Low for random sets: the story

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Abstract. A set $A \subseteq \mathbb{N}$ is low for random if every random set already is A-random. This preprint contains some proofs which have been superseded, but still provide interesting methods which can be used elsewhere. 1. We prove that each such set is Δ^0_2, thereby answering to the negative a question of Kucera and Terwijn [**4**]. 2. Subsequently we strengthen this by showing that low for random sets are low for K. 3. Then we use martingales to obtain yet a stronger result (the final one in [**7**]): if each (Martin-Löf-) random set is computably random in A, then A is low for K.

1. Introduction

The most commonly accepted notion of algorithmic randomness is the one introduced by Martin-Löf [**5**]. A *Martin-Löf test* is a uniformly r.e. sequence (U_n) of open sets in Cantor Space 2^ω such that $\mu(U_n) \le 2^{-n}$, where μ is the usual Lebesgue measure on 2^ω. A real X is *Martin-Löf random* if it passes each test in the sense that $X \notin \bigcap_n U_n$. Schnorr [**9**] proved that a real X is random in this sense if and only if the algorithmic prefix complexity K of all its initial segments is large, namely $\forall n\, K(X \upharpoonright n) \ge n - \mathcal{O}(1)$.

A *lowness property* of a real A says that, in some sense, A has low computational power when used as an oracle. We require that such a property be downward closed under $\le_T$. The usual lowness, $A' \equiv_T \emptyset'$, is an example. The lowness property Low(MLRand)) is itself based on relative randomness: A is *low for random* if each random real X is already random relative to A, i.e. X passes all A-r.e. tests, which means the class of A–random sets is as large as possible, namely it coincides with the class of random sets. Intuitively, this means that A has low computational power, because A is not able to recognize any more sets as nonrandom than a recursive oracle.

1. Kucera and Terwijn [**4**] construct a nonrecursive low for random set which is r.e. They ask whether there exists such a set outside Δ^0_2. In our first result we answer their question to the negative. Nies and Stephan gave a characterization of low for random sets in order to prove that the set of indices of r.e. low for random sets is Σ^0_3. We use a variant of this characterization.

Key words and phrases. Kolmogoroff complexity, randomness, low for random.

A result of Kucera [**3**] implies that each low for random A satisfies $A' \le_T A \oplus \emptyset'$. Thus, by our result A is low (namely, $A' \le_T \emptyset'$). This supports the intuition that A is close to being recursive.
A further notion of being almost recursive is the notion of an K–trivial set: B is K–trivial if, for some constant d, $\forall n H(B \upharpoonright n) \le H(n)+d$, that is, up to a constant, the K–complexity of all the initial segments of B is as low as possible. The study of low for random sets parallels the investigation of K–trivial sets (in fact, it is unknown at present if the two concepts differ). Using a similar construction as in [**4**], one can construct a nonrecursive r.e. K–trivial set. Zambella [**10**] proved that all K–trivials are Δ^0_2, by showing that the Δ^0_2 tree $T_d = \{y \in 2^{<\omega} : \forall z \subseteq y H(z) \le |z| + d\}$ has only finitely many (in fact, $O(2^d)$) infinite paths (and clearly B is one of them). Our basic approach is similar: based on the two parameters from our characterization of low for random sets A, we define a Π^0_2 tree which has only finitely many paths, and A is one of them. However, the argument is more complex than in Zambella's proof, since the tree is defined dynamically.
2. We consider the reals which, when used as an oracle, do not decrease K.

DEFINITION 1.1. *A is* low for K *if* $\forall y\ K(y) \le K^A(y) + \mathcal{O}(1)$.
Let $\mathcal{L}K$ *denote this class of reals.*

In the second (stronger) theorem, we show that $\mathrm{Low}(\mathsf{MLRand}) \subseteq \mathcal{L}\mathrm{K}$.
3. A real Z is *computably random* if no computable betting strategy (martingale) which is monotone, i.e. bets on the bit positions in their natural order, succeeds on Z. If no strategy betting in *any* order succeeds, the real is called *Kolmogorov-Loveland random*. Denoting the classes of such reals by CRand and KLRand, repectively, the inclusions $\mathsf{MLRand} \subseteq \mathsf{KLRand} \subset \mathsf{CRand}$ hold. A persistent open question is whether the first inclusion is strict as well [**1, 6**].
Given randomness notions $\mathcal{C} \subseteq \mathcal{D}$, let $\mathrm{Low}(\mathcal{C}, \mathcal{D})$ denote the class of oracles A such that $\mathcal{C} \subseteq \mathcal{D}^A$. We write $\mathrm{Low}(\mathcal{C})$ for $\mathrm{Low}(\mathcal{C}, \mathcal{C})$ We prove in fact that $\mathrm{Low}(\mathsf{MLRand}, \mathsf{CRand}) \subseteq \mathcal{L}\mathrm{K}$, which implies both $\mathrm{Low}(\mathsf{MLRand})) \subseteq \mathcal{L}K$ and $\mathrm{Low}(\mathsf{KLRand}) \subseteq \mathcal{L}\mathrm{K}$. However, it is unknown if non-recursive reals in $\mathrm{Low}(\mathsf{KLRand})$ exist. If not, then at least for some oracle X, the relativized classes KLRand^X and MLRand^X are distinct.
We introduce some notation, but also rely on [**4**] for further background and terminology. A p.r. function $M : 2^{<\omega} \to 2^{<\omega}$ (also called machine) is *prefix free* if any two strings in its domain are incompatible. A p.r. functional $M : 2^\omega \times 2^{<\omega} \to 2^{<\omega}$ (also called oracle machine) is *prefix free* if $M^X = \lambda y.M(X, y)$ is a prefix free function for each set $X \subseteq \mathbb{N}$. Let $(M_d)_{d>0}$ be an effective listing of all oracle prefix free machines. We work with the universal machine V given by $V^X(0^{d-1}1w) = M_d(w)$. We define the prefix free Kolmogoroff complexity relative to X to be $H^X(y) = \min\{|p| : V^X(p) = y\}$.
We use the usual topological notions for Cantor space 2^ω, and denote the Lebesgue measure on 2^ω by μ. For a string y, $[y]$ denotes the clopen set $\{X : y \subseteq X\}$ (so that $\mu[y] = 2^{-|y|}$). A set U is Σ^0_1 if $U = \bigcup_{y \in W}[y]$ for some r.e. set of strings W (in particular, U is open).

2. The Kraft-Chaitin Theorem

In this Section we provide an important tool for our constructions.

DEFINITION 2.1. *An r.e. set* $W \subseteq \mathbb{N} \times 2^{<\omega}$ *is a* Kraft-Chaitin set (KC set) *if*
$$\sum_{\langle r,y\rangle \in W} 2^{-r} \le 1.$$

Given W, for any $E \subseteq W$, let the weight of E be $wt(E) = \sum\{2^{-r} : \langle r, n\rangle \in E\}$. If $X \subseteq \mathbb{N}$, the weight (in the context of W) is

$$wt(X) = \sum_{n \in X} \sum\{2^{-r} : \langle r, n\rangle \in W\}.$$

The pairs enumerated into such a set W are called *axioms*.

THEOREM 2.2 (Chaitin, [**2**], Thm 3.2). *From a Kraft-Chaitin set W one can effectively obtain a prefix machine M such that*

$$\forall \langle r, y\rangle \in W \exists w\ (|w| = r\ \&\ M(w) = y)$$

We say that M is a prefix machine for W.

For later reference, we give a quick review of the proof (based on [**2**, Thm 3.2]).

PROOF. Let $\langle r_n, y_n\rangle_{n\in\mathbb{N}}$ be an effective enumeration of W. At stage n, we will find a string w_n of length r_n, and we set $M(w_n) = y_n$. We let $D_{-1} = \{\lambda\}$. At each stage $n \geq 0$ we have a finite set D_{n-1} of strings all of whose extensions are unused. It is useful to think of a string x as the half-open subinterval $I(x) \subseteq [0, 1)$ of real numbers whose binary representation extends x. Let z_n be the longest string in D_n of length $\leq r_n$. Choose w_n so that $I(w_n)$ is the leftmost subinterval of $I(z_n)$ of length 2^{-r_n}, i.e., let $w_n = z_n 0^{r_n - |z_n|}$. To obtain D_n, first remove z_n from D_{n-1}. If $w_n \neq z_n$ then also add the strings $z_n 0^i 1$, $0 \leq i < r_n - |z_n|$.
One checks inductively that for each $n \geq 0$ the following hold:

(a) z_n exists
(b) all the strings in D_n have different lengths
(c) $\{I(z) : z \in D_n\} \cup \{I(w_i) : i \leq n\}$ is a partition of $[0, 1)$

We prove (a) for $n \geq 0$, assuming (b) and (c) for $n-1$ (these are trivial statements for $n = 0$). If z_n fails to exist, then r_n is less than the length of each string in D_{n-1}, so that $2^{-r_n} > \sum\{2^{-|z|} : z \in D_{n-1}\}$ by (b) for $n-1$. Then $\sum_{i=0}^{n} 2^{-r_i} > 1$ since $\sum\{2^{-|z|} : z \in D_{n-1}\} + \sum_{i=0}^{n-1} 2^{-r_i} = 1$ by (c) for $n-1$. This contradicts the assumption that W is a KC-set.
Next, (b) for n holds if $w_n = z_n$. Otherwise $|z_n| < |w_n|$ but also $|w_n|$ is less than the next shortest string in D_{n-1}, so (b) holds by the definition of D_n. Finally, (c) is satisfied by the definition of D_n. □

3. Two preliminary results

We first provide an oracle version of the Kraft-Chaitin Theorem, and a characterization of the low for random sets.

DEFINITION 3.1. *Consider an r.e. set $L \subseteq \mathbb{N} \times 2^{<\omega} \times 2^{<\omega}$. The elements $\langle r, z, \gamma\rangle$ of L (also called axioms) will be written in the form $\langle r, z\rangle^\gamma$. L is called an oracle Kraft–Chaitin (oracle KC) set if, for all $\rho \in 2^{<\omega}$,*

$$L^\rho = \{\langle r, y\rangle : \exists \gamma \subseteq \rho\ \langle r, z\rangle^\gamma \in L\} \tag{1}$$

is a Kraft–Chaitin set.

PROPOSITION 3.2. *From an index for an oracle KC set L, one can effectively obtain an index d for an oracle prefix machine M_d^X such that*

$$\forall X \subseteq \mathbb{N}\ \forall \langle r, z\rangle^\gamma \in L\ [\gamma \subseteq X \ \Rightarrow\ \exists w(|w| = r\ \&\ M_d^X(w) = z)].$$

W. Merkle has pointed out that one can in fact obtain the result with $b = 1$.

Proof. For each real X, $L^X = \{\langle r, y\rangle : \exists\gamma \subseteq X\ \langle r, z\rangle^\gamma \in L\}$ is a KC set relative to X. Applying the construction in the proof of Theorem 2.2, we obtain an index d (which only depends on an r.e. index for L) such that M_d^X is an oracle prefix machine as desired. $\square$

Theorem 3.3 (with F. Stephan, also see [**8**]). *A is low for random* $\Leftrightarrow$
$\exists b \in \mathbb{N}\ \exists R \subseteq 2^\omega\ (\ R \in \Sigma^0_1\ \&\ \mu R < 1\ \&$

$$\forall z \in 2^{<\omega}[K^A(z) \leq |z| - b\ \Rightarrow\ [z] \subseteq R]). \tag{2}$$

Proof. To gain insight we reformulate the condition in Proposition 3.3. For each $X \subseteq \mathbb{N}$ and $b \in \mathbb{N}$, let $R_b^X = \{z : \exists w \subseteq z\ K^X(w) \leq |w| - b\}$, so that $(R_b^X)_{b\in\mathbb{N}}$ is a universal Martin-Löf test relative to X (namely, $\bigcap_b R_b^X = 2^\omega - \mathrm{RAND}(X)$). Then A is low for random iff $\bigcap_b R_b^A \subseteq \bigcap_n S_n$ for some (unrelativized) Martin Löf test (S_n). By a method of Kucera (see e.g. [**4**, Lemma 1.5]), this is equivalent to $\bigcap_b R_b^A \subseteq R$ for some Σ^0_1 set $R \subseteq 2^{<\omega}$ such that $\mu R < 1$ (to obtain (S_n), one "iterates" R). The condition in Proposition 3.3 states that already for some b, R_b^A is contained in such a set R. The direction from right to left follows.
To prove the converse direction, suppose $A \in \mathrm{Low}(\mathsf{MLRand}))$, and fix a Σ^0_1 set $R_0 \subset 2^\omega$ of measure less that 1 containing all the nonrandom sets (say R_0 is the first component of a universal Martin Löf test). We claim that, for some string x such that $[x] \not\subseteq R_0$ and some $m \in \mathbb{N}$,

$$\forall y \supseteq x\ (K^A(y) \leq |y| - m\ \Rightarrow\ [y] \subseteq R_0). \tag{3}$$

Suppose otherwise. Define a sequence of strings (y_m) as follows: let y_0 be the empty string, and let y_{m+1} be some proper extension of y_m such that $K^A(y) \leq |y| - m$ but $[y_{m+1}] \nsubseteq R_0$. Then $Y = \bigcup_m y_m$ is not A–random, but Y is random since $Y \notin R_0$. Now fix x, m such that (3) holds, and let $R = \{z : xz \in R_0\}$. Then $R \neq 2^\omega$. Since the Π^0_1 set R^c is contained in the random sets and no random set can be in a Π^0_1 set of measure 0, $\mu R < 1$.
Since $K^A(xz) \leq K^A(x) + K^A(z) + \mathcal{O}(1)$, letting $b = K^A(x) + m + \mathcal{O}(1)$, we obtain, for each y,

$$K^A(z) \leq |z| - b\ \Rightarrow\ K^A(xz) \leq |xz| - m\ \Rightarrow\ [xz] \subseteq R_0\ \Rightarrow\ [z] \subseteq R.$$

$\square$

4. A first restriction: Any low for random set is Δ^0_2

Theorem 4.1. *Any low for random set is Δ^0_2.*

Proof. Suppose A is low for random. We will define a Π^0_2 subtree T of $2^{<\omega}$ which has only finitely many paths (a path is a maximal linearly ordered subset), A being one of them. Relativizing a standard argument for Π^0_1 trees to $\emptyset'$, we conclude $A \leq_T \emptyset'$: fix a string σ such that A is the only path extending σ. We use that the set of strings not on T is r.e. in $\emptyset'$. For an input $p \geq lg(\sigma)$, use the oracle $\emptyset'$ to compute s such that only one string of length p extending σ has not appeared in the complement of T, and output that string.
Fix b, U as in Proposition 3.3. We will define an effective sequence of finite trees $(T_s)_{s\in\mathbb{N}}$ such that

(4) $\qquad T = \{\sigma : \exists^\infty t \ \sigma \in T_t\}$.

T clearly is Π^0_2. Along with (T_s) we will enumerate an OKC set L. The following is not essential, but reduces the complexity of the construction below. By the Recursion Theorem, we can assume that an r.e. index for L is given. Then, in fact, we can assume that an index d for a machine M_d corresponding to L is given, because, for any r.e. set $B \subseteq \mathbb{N} \times 2^{<\omega} \times 2^{<\omega}$, we can effectively obtain an index for an OKC set $\widetilde{B}$ such that $\widetilde{B} = B$ in case B already is an OKC set. Let M_d be the machine effectively obtained from $\widetilde{B}$ via Proposition 3.2. Our construction effectively produces an OKC set L from d. Thus, if $B = L$, then B is an OKC set and M_d is a machine for L.
Let $c = b + d$. We define T_t and L_t by recursion on t. We will enumerate finitely many axioms $\langle r, y\rangle^\sigma$, $r = |y| - c$, into L at stages t.
Let T_0 contain only the empty string and let $L_0 = \emptyset$. Suppose $t > 0$ and T_s and L_s have been determined for $s < t$. We define T_t by a subrecursion on the length of strings. We begin by putting the empty string into T_t. Suppose currently the string γ, $|\gamma| < t$, is a leaf of T_t. For $i = 0, 1$, let $s_i < t$ be the greatest s such that $\gamma\widehat{\ }i \in T_s$ or $s_i = 0$ if there is no such stage. For $i = 0$ or $i = 1$, if

$$\forall y[\langle |y| - c, y\rangle^{\gamma\widehat{\ }i} \in L_{s_i} \Rightarrow [y] \subseteq U_t]],$$

then put $\gamma\widehat{\ }i$ into T_t.
It remains to define L_t by enumerating finitely many axioms at stage t. We first show that, no matter how we do this, as long as $L = \bigcup_t L_t$ is an OKC set (and $B = L$), A will be a path of T.

Lemma 4.2. *Suppose L is an OKC set. Then each set A satisfying (2) is a path of T.*

Proof. If $\langle |y| - c, y\rangle^\sigma \in L$ then, for each set X extending σ, $M_d^X(w) = y$ for some w of length $|y| - c$. Hence $V^X(0^{d-1}1w) = y$ and $H^X(y) \le |y| - b$ (recall that $c = b + d$).
Suppose A satisfies (2). We show by induction on p that $A \restriction p$ is on T for each p. We can suppose that $p > 0$. By inductive hypothesis, there are infinitely many s such that $A \restriction p - 1 \in T_s$. Suppose for a contradiction that t is greatest such that $\sigma = A \restriction p \in T_t$. Then, by the above remarks (for $\sigma = A|p$ and $X = A$), there is $v > t$ such that $[y] \subseteq U_v$ for each of the finitely many y such that $\langle |y| - c, y\rangle^\sigma \in L_t$. Then at a stage $s \ge v$ such that $A \restriction p - 1 \in T_s$, we define T_s to contain σ, contrary to the choice of t. $\diamondsuit$

We now describe how to enumerate L in a way that T has few incompatible strings. For each σ, if $g = \sum\{2^{-r} : \langle r, y\rangle^\sigma$ enters L at $s\}$, then we say we *put measure g on σ at s*. We view this as a cost, as it conflicts with our goal to make K an OKC set. Let $k = 2^{c+3}$. We view k–element subsets of $2^{<\omega}$ as strategies whose goal it is to increase $\mu(U)$ (U belongs to the opponent). Let α, β denote such strategies, and let α_i denote the i-th element of α in lexicographical order. If $\alpha \subseteq T_t$, we say α is *available at t*. Slightly simplifying, when α *acts* at s, it puts axioms $\langle |y_i| - c, y_i\rangle^{\alpha_i}$ into L for each $i < k$, where the y_i's are distinct strings of the same lenght r such that $[y_i] \cap U_s = \emptyset$. Suppose α is available again at a stage $t > s$. Our cost at s was to put measure 2^{-r+c} on each α_i, while the opponent now enumerated each $[y_i]$

into U, so that he needed to increase $\mu(U)$ by $k2^{-r}$. Since $k = 2^{c+3}$, he puts the eightfold amount into U than we put on each α_i. If some α is available infinitely often, this leads to a contradiction.

Instead of the single strings y_i we actually use pairwise disjoint clopen sets C_i which are disjoint from U_s and sets previously reserved by other strategies, and have measure $2^{-c}p_\alpha$, where p_α is a *fixed* small quantity. Thus our strategy α puts measure p_α on each α_i when acting, and the opponent puts $8p_\alpha$ into U. Since $\mu(U) < 2$, we can only put $1/4$ in total on each path that way. Thus any α which is even available 2^{l-2} times (where p_α was chosen to be 2^{-l}) leads to a contradiction. Then, by definition of T, there are no k incomparable strings on T.

In the construction below, we have to ensure that clopen sets C_i as specified can be chosen, and we need to limit the negative effects of strategies which do not reappear on the tree after they act. Let $n_\alpha \in \mathbb{N} - \{0\}$ be a code number assigned to α in some effective way.

Definition of L_t

We assume T_s $(s \leq t)$ and L_s $(s < t)$ have been defined. At stage t, for any α which is available, i.e., a subset of T_t, do the following in order of n_α. Pick clopen sets $C_0, \ldots, C_{k-1}$ which are pairwise disjoint, disjoint from U_t and from sets previously reserved by other strategies, and have measure $2^{-c}p_\alpha$ (the precise value of p_α is defined below). For each $i < k$ and each string $y \in C_i$ which is minimal under inclusion of strings, enumerate an axiom $\langle |y| - c, y\rangle^{\alpha_i}$ into L. Note that this puts measure p_α on each α_i. We say α *acts via* $C_0, \ldots, C_{k-1}$. This completes the definition.

If α acts at t via $C_0, \ldots, C_{k-1}$ and is not available at any later stage, then this action has the following negative effects:

(a) It jeopardizes (1) for each i and each ρ extending α_i, by wasting measure on α_i
(b) It keeps all strings y, $[y] \subseteq C_i$ for some i, permanently away from assignment to other strategies.

We will define p_α (in advance) so small that these effects can be tolerated. To limit (a), we want to ensure that the total measure put on each σ by all such strategies together is at most $2^{-|\sigma|-2}$. Thus we require that

$$p_\alpha \leq 2^{-q(\alpha)-2}2^{-n_\alpha},$$

where $q(\alpha)$ is the maximum length of a string in α.

Next, by hypothesis on U, fix a rational $q > 0$ such that $\mu U < 1 - q$. To limit (b), we want that the total measure of the strings kept away is less that $q/2$. Thus we require that

$$k2^{-c}p_\alpha \leq 2^{-n_\alpha}q/2,$$

Let p_α be the greatest number of the form 2^{-l} satisfying those two conditions.

We notice some features of the construction and verify that L is an OKC set. By the conditions on the numbers p_α and since $\mu U^c > q$, at any stage we have a clopen set of measure $\geq q/2$ at our disposal. Since $k2^{-c}p_\alpha \leq q/2$ by the second condition, this gives enough space to choose new C_i's for a strategy α which wants to act. (Note, however, that we may not be able to choose singleton C_i's.)

Let $\langle r_j, y_j\rangle^{\sigma_j}_{j\in\mathbb{N}}$ be the effective list of axioms in L produced by the construction (putting the finitely many enumerated at each stage in some order). For each string ρ, let $S_\rho = \sum\{2^{-r_j} : \sigma_j \subseteq \rho \;\&\; \langle r_j, y_j\rangle^{\sigma_j}$ is enumerated by a strategy α which is available again at a later stage$\}$. We claim that

$$S_\rho < 1/4.$$

For when α acts at s via $C_0, \ldots, C_{k-1}$, it puts measure p_α on σ_j. When α is available at a later stage t, then $C_i \subseteq U_t$ for each i. Since the C_i were chosen pairwise disjoint and disjoint from U_s and sets chosen by other strategies, the measure of U increases by at least $k2^{-c}p_\alpha = 8p_\alpha$ due to any action of an α which becomes available again. Since $\mu U < 2$, we conclude that $S_\rho < 1/4$.

To see that L is an OKC set, fix ρ. We need to show that, for each ρ, $\sum_{\langle r,y\rangle \in L^\rho} 2^{-r} \leq 1$, where L^ρ is defined in (1). This sum equals $S_\rho + \sum\{2^{-r_j} : \sigma_j \subseteq \rho \;\&\; \langle r_j, y_j\rangle^{\sigma_j}$ is enumerated by a strategy α which is *not* available again at a later stage$\}$. By the definition of the numbers p_α, the second sum is at most $1/4$. Note that the arguments above work regardless whether or not the set L we construct equals the given r.e. set. Thus we always produce some OKC set, as required for our application of the Recursion Theorem.

We now verify that T has no k–element subset α of incomparable strings, assuming that a correct index for L was given. For a contradiction, suppose there is such an α. Then α is always available again at some stage after acting. Let ρ be any string in α. Each time α acts, it causes an increase of S_ρ by $p_\alpha = 2^{-l}$. Thus if it acts more than 2^{l-2} times, we contradict $S_\rho < 1/4$.

5. A stronger restriction: low for K

THEOREM 5.1. *Any low for random real is low for K, in a uniform way.*

PROOF. Suppose A is low for random. The proof will be uniform: a constant for the strong K–triviality of A is obtained effectively in b, R, q, where b, R are as in (2) and $q > 0$ is a rational such that $\mu R \leq 1 - q$. We define an effective sequence $(T_s)_{s\in\mathbb{N}}$ of finite subtrees of $2^{<\omega}$ (viewed as characteristic functions) such that the limit tree T given by $T(\gamma) = \lim_s T_s(\gamma)$ exists. The real A is a path of T, and each path of T is K-trivial, via a constant which can be determined effectively from b, R and q. To ensure this, we enumerate a KC set W such that, for some constant c determined below, if $\gamma \in T$ and $K^\gamma(y) = p$, then $\langle p + c, y\rangle \in W$ (so that $K^\gamma(y) \leq p + \mathcal{O}(1)$ by the Kraft-Chaitin Theorem). Of course, the condition "$\gamma \in T$ and $K^\gamma(y) = p$" is only Δ^0_2, so we need to work with approximations. At stage t, if $\gamma \in T_t$, $K^\gamma_t(y) = p$ and some further conditions hold, then we plan to enumerate $\langle p + c, y\rangle$ into W. While defining (T_s) we enumerate an auxiliary oracle KC set L, which ensures that we do not make too many errors in this enumeration of W (putting axioms for strings $\gamma \notin T$), so that W is indeed a KC set. Our enumeration of L at stage t exploits (2) in a way which makes it harder for a string $\gamma \in T_t$ to reappear on T_s at a later stage s.

Preliminaries and the general framework. We may assume that an index d for the oracle prefix machine M_d corresponding to L is given (d can even be obtained effectively in the parameters b, q and a Σ^0_1–index for R). The reason is that, for any index of an r.e. set $Q \subseteq \mathbb{N} \times 2^{<\omega} \times 2^{<\omega}$, we can effectively obtain an index for an oracle KC set $\widetilde{Q}$ such that $\widetilde{Q} = Q$ in case Q already is an oracle KC set. Let d be an index for the oracle prefix machine effectively obtained from $\widetilde{Q}$ via Proposition 3.2. Our construction will effectively produce an oracle KC set L from d (for any

$d \in \mathbb{N}$). By the Recursion Theorem with parameters, we can assume that $Q = L$. Thus Q is an oracle KC set, and M_d is a machine for L.
Let $c \in \mathbb{N}$ be least such that $c \geq b + d$ and $2^{-c} \leq q/2$. We define T_t and L_t by recursion on t. For strings $\gamma \in T_t$ we will enumerate finitely many axioms $\langle r, z\rangle^\gamma$, $r = |z| - c$, into L at stages t. Such an enumeration will cause $z \in R_s$ in case $\gamma \in T_s$ at a later stage s.
Let T_0 contain only the empty string and let $L_0 = \emptyset$. Suppose $t > 0$ and T_s and L_s have been determined for $s < t$. We define T_t by a subrecursion on the length of strings. We begin by putting the empty string into T_t. Suppose currently the string γ, $|\gamma| < t$, is a leaf of T_t. For $i = 0, 1$, let $s_i < t$ be the greatest s such that $\gamma\widehat{\ }i \in T_s$ or $s_i = 0$ if there is no such stage. For $i = 0$ or $i = 1$, if

$$\forall z[\langle |z| - c, z\rangle^{\gamma\widehat{\ }i} \in L_{s_i} \Rightarrow [z] \subseteq R_t]],$$

then put $\gamma\widehat{\ }i$ into T_t.
It remains to define L_t, by enumerating finitely many axioms at stage t. We first show that, no matter how we do this, as long as $L = \bigcup_t L_t$ is an oracle KC set (and $Q = L$), A will be a path of T. Sometimes in variants the limit tree may fail to exists, but we can as well work with the Π^0_2-tree $T = \{\gamma : \exists^\infty t\ \gamma \in T_t\}$.

LEMMA 5.2. *Suppose L is an oracle KC set and M_d is an oracle machine for L in the sense of 3.2. Then each real A satisfying* (2) *is a path of T.*

PROOF. If $\langle |z| - c, z\rangle^\gamma \in L$ then, since M_d is a machine for L, for each set X extending γ, $M_d^X(w) = z$ for some w of length $|z| - c$. Hence $U^X(0^{d-1}1w) = z$ and $K^X(z) \leq |z| - b$ (recall that $c = b + d$).
Suppose A satisfies (2). We show by induction on m that $A \restriction m$ is on T for each m. We may suppose that $m > 0$. By inductive hypothesis, there are infinitely many s such that $A \restriction m - 1 \in T_s$. Suppose for a contradiction that t is greatest such that $\gamma = A \restriction m \in T_t$. Then, by the above remarks (for $\gamma = A \restriction m$ and $X = A$), there is $v > t$ such that $[z] \subseteq R_v$ for each of the finitely many z such that $\langle |z| - c, z\rangle^\gamma \in L_t$. Then at a stage $s \geq v$ such that $A \restriction m - 1 \in T_s$, we put γ into T_s, contrary to the choice of t. $\square$

For each γ, if $g = \sum\{2^{-r} : \langle r, z\rangle^\gamma$ *enters* L *at* $s\}$, then we say we *put measure g on γ at s*. We view this as a cost, as it conflicts with our goal to make L an oracle KC set, which requires that, for each ρ, the total measure put on substrings of ρ be at most 1.
Some more intuition. Recall that if $\gamma \in T_t$ and $K_t^\gamma(y) = p$, then we want to enumerate $\langle p + c, y\rangle$ into W. A *strategy* α is a triple $\langle \sigma, y, \gamma\rangle$, where $\sigma, y, \gamma \in 2^{<\omega}$, $|y| < |\gamma|$ and $|\sigma| \leq |y| + 2\log|y| + c^*$ (σ will be a U^γ-description of y). We start α at a stage t which is least such that $\gamma \in T_t$ & $U_t^\gamma(\sigma) = y$, and γ is the shortest among such strings at t.
Let $p = |\sigma|$. Simplifying, the idea is to choose a clopen set $C = C(\alpha)$, $\mu C = 2^{-(p+c)}$, which is disjoint from R and the sets chosen by other strategies. The strategy α puts an axiom $\langle |z| - c, z\rangle^\gamma$ into L for each string $z \in C$ of minimal length. If at a stage $s > t$, once again $\gamma \in T_s$, then $C \subseteq R_s$. At this stage, we put $\langle p + c, y\rangle$ into W. Using that $\mu R \leq 1$ and that the sets belonging to different strategies are disjoint, we want to argue that W is a KC set. Moreover, L is an oracle KC set, since the measure put on any substring γ of a string ρ is a sum of quantities $2^{-|\sigma|}$, where $U^\gamma(\sigma) = y$ for some y. Then each set L^ρ in (1) is a KC-set.

The problem is to make the sets C chosen by strategies at different stages disjoint. Suppose $\beta \neq \alpha$ is a strategy which chose its set $C(\beta)$ at a stage before stage t. If β, or rather, its last component, has reappeared on the tree, then $C(\beta) \subseteq R_t$, so there is no problem since α chooses its set disjoint from R. However, if β has not reappeared (and it possibly never will), then β keeps away its set from assignment to other strategies. The solution is to build up the set $C(\alpha)$ in small pieces D_α, whose measure is a fixed fraction of $2^{-(p+c)}$. Recall $\alpha = \langle \sigma, y, \gamma \rangle$ and $p = |\sigma|$. If α always reappears after assigning such a set, then eventually $C(\alpha)$ reaches the required measure $2^{-(p+c)}$, in which case we are allowed to enumerate the axiom $\langle p+c, y \rangle$ into W. Otherwise, α only keeps away one single set D_α, whose measure is so small that the union (over all strategies) of sets kept away is at most $q/2$. Thus there is always a clopen set of measure $\geq 1 - \mu R - q/2 \geq q/2$ available for other strategies.

For a strategy $\alpha = \langle \sigma, y, \gamma \rangle$, let $n_\alpha \geq |\sigma|$ be a natural number assigned to α in some effective one-one way.

Inductive definition of L_t and of the sets $C_t(\alpha)$.

Let $L_0 = \emptyset$ and $C_0(\alpha) = \emptyset$ for each strategy α. Suppose $t > 0$, and T_s $(s \leq t)$ and L_s $(s < t)$ have been defined.

1. For each $\gamma \in T_t$, if $\alpha = \langle \sigma, y, \gamma \rangle$ is a strategy, $V_t^\gamma(\sigma) = y$, $V_{t-1}^\gamma(\sigma)$ is undefined and, for σ, y, the string γ is the shortest such string, then start the strategy $\langle \sigma, y, \gamma \rangle$.
2. For each strategy $\alpha = \langle \sigma, y, \gamma \rangle$ which is now running, if $\gamma \in T_t$, then do the following. If $\mu C_{t-1}(\alpha) = 2^{-(|\sigma|+c)}$, then let α end. For the remaining such strategies α, pick pairwise disjoint clopen sets D_α such that $\mu D_\alpha = 2^{-(n_\alpha + c)}$, and

$$D_\alpha \cap R_t = \emptyset \ \& \ \forall \beta \neq \alpha \ [D_\alpha \cap C_{t-1}(\beta) = \emptyset]$$

(we will verify that this is possible). Put D_α into $C(\alpha)$ and, for each string $z \in D_\alpha$ which is minimal under inclusion of strings, enumerate an axiom $\langle |z| - c, z \rangle^\gamma$ into L (this puts measure 2^{-n_α} on γ). We say α *acts via* D_α. This completes the definition of L_t.

Verification. Note that, by definition of T_t, for each $\alpha = \langle \sigma, y, \gamma \rangle$, if $\gamma \in T_t$, then $C_{t-1}(\alpha) \subseteq R_t$. Thus for each strategy β, $\mu(C_{t-1}(\beta) - R_t) \leq 2^{-(n_\beta + c)}$. Then the union S of all such sets, which represents the strings outside R being kept away for assignment for other strategies, has measure at most $q/2$ (recall that $2^{-c} \leq q/2$). Thus we always have a clopen set of measure at least $q/2$ at our disposal at a stage t, which suffices for the strategies α which want to choose sets D_α at stage t.

Let $C(\alpha) = \bigcup_t C_t(\alpha)$. Clearly $\alpha \neq \beta$ implies $C(\alpha) \cap C(\beta) = \emptyset$.

To see that L is an oracle KC set, fix ρ. We need to show that, for each ρ, $\sum_{\langle r,z \rangle \in L^\rho} 2^{-r} \leq 1$, where L^ρ is defined in (1). For each $\gamma \subseteq \rho$, a strategy $\alpha = \langle \sigma, y, \gamma \rangle$ puts measure at most $2^{-|\sigma|}$ on γ, since the maximum measure $C(\alpha)$ can reach is $2^{-(|\sigma|+c)}$. Then, the total put on all substrings of ρ is bounded by $\mu(\mathrm{dom}(U^\rho)) \leq 1$. (Note that we did not assume M_d is an oracle machine for L, as required. Such an assumption is only needed in the proof of Fact 5.2.)

Defining a KC set W which shows that each path of T is K-trivial. We first verify that $\lim_s T_s(\gamma)$ exists. There are only finitely many strategies $\alpha = \langle \sigma, y, \gamma \rangle$. Each time such a strategy acts and then γ reappears on the tree, we increased $\mu C(\alpha)$ by at least $2^{-(n_\alpha + c)}$. So eventually the strategy ends, and the limit exists.

Define W as follows. For each $\alpha = \langle \sigma, y, \gamma \rangle$, if α ends at t, then put $\langle |\sigma| + c, y \rangle$ into W. To verify that W is a KC set, we note that

$$\sum_t \sum\{2^{-(|\sigma|+c)} : \langle|\sigma|+c, y\rangle \text{ is put into } W \text{ via } \langle\sigma, y, \gamma\rangle \text{ at stage } t\} \leq \mu R.$$

For, when α ends at t then $\mu C_{t-1}(\alpha) = 2^{-(|\sigma|+c)}$ and $C_{t-1}(\alpha) \subseteq R$. Since the sets $C(\alpha)$ are pairwise disjoint, the required inequality holds.
Let M_e be a prefix machine for W according to the Kraft-Chaitin Theorem 2.2. We claim that, for each path X of T and each string y, $K(y) \leq K^X(y)+c+e$. For choose a shortest U^X-description σ of y, and choose $\gamma \subseteq X$ shortest such that $|\gamma| > y$ and $U^\gamma(\sigma) = y$. Then at some stage t, we start the strategy $\langle\sigma, y, \gamma\rangle$. Since $\gamma \in T$, the strategy ends and we put $\langle|\sigma|+c, y\rangle$ into W, causing $K(y) \leq K^X(y) + c + e$.
We obtained the constant $c+e$ effectively from the parameters b, R and q, since we used the Recursion Theorem with parameters in the proof. □

6. The final result: each Low(MLRand, CRand) real is low for K

Recall that, if $\mathcal{C} \subseteq \mathcal{D}$ are randomness notions, then $\mathrm{Low}(\mathcal{C}, \mathcal{D})$ denotes the class of oracles A such that $\mathcal{C} \subseteq \mathcal{D}^A$. We review the definition of CRand, but see [**1**] for more details, and also for a definition of Kolmogorov-Loveland randomness. A *martingale* is a function $M : \{0,1\}^* \mapsto \mathbb{R}_0^+$ such that, for all strings x, $M(x0)+M(x1) = 2M(x)$. M *succeeds* on a real Z if $\limsup_n M(Z \upharpoonright n) = \infty$. We write $S(M)$ for this success set. Z is computably random (CRand) if *no* computable martingale M succeeds on Z. By a result of Schnorr [**9**], we can restrict ourselves to $\mathbb{Q}$-valued martingales.

Theorem 6.1. *Each* Low(MLRand, CRand) *real is low for* K*.*

If $\mathcal{C} \subseteq \widetilde{\mathcal{C}} \subseteq \widetilde{\mathcal{D}} \subseteq \mathcal{D}$ are randomness notions, then $\mathrm{Low}(\widetilde{\mathcal{C}}, \widetilde{\mathcal{D}}) \subseteq \mathrm{Low}(\mathcal{C}, \mathcal{D})$. So the following are immediate consequences of the theorem.

Corollary 6.2. *Each* Low(MLRand) *real is low for* K*.*

Corollary 6.3. *Each* Low(KLRand) *real is low for* K*.*

Proof. To prove Theorem 6.1, we apply the usual topological notions for Cantor space 2^ω. For a set S of strings, $[S]$ denotes the open set $\{X : \exists y \in S\ y \prec X\}$, which is identified with the set of strings extending a string in S. For a string y, we write $[y]$ instead of $[\{y\}]$ (so that $\mu[y] = 2^{-|y|}$). An open set $R \subseteq 2^\omega$ is Σ^0_1 if $R = [W]$ for some r.e. set of strings W. Given a string v, we let

$$\mu_v(X) = 2^{-|v|}\mu(X \cap [v]).$$

A *martingale operator* is a Turing functional L such that, for each oracle X, L^X is a total martingale. Let R be any r.e. open set such that $\mu R < 1$ and Non-MLRand $\subseteq R$ (for instance, let $R = [\{z : K(z) \leq |z| - 1\}]$. We will define a martingale operator L. If $A \in$ Low(MLRand, CRand) then $S(L^A) \subseteq$ Non-MLRand, and we can apply the following to $N = L^A$.

Lemma 6.4. *Let N be any martingale such that $S(N) \subseteq$ Non-MLRand.*
Then there are $v \in 2^{<\omega}$ and $d \in \mathbb{N}$ such that $v \notin R$ and

$$\forall x \succeq v[\ N(x) \geq 2^d \Rightarrow x \in R]. \tag{5}$$

Proof. Suppose the Lemma fails. Define a sequence of strings (v_m) as follows: let v_0 be the empty string, and let v_{m+1} be some proper extension y of v_m such that $N(y) \geq 2^m$ but $y \notin R$. Then N succeeds on $Z = \bigcup_n v_n$ but $Z \notin R$. □

Note that $v \notin R$ implies that $\mu_v(R) < 1$ (otherwise let $X \notin R$ be a real extending v, then X is a random real in a Π^0_1 class of measure 0, which is impossible). In the following we fix an enumeration $(R_s)_{s\in\mathbb{N}}$ of R (viewed as a set of strings) such that R_s contains only strings up to length s and is closed under extension within those strings.

We will independently, but uniformly in m, build martingale operators L_m for each $m \geq 1$ which have value 2^{-m} on any input of length $\leq m$. Then $L = \sum_{m\geq 1} L_m$ is a martingale operator (L is $\mathbb{Q}$-valued since the contributions of L_m, $m > |w|$, add up to $2^{-|w|}$). We define L in order to ensure that for each A, if $N = L^A$ and $S(N) \subseteq \mathsf{Non\text{-}MLRand}$, then A is low for K. *Fix an effective listing $(\delta_m)_{m\geq 1}$ of all triples $\delta_m = \langle v, d, u\rangle$, where v is a string, and $d, u \in \mathbb{N}$. Given δ_m, we let $q = 2^{-u}$.* If δ_m represents witnesses v, d in Lemma 6.4 and $0 < q < 1 - \mu_v(R)$, the we can define a KC set W based on L_m showing A is low for K. So only L_m matters in the end. However, we need to consider all δ_m together, since we do not know the witnesses in advance.

Fix m. We define an effective sequence $(T_{m,s})_{s\in\mathbb{N}}$ of finite subtrees of $2^{<\omega}$ (viewed as characteristic functions). The limit tree T_m given by $T_m(\gamma) = \lim_s T_{m,s}(\gamma)$ exists, and if δ_m is a witness, the real A is a path of T_m. Roughly speaking, γ is on $T_{m,s}$ if the condition (5) looks correct at stage s for $N = L^{\gamma}_m$ (the martingale operator where only γ is used as an oracle). Each path of T_m is low for K, since we enumerate a KC set W such that, for some constant c determined below, if $\gamma \in T_m$ and $K^{\gamma}(y) = r$, then $\langle r + c, y\rangle \in W$ (so that $K^{\gamma}(y) \leq r + \mathcal{O}(1)$ by the Kraft-Chaitin Theorem).

Given $\delta_m = \langle v, d, u\rangle$, the value of c is $m + d + u + 3$. A *procedure* α is a triple $\langle \sigma, y, \gamma\rangle$, where $\sigma, y, \gamma \in 2^{<\omega}$, $|y| < |\gamma|$ and $|\sigma| \leq |y| + 2\log|y| + c^*$. We start α at a stage s which is least such that $\gamma \in T_s$ & $U^{\gamma}_s(\sigma) = y$, and γ is the shortest among such strings at s. Now α wants to put $\langle r + c, y\rangle$ into W, where $r = |\sigma|$. But it first needs to cause a clopen set $C \subseteq [v]$ of measure $\mu_v(C) = 2^{-(r+c)}$ into R. Simplifyingly, α chooses such a clopen set $\widetilde{C} = \widetilde{C}(\alpha)$ of that measure, which is disjoint from R_s and the sets chosen by other procedures, and causes (in a way to be specified) $L^X_m(z) \geq 2^d$ for each $X \succeq \gamma$ and each string $z \in \widetilde{C}$ of minimal length. If at a stage $t > s$, once again $\gamma \in T_{m,t}$, then $\widetilde{C} \subseteq R_t$, and α now has permission to put $\langle r + c, y\rangle$ into W. If the sets belonging to different procedures are disjoint, then W is a KC set.

To guarantee disjointness, suppose $\beta \neq \alpha$ is a procedure which chose its set $\widetilde{C}(\beta)$ at a stage before stage s. If $(\beta)_2$, the third component of β, has reappeared on the tree, then $\widetilde{C}(\beta) \subseteq R_s$, so there is no problem since α chooses its set disjoint from R. However, if $(\beta)_2$ has not reappeared (and it possibly never will), then β keeps away its set from assignment to other procedures. The solution is to build up the set $\widetilde{C}(\alpha)$ in small pieces $\widetilde{D}$, whose measure is a fixed fraction of $2^{-(r+c)}$. If α always reappears after assigning such a set, then eventually $\widetilde{C}(\alpha)$ reaches the required measure $2^{-(r+c)}$, in which case α is ‘ allowed to enumerate the axiom $\langle r + c, y\rangle$ into W. Otherwise, α only keeps away one single set $\widetilde{D}$, whose measure is so small that the union (over all procedures) of sets kept away is at most $q/4$. In the formal construction, E_t denotes the union of sets of strings appointed by procedures by stage t. Then the measure of $E_t - R_t$ is at most $q/4$ at any stage.

We discuss how $\alpha = \langle \sigma, y, \gamma\rangle$ ensures $L^X_m(z) \geq 2^d$ for a string z, and each $X \succeq \gamma$. The procedure α “owns” the amount $2^{-(r+m)}$ of the initial capital 2^{-m} of L^X_m,

for any X. So the total capital owned by all procedures is $2^{-m}\Omega^X < 2^{-m}$, for each oracle X. Unless α decides otherwise, its capital is kept constant along both sucessors of a string. The procedure chooses its strings z of the form $y0^{1+r+m+d}$, and "withdraws" its capital at y, increasing $L^X_m(y0)$ by ϵ for oracles $X \succeq \gamma$. To maintain the martingale property, it also has to decrease $L^X_m(y1)$ by ϵ. Now it doubles the capital along z and reaches an increase of 2^d at z.
Let $C_t(\alpha)$ denote the set of strings y' used by α up to stage t. We must ensure $y \notin C_{s-1}(\alpha)$ so that α's capital is still available at y. Such a choice is possible for sufficiently many y, since for all t, $\mu_v \widetilde{C}_t(\alpha) \le 2^{-(r+c)}$, so that $\mu_v C_t(\alpha) \le 2^{-(r+c)}2^{1+r+d+m} = q/4$. We have to choose the extension z outside $[E_{s-1}]$, where E_{s-1} is the set of strings previously appointed by other procedures β, but there is no conflict with such a β as far as the capital is concerned: if $\gamma' = (\beta)_2$ is incomparable with γ then γ and γ' can only be extended by different oracles X. Otherwise α and β own different parts of the initial capital of L^X_m.
We are now ready for the formal definition of the martingale operator L_m. For a procedure $\alpha = \langle \sigma, y, \gamma\rangle$, let $n_\alpha > \max(|\sigma| + m + d + 1, |\gamma|, |v|)$ be a natural number assigned to α in some effective one-one way. Each procedure α defines an auxiliary function $F_\alpha : 2^{<\omega} \mapsto \mathbb{Q}$. The set $\widetilde{C}(\alpha)$ discussed above coincides with the set of minimal strings in $\{w : F_\alpha(w) \ge 2^d\}$. For each oracle X, let

$$L^X_m(w) = 1 + \sum\{F_\alpha(w) : (\alpha)_2 \preceq X\}. \tag{6}$$

Given $\alpha = \langle \sigma, y, \gamma\rangle$, let $r = |\sigma|$. We ensure

(F1) $F_\alpha(w) = 0$ if $|w| \le |\gamma|$,
(F2) $F_\alpha(w) \ge -2^{-(r+m)}$, and
(F3) $\forall w\ F_\alpha(w0) + F_\alpha(w1) = 2F_\alpha(w)$.

Based on those properties, we check that L^X_m is a martingale operator. Firstly, L^X_m is total for each X, and the use of $L^X_m(w)$ is $|w|$, since by (F1) only the procedures α such that $|(\alpha)_2| < |w|$ contribute to the sum in (6). Next, for $p = |w|$,

$$\begin{aligned} L^X_m(w0) + L^X_m(w1) &= 2 + \sum\{F_\alpha(w0) + F_\alpha(w1) : (\alpha)_2 \preceq X \restriction p + 1\} \\ &= 2(1 + \sum\{F_\alpha(w) : (\alpha)_2 \preceq X \restriction p + 1\} \\ &= 2L^X_m(w) \end{aligned}$$

(for the last equality we used (F1)). Finally, $L^X_m(w) \ge 0$, since $F_\alpha(w) \ge -2^{-(r+m)}$, and each α occuring in the sum (6) is based on a computation $U^\gamma(\sigma) = y$ where $\gamma \preceq X$. So, for each w, $L^X_m(w) \ge 2^{-m}(1 - \Omega^X) \ge 0$.
We run a construction for each m. Let $\delta_m = \langle v, d, u\rangle$, $q = 2^{-u}$. The construction works at stages which are powers of 2; letters s, t denote such stages. At stage s we define $T_{m,s}$ and extend the functions $F_\alpha(w)$ to all w such that $s \le |w| < 2s$. For each w such that $s \le |w| < 2s$ and each string η (which may be shorter that w), by the end of stage s we may calculate

$$\overline{L}_m(\eta, w) = 1 + \sum\{F_\alpha(w) : (\alpha)_2 \preceq \eta\}.$$

Stage 1. Let $T_{m,1}$ contain only the empty string and let $F_\alpha(w) = 0$ for each α and each w, $|w| \le 1$. Let $E_1 = \emptyset$.
Stage $s > 1$. Suppose $T_{m,t}$ has been determined for $t < s$, and the functions $F_\alpha(w)$ have been defined for all w, $|w| < s$. Let

$$T_{m,s} = \{\gamma : \forall w \succeq v[(|w| < s \ \&\ \overline{L}_m(\gamma, w) \geq 2^d) \Rightarrow w \in R_s]\}.$$

(1.) If $\mu_v(R_s) > 1 - q$ goto (4.) (If δ_m is a witness this case does not occur.)
(2.) For each $\alpha = \langle \sigma, y, \gamma \rangle$, $n_\alpha < s$, if $U_s^\gamma(\sigma) = y$, $U_{s/2}^\gamma(\sigma)$ is undefined and, for σ, y, the string γ is the shortest such string, then *start* the procedure α.
(3.) Carry out the following for each procedure $\alpha = \langle \sigma, y, \gamma \rangle$ in the order of $n_\alpha < s$. Let $r = |\sigma|$.

(3a.) If α is has been started and $\gamma \in T_{m,s}$, first check if the goal has been reached, namely $\mu_v \widetilde{C}_{s/2}(\alpha) = 2^{-(|\sigma|+c)}$. In that case we put $\langle |\sigma|+c, y \rangle$ into W, and we say that α *ends*. Otherwise we say that α *acts*, and we choose a set $D = D_\alpha \subseteq [v]$ of strings of length s such that $\mu_v D = 2^{-(n_\alpha+u+2)}$ and

$$[D] \cap [R_s \cup E_{s/2} \cup G \cup C_{s/2}(\alpha)] = \emptyset,$$

where $G = \bigcup\{D_\beta : \beta$ has acted at stage s so far$\}$. (We will verify that D exists.) Let $\widetilde{D} = \{y0^{m+d+r+1} : y \in D\}$, put D into $C_s(\alpha)$, and put $\widetilde{D}$ into $\widetilde{C}_s(\alpha)$ and E_s. Note that $|x| < 2s$ for all strings in $\widetilde{D}$, since $m + d + r + 1 < n_\alpha < s$.

(3b.) For each $y \in D$, let $F_\alpha(y) = 0, F_\alpha(y1) = -\epsilon$ and $F_\alpha(y0) = \epsilon$, where $\epsilon = 2^{-(r+m)}$. Now double the capital along $y0^{r+d+m+1}$: for each x, $|x| \leq r + m$, let $F_\alpha(y0x) = \epsilon 2^l$ if $x = 0^l$, and $F_\alpha(y0x) = 0$ otherwise. (This causes $\overline{L}_m(\gamma, w) \geq 2^d$ for each $w \in \widetilde{D}$.)

Go on to the next α.
(4.) For each string w, $s \leq |w| < 2s$ such that $F_\alpha(w)$ is still undefined, let $F_\alpha(w) = F_\alpha(w')$, where $w' \preceq w$ is longest such that $F_\alpha(w')$ is defined. *End of Stage s.*
Verification. We go through a series of Claims. Let $\alpha = \langle \sigma, y, \gamma \rangle$.
Claim 1. *The properties (F1)-(F3) are satisfied.* (F1) holds because when we assign a non-zero value to $F_\alpha(w)$ at stage s, then $|w| \geq s > n_\alpha > |\gamma|$. (F2) and (F3) are satisfied since each y chosen in (3b.) goes into $C(\alpha)$. So by choice of D in (3a.), no future definition of F_α on extensions of y is made except for by (4.)
Claim 2. *α is able to choose D_α in (3a.)* Firstly, by definition of $T_{m,s}$, for each $\beta = \langle \sigma', y', \gamma' \rangle$, if $\gamma' \in T_t$, then $\widetilde{C}_{t/2}(\beta) \subseteq R_t$. Thus for each procedure β, $\mu_v(\widetilde{C}_t(\beta) - R_t) \leq 2^{-(n_\beta+u+2)}$ as $\widetilde{C}_t(\beta) - R_t$ consists of a single set $\widetilde{D}_\beta$. Then, letting $t = s/2$, $\mu_v(E_{s/2} - R_s) \leq 2^{-(u+2)} = q/4$. Secondly, each set D_β chosen during stage s satisfies $\mu_v(D_\beta) \leq 2^{-(n_\beta+u+2)}$, hence $\mu_v G$ never exceeds $q/4$. Thirdly, for each s, $\mu_v \widetilde{C}_s(\alpha) \leq 2^{-(r+c)}$, and hence $\mu_v C_s(\alpha) \leq 2^{r+d+m+1} 2^{-(r+c)} = q/4$.
Since the test in (1.) failed, inside $[v]$ a measure of $q/4$ is available outside $[R_s \cup E_{s/2} \cup G \cup C_{s-1}]$ for choosing D_α.
Claim 3. *Each procedure α acts finitely often.* Each time α acts at s and $s' > s$ is least such that $\gamma \in T_{m,s'}$, we have increased $\mu(\widetilde{C}(\alpha))$ by a fixed amount $2^{-(n_\alpha+c+r)}$. So eventually α ends.
Claim 4. *For each string η, there is a stage s_η such that no procedure α, $(\alpha)_2 \preceq \eta$, acts at any stage $\geq s_\eta$. Moreover, for each w, $|w| \geq s_\eta$, $\overline{L}_m(\eta, w) = \overline{L}_m(\eta, w')$ for some $w' \preceq w$ of length $< s_\eta$.* This follows because there are only finitely many procedures α such that $(\alpha)_2 \preceq \eta$. By (3.) there is a stage s_η by which those procedures have stopped acting, and further definitions $F_\alpha(w)$ are only made in (4.)

Claim 5. *$T_m(\eta) = lim_s T_{m,s}(\eta)$ exists.* Suppose $s \geq s_\eta$ is least such that $\eta \in T_{m,s}$. We show $\eta \in T_{m,s}$ for each $t \geq s$. Suppose $|w| \leq t$ and $\overline{L}_m(\eta, t) \geq 2^d$. By (4.), $_m(\eta, w) = \overline{L}_m(\eta, w')$ for some $w' \preceq w$ of length $< s_\eta$. Then $w' \in R_s$ since $\eta \in T_{m,s}$, and hence $w \in R_t$.

In the following we assume δ_m is a witness for Lemma 6.4 where $N = L^A$.

Claim 6. *A is on T_m.* Given l, let $\eta = A \upharpoonright l$. Suppose $|w'| < s_\eta$ and $\overline{L}_m(w', \eta) \geq 2^d$. Then $L^A(w') \geq 2^d$, since $L^A(w') \geq L^A_m(w') \geq \overline{L}_m(w, \eta)$. By (5), $w' \in R$. Let s be a stage so that all such w' are in R_s. Then by Claim 4, $\eta \in T_{m,t}$ for all $t \geq s$.

Claim 7. *Each path of T_m is low for K.* We first verify that W is a KC set. Note that

$$\sum_s \sum \{2^{-(|\sigma|+c)} : \langle |\sigma| + c, y\rangle \text{ is put into } W \text{ by } \langle \sigma, y, \gamma\rangle \text{ at stage } s\} \leq \mu_v R.$$

For, when α ends at s then $\mu \widetilde{C}_{s/2}(\alpha) = 2^{-(|\sigma|+c)}$ and $\widetilde{C}_{t/2}(\alpha) \subseteq R$. The sets $[\widetilde{C}(\alpha)]$ are pairwise disjoint by the choice of D in (3a.). Hence the required inequality holds.

Let M_e be a prefix machine for W according to Theorem 2.2. We claim that, for each path X of T_m and each string y, $K(y) \leq K^X(y) + c + e$. For choose a shortest U^X-description σ of y, and choose $\gamma \subseteq X$ shortest such that $|\gamma| > y$ and $U^\gamma(\sigma) = y$. Since $\gamma \in T_m$, at some stage t, we start the procedure $\langle \sigma, y, \gamma\rangle$, and the procedure ends. At this stage we put $\langle |\sigma| + c, y\rangle$ into W, causing $K(y) \leq K^X(y) + c + e$. $\square$

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