (also called Kleene-Brouwer) ordering, one can effectively "linearize" Ψ^Z_R (see [**11**, proof of Thm 3.5.III]). Thus there is a Turing functional Φ such that for each Z, Φ^Z is a real which is a code for a linear order with domain ω, such that

$$S(Z) \;\Leftrightarrow\; \Phi^Z \text{ is well-ordered.}$$

By [**11**, Exercise 1.11.IV], we have

LEMMA 5.1. *The binary relation "$\lambda S > q$" is Π^1_1, where S is an (index for a) Π^1_1-Set and q a rational.*

In particular, λS is a left-Π^1_1 real. A Δ^1_1 Set B is given by Π^1_1-indices for B and $2^\omega - B$. By the Lemma, the function which assigns to a Δ^1_1 set its measure is Δ_1 over $L(\omega_1^{\mathrm{ck}})$.

A randomness notion based on Π^1_1*-Sets of reals.* Recall that to introduce ML-randomness, both in the classical (r.e. case and in the form of Subsection 3.4, we used a test concept based on uniformly r.e., or Π^1_1, open sets. In both cases there is a universal test $(\mathbf{R}_b)_{b\in\omega}$, namely, $\bigcap_k S_k \subseteq \bigcap_b \mathbf{R}_b$ for each ML-test $(S_k)_{k\in\omega}$. (Naively one would want to use tests of uniformly Π^1_1 Sets $\mathcal{S}_k$ in place of the open sets, $\lambda \mathcal{S}_k \leq 2^{-k}$. However, for such a test, $\bigcap_k \mathcal{S}_k$ is a null Π^1_1-Set, and trivially each null set also induces a test. Thus the null sets are the analog of tests.)

We show that there is a largest one. There is a topological counterpart: Kechris [**4**] shows that there a *largest thin* Π^1_1-Set (a Set is thin if it has no perfect subset). See also [**7**, Thm 4F.4].

Theorem 5.2. *There is a null* Π^1_1*-set* Q *such that* $S \subseteq Q$ *for each null* Π^1_1*-Set* S.

Proof. We claim that one may effectively assign to each Π^1_1 class S a Π^1_1 class $\widehat{S} \subseteq S$ such that $\lambda(\widehat{S}) = 0$ and if $\lambda(S) = 0$ then $\widehat{S} = S$. Then to obtain Q we take the union of all $\widehat{S}$, as S ranges over the Π^1_1 sets.

To prove the claim, let Φ be a functional representing S in the sense above. At each α let S_α be the collection of all $Z \in S$ for which the corresponding well ordering Φ^Z has rank less than α. Let $\widehat{S}$ be the set of all Z such that there exists some $\alpha < \omega_1^Z$ with

$$Z \in S_\alpha$$

$$\lambda(S_\alpha) = 0.$$

Following 5.1 membership of Z in $\widehat{S}$ is uniformly $\Sigma_1(Z)$ over $L(\omega_1^Z)[Z]$. Thus by Spector-Gandy $\widehat{S}$ is Π^1_1. Since the set of Z such that $\omega_1^Z = \omega_1^{\mathrm{ck}}$ is conull, $\widehat{S}$ is the union of a null set and all S_α, $\alpha < \omega_1^{\mathrm{ck}}$ which are null, hence $\widehat{S}$ is null. When S is null every S_α, $\alpha < \omega_1^Z$, will be null, and hence we will have $\widehat{S} = S$. □

The Set Q has the interesting property that $Q \cap R \neq \emptyset$ for each non-empty Π^1_1-Set R. For if $\lambda R > 0$ then R has a hyperarithmetic member X by [**11**, Thm 2.2.IV], so that $\{X\}$ is a Π^1_1 Set of measure 0.

Definition 5.3. $Z \in 2^\omega$ *is* strongly random *if it avoids every null* Π^1_1 *set. Or, equivalently, if it is not an element of the largest null* Π^1_1 *set. Let* $\mathcal{S}$ *denote the class of strongly random sets.*

This notion is called Σ^1_1-random in [**11**]. Of course it implies the Π^1_1 version of ML-randomness, and is in fact much stronger. For instance, each strongly random set Z satisfies $\omega_1^Z = \omega_1^{\mathrm{ck}}$, since the class $\{Z : \omega_1^Z = \omega_1^{\mathrm{ck}}\}$ is Σ^1_1 and has measure 1. By Gandy's basis theorem, some strongly random set satisfies $\mathcal{O}^Z \leq_T \mathcal{O}$.

The analog of van Lambalgen's Theorem [**12**] holds:

Proposition 5.4. *For any sets* X, Y,

$$X \oplus Y \in \mathcal{S} \ \Leftrightarrow\ X \in \mathcal{S}^Y \ \&\ Y \in \mathcal{S}.$$

Proof. For the "⇒" direction, note that the Set $\mathcal{L} = \{X \oplus Y : X \in \mathcal{S}^Y\}$ is Σ^1_1. Since $\lambda \mathcal{S}^Y = 1$ for each Y, by Fubini's Theorem $\mathcal{L}$ has measure 1. Hence $\mathcal{S} \subseteq \mathcal{L}$.

For the "$\Leftarrow$" direction, let $\mathcal{S}[B] = \{A : A \oplus B \in \mathcal{S}\}$. Then the Set $\{B : \lambda\mathcal{S}[B] = 1\}$ is Σ^1_1 and has measure 1, again by Fubini's Theorem (otherwise there are rationals $\epsilon > 0$ and $q < 1$ such that $\lambda\{B : \lambda\mathcal{S}[B] \leq q\} \geq \epsilon$, so that $\lambda\mathcal{S} = \int_Y (\lambda\mathcal{S}[Y])d\lambda \leq \epsilon q + (1 - \epsilon) < 1$). Thus if $Y \in \mathcal{S}$ then $\lambda\mathcal{S}[Y] = 1$. Since $\mathcal{S}[Y]$ is Σ^1_1 relative to Y, $X \in \mathcal{S}^Y$ implies $X \in \mathcal{S}[Y]$, that is, $X \oplus Y \in \mathcal{S}$. □

It is unknown if there is a low for strongly random set which is not hyper-arithmetical.

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