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INITIAL SEGMENTS OF THE LATTICE OF Π_1^0 CLASSES

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Abstract. We show that in the lattice $\mathscr{E}_\Pi$ of Π_1^0 classes there are initial segments $[\emptyset, P] = \mathscr{L}(P)$ which are not Boolean algebras, but which have a decidable theory. In fact, we will construct for any finite distributive lattice L which satisfies the dual of the usual reduction property a Π_1^0 class P such that L is isomorphic to the lattice $\mathscr{L}(P)^*$, which is $\mathscr{L}(P)$, modulo finite differences. For the 2-element lattice, we obtain a *minimal* class, first constructed by Cenzer, Downey, Jockusch and Shore in 1993. For the simplest new Π_1^0 class P constructed, P has a single, non-computable limit point and $\mathscr{L}(P)^*$ has three elements, corresponding to $\emptyset$, P and a minimal class $P_0 \subset P$. The element corresponding to P_0 has no complement in the lattice. On the other hand, the theory of $\mathscr{L}(P)$ is shown to be decidable.

A Π_1^0 class P is said to be decidable if it is the set of paths through a computable tree with no dead ends. We show that if P is decidable and has only finitely many limit points, then $\mathscr{L}(P)^*$ is always a Boolean algebra. We show that if P is a decidable Π_1^0 class and $\mathscr{L}(P)$ is not a Boolean algebra, then the theory of $\mathscr{L}(P)$ interprets the theory of arithmetic and is therefore undecidable.

§1. Introduction. The study of the lattice $\mathscr{E}$ of computably enumerable sets under inclusion has been one of the central tasks of computability theory since the 1960s. We investigate here initial segments of the lattice $\mathscr{E}_\Pi$ of Π_1^0 classes under inclusion and we compare this lattice with $\mathscr{E}$. For an introduction to Π_1^0 classes, see Cenzer [2]. For more on recent results and open problems, see Cenzer and Jockusch [4]. Some terminology and definitions are given at the end of this section.

It was proved in Nies [13] that the theory of each interval of the lattice $\mathscr{E}$ which is not a Boolean algebra interprets true arithmetic (and is therefore certainly undecidable). However, we will show that in $\mathscr{L}$ there are initial segments $[\emptyset, P] = \mathscr{L}(P)$ which are not Boolean algebras, but which have a decidable theory.

We will construct for any finite distributive lattice L which satisfies the dual of the usual reduction property a Π_1^0 class P such that L is isomorphic to the lattice $\mathscr{L}(P)^*$, which is $\mathscr{L}(P)$, modulo finite differences. We will show that $\mathscr{L}(P)$ is isomorphic to a sublattice of $\mathscr{P}(\mathbb{N})$ which is closed under finite differences and then apply a theorem of Lachlan [10] to conclude that the theory of $\mathscr{L}(P)$ is many-one reducible to the theory of the finite lattice L and is therefore decidable.

The construction of the Π_1^0 class corresponding to a given lattice builds on the construction of a *minimal* Π_1^0 class in [3]. The simplest minimal Π_1^0 class P has a single limit point together with countably many isolated points. P has the property that every Π_1^0 subclass Q of P is either finite or is cofinite in P – furthermore, Q is

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the intersection of P with a clopen set. Thus the lattice $\mathscr{L}(P)$ of Π_1^0 subclasses of P is isomorphic to the class of finite/cofinite subsets of ω and is a Boolean algebra. Such a class plays a role in the lattice $\mathscr{L}$ corresponding to the dual of the role played by a maximal c.e. set in the lattice $\mathscr{E}$.

For the simplest new Π_1^0 class P constructed, P includes a minimal subclass P_0, has a single, non-computable limit point and P has three types of subclasses: (i) finite classes, (ii) cofinite classes, and (iii) classes which are cofinite in P_0 and finite in $P - P_0$. The third type of subclass has no complement in the lattice, which is why the lattice is not a Boolean algebra.

This lattice is isomorphic to the lattice L of subsets of ω containing all finite and cofinite sets together with all sets S containing cofinitely many even numbers and finitely many odd numbers. We observe that L is isomorphic to the dual lattice of complementary sets. The theory of L is seen to be decidable by Lachlan's result, as explained above. It is not hard to see that this lattice may not be realized as the class of c.e. subsets of any c.e. set. Indeed, let $A \subset B$ be c.e. sets and let $\mathscr{L}$ be the interval $[A, B]$ of c.e. sets C such that $A \subset C \subset B$, modulo finite difference. If some set C is not complemented in $\mathscr{L}$, then it follows from repeated applications of the Owings Splitting Theorem ([14], p. 183) that $\mathscr{L}$ is infinite.

There is another notion of decidability. A Π_1^0 class P is commonly defined to be the set $[T]$ of infinite paths through a computable tree T. A node σ of T is said to be *extendible* if there is an infinite path which passes through σ. Then $P = [T]$ is said to be decidable if the set of extendible nodes is computable.

The original construction of a minimal thin class in Theorem 2.2 of [3], p. 88, provides a decidable Π_1^0 class P such that $\mathscr{L}(P)^*$ is the trivial Boolean algebra $\{0, 1\}$.

We will show that if P is decidable and has only finitely many limit points, then $\mathscr{L}(P)^*$ is always a Boolean algebra. Thus if P is a decidable Π_1^0 class and $\mathscr{L}(P)^*$ is not a Boolean algebra, then P has infinitely many limit points.

Finally, we will show that if P is a decidable Π_1^0 class and $\mathscr{L}(P)$ is not a Boolean algebra, then the theory of $\mathscr{L}(P)$ interprets the theory of arithmetic and is therefore undecidable.

As usual, we say that sets A and B are equal modulo finite difference (written $A =^* B$) if the symmetric difference $(A - B) \cup (B - A)$ is finite. For a lattice L of sets, let L^* be the quotient lattice of L modulo the equivalence relation $=^*$. We note here that if A and B are Π_1^0 classes and $A - B$ is finite, then any element of $A - B$ is computable, so that $A - B$ is also a Π_1^0 class. However, the lattice $\mathscr{E}_P$ is *not* closed under finite differences, since if x is a computable element of the Π_1^0 class P and is a limit point of P, then $\{x\}$ is also a Π_1^0 class, but $P - \{x\}$ is not even a closed set and thus is not a Π_1^0 class.

Here are a few basic definitions needed for the discussion of Π_1^0 classes.

Let $\omega^{<\omega}$ be the set of *strings*, that is, functions from a finite initial segment of ω into ω. Similarly, $2^{<\omega}$ is the set of strings with values in $\{0, 1\}$. We write $\sigma \preceq \tau$ for $\sigma \subseteq \tau$. For $x \in \omega^\omega$ and $n \in \omega$, let $x\lceil n = (x(0), x(1), \dots, x(n-1))$. Let $\sigma \prec x$ if $\sigma = x\lceil n$ for some n. We write $|\sigma|$ for the cardinality of the domain of the string σ and often identify σ with the sequence $(\sigma(0), \sigma(1), \dots, \sigma(|\sigma| - 1))$.

A subset T of $\omega^{<\omega}$ is a *tree* if whenever $\tau \in T$ and $\sigma \preceq \tau$, it is the case that $\sigma \in T$. For any tree T, $[T]$ denotes the set of infinite paths through T, that is,

$$[T] = \{x \in \omega^\omega : (\forall n)[x\lceil n \in T]\}.$$

The set of *extendible* nodes of T is defined by

$$Ext(T) = \{\sigma : (\exists x \in [T])[\sigma \prec x]\}.$$

The usual product topology on the space ω^ω has a sub-basis of *intervals*

$$I(\sigma) = \{x : \sigma \prec x\}.$$

With this topology, the closed subsets of ω^ω are exactly those of the form $[T]$ for some tree T. For the subspace 2^ω, the clopen sets are just finite unions of intervals.

A Π_1^0 *class* in ω^ω is an effectively closed set, i.e., one of the form $[T]$ for some *computable* tree T. It is easily seen that an equivalent definition is obtained by requiring T to be primitive recursive, or only co-c.e., instead of computable. A Π_1^0 class is called *decidable* if it has the form $[T]$ for some computable tree T such that $Ext(T)$ is computable. A Π_1^0 class $P \subseteq 2^\omega$ is called a Π_1^0 *class of sets* and clearly has the form $[T]$ for some computable tree $T \subseteq 2^{<\omega}$.

An element x of a Π_1^0 class P is said to be *isolated* if there is some σ such that $P \cap I(\sigma) = \{x\}$. The Cantor-Bendixson derivative $D(P)$ is the set of nonisolated points of P; the αth iteration of D is D^α. The *rank* of x in P is the least α such that $x \in D^\alpha(P) \setminus D^{\alpha+1}(P)$. If P is countable, the *rank* of P is the least ordinal α such that $D^\alpha(P) = \emptyset$.

An infinite Π_1^0 class $P \subseteq 2^\omega$ is said to be *thin* if every Π_1^0 subclass Q of P is equal to $U \cap P$ for some clopen set U. P is said to be *minimal* if every Π_1^0 subclass of P is either finite or cofinite in P.

The first example of a thin Π_1^0 class is due implicitly to D. Martin and M. Pour-El in [11]. They constructed an axiomatizable, essentially undecidable theory T such that every axiomatizable extension of T is finitely axiomatizable over T. It is easy to see that the class of complete extensions of such a theory T is a thin Π_1^0 class, and it is perfect because it contains no computable element. Information on the degrees of such theories may be found in [6] and [7]. Countable thin Π_1^0 classes of arbitrary computable rank, including minimal classes, were constructed in [3].

For more on Π_1^0 classes and the dual concept of c.e. ideals of computable Boolean algebras, see the survey papers by Cenzer [2] and Cenzer and Remmel [5].

§2. Representation of finite lattices. For any Π_1^0 class P, the family $\mathscr{L}(P)$ of Π_1^0 subclasses of P is an initial segment of the lattice of Π_1^0 classes. It is clear that each such initial segment is a sublattice of the full lattice of Π_1^0 classes with least member $\emptyset = 0$ and greatest element $P = 1$, and is distributive. The quotient lattice $\mathscr{L}(P)$ is likewise a distributive lattice. In this section, we characterize the family of finite lattices L which are isomorphic to $\mathscr{L}(P)^*$ for some Π_1^0 class P and also the family of finite lattices L which are isomorphic to $\mathscr{L}(P)^*$ for some decidable Π_1^0 class P.

We will show that $\mathscr{L}(P)$ satisfies the following *Dual Reduction Property*.

Definition 2.1. The lattice $(L, \leq)$ satisfies the dual reduction property if for any $a, b \in L$, there exist $a_1 \geq a$ and $b_1 \geq b$ such that $a_1 \vee b_1 = 1$ and $a_1 \wedge b_1 = a \wedge b$.

Let $\mathscr{L}(P)^*$ denote the lattice $[\emptyset, P]$ modulo finite difference. This lattice will also be distributive and satisfy the dual reduction property.

Proposition 2.2. *For any Π^0_1 class P, the lattices $\mathscr{L}(P)$ and $\mathscr{L}^*(P)$ satisfy the dual reduction property.*

Proof. Let P_1 and P_2 be (nonempty) Π^0_1 subclasses of P and, for $i = 0, 1$, let T_i be a computable tree such that $P_i = [T_i]$ is the set of infinite paths through T_i. We define computable trees $S_i \supset T_i$ such that $S_1 \cap S_2 = T_1 \cap T_2$ and $S_1 \cup S_2 = \{0,1\}^{<\omega}$ and let $Q_i = [S_i]$. It will follow that $Q_1 \cap Q_2 = P_1 \cap P_2$ and that $Q_1 \cup Q_2 = \{0,1\}^\omega$; the desired classes are $Q_1 \cap P$ and $Q_2 \cap P$. For the first condition, suppose that $x \in Q_1 \cap Q_2$. Then $x\lceil n \in S_1 \cap S_2$ for each n, so that $x\lceil n \in T_1 \cap T_2$ for each n, and therefore $x \in P_1 \cap P_2$. For any x, we have that for each n, either $x\lceil n \in S_1$ or $x\lceil n \in S_2$. Thus without loss of generality $x\lceil n \in S_1$ for infinitely many n. Since S_1 is a tree, $x\lceil n \in S_1 \rightarrow x\lceil m \in S_1$ for $m < n$, so that $x\lceil n \in S_1$ for all n and therefore $x \in Q_1$.

The definition of the trees S_i is by recursion on the length of $\sigma \in \{0,1\}^{<\omega}$. First put the empty string in both S_1 and S_2 since it is in $T_1 \cap T_2$. Now assume by induction that for strings σ of length $\leq n$, we have

(i) $\sigma \in S_1 \cup S_2$ and
(ii) $\sigma \in S_1 \cap S_2 \iff \sigma \in T_1 \cap T_2$.

Now for $\tau = \sigma^\frown 0$ or $\sigma^\frown 1$, there are 4 cases; the final case is most important.

(a) If $\tau \in T_1 \cap T_2$, then we put $\tau \in S_1 \cap S_2$.
(b) If $\tau \in T_1 - T_2$, then we put $\tau \in S_1 - S_2$.
(c) If $\tau \in T_2 - T_1$, then we put $\tau \in S_2 - S_1$.
(d) If $\tau \notin T_1 \cup T_2$, then we consider whether $\sigma \in S_1$ or S_2. If $\sigma \in S_2 - S_1$, then we put $\tau \in S_2 - S_1$ and otherwise, we put $\tau \in S_1 - S_2$.

It is easy to check that in each case, if $\tau \in S_i$, then $\sigma \in S_i$, so that each S_i is a tree. The conditions (i) and (ii) follow from the construction by induction on the length of σ. ⊣

In this section, we obtain a converse result.

Theorem 2.3. *For any finite distributive lattice L which satisfies the dual reduction property, there exists a Π^0_1 class Q such that $\mathscr{L}(Q)^*$ is isomorphic to L. Furthermore, the theory of $\mathscr{L}(Q)$ is decidable.*

Proof. Notice that for any finite class Q, $\mathscr{L}(Q)^*$ will be the one-point lattice. For the simplest non-trivial example, the two-point lattice $L = \{0,1\}$, Q must be a *minimal* Π^0_1 class, meaning that every Π^0_1 subclass is either finite or is cofinite in Q. Such a class was constructed in [3]. The construction given below is based on the construction of a minimal Π^0_1 class. Let T_e be a standard enumeration of the primitive recursive trees, so that $P_e = [T_e]$ enumerates the Π^0_1 classes as in [5].

We need the following characterization of the finite distributive lattices satisfying the dual reduction property, which follows from Hermann [8].

Lemma 2.4. *Suppose L is a finite lattice of sets. Then L satisfies the dual reduction property if and only if there exists a tree S with root $\emptyset$ which generates L in the sense that every element of L is a join of a set of nodes.*

Note that S is uniquely determined from L as the set of join-irreducible elements of L.

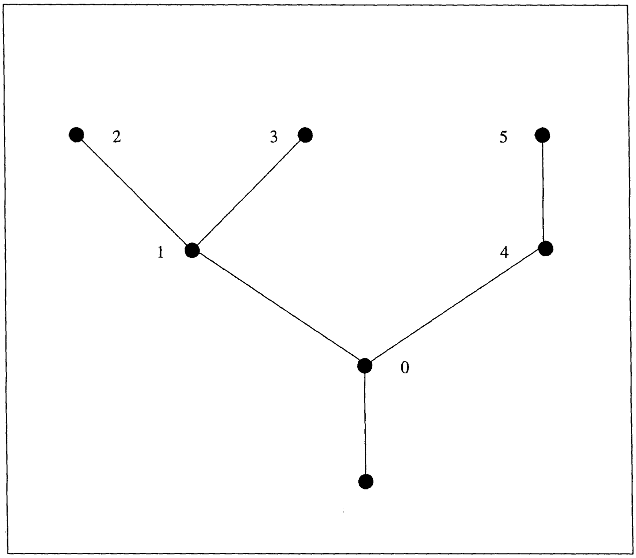

FIGURE 1. S.

To illustrate this idea, let S consist of the following subsets of $\{0,1,2,3\}$: $\emptyset, \{0\}, \{0,1\}, \{0,1,2\}, \{0,1,3\}, \{0,4\}, \{0,4,5\}$. Here a set B is a successor of a set A if $B = A \cup \{b\}$ for some b. S is a lower semi-lattice under the operation of intersection and generates a lattice with the operation of union as follows. The leaves of S are the sets $\{0,1,2\}$, $\{0,1,3\}$ and $\{0,4,5\}$. Thus S generates a lattice $L(S)$ with the addition of 8 sets: $\{0,1,4\}$, $\{0,1,4,5\}$, $\{0,1,2,4\}$, $\{0,1,3,4\}$, $\{0,1,2,3\}$, $\{0,1,2,4,5\}$, $\{0,1,3,4,5\}$ and $\{0,1,2,3,4,5\}$ – the maximum element of $L(S)$. A sketch of the tree S is given above in Figure 1.

Suppose now that the lattice L is generated by a tree S of finite sets with B a successor of A in S if and only if $B = A \cup \{b\}$ for some b as in the above example, so that the new elements are ordered as usual from left to right in the tree.

Each $b \leq m$ may be identified with the unique $B(b) \in S$ such that $B(b) = A \cup \{b\}$ for some $A \in S$. Then we define a partial ordering on $\{0,1,\dots,m\}$ by

$$a \leq_* b \iff B(a) \subset B(b).$$

We may assume that $a \leq_* b$ implies $a \leq b$ (by renumbering if necessary). We may also simplify the problem by assuming, without loss of generality, that there is only one atom $\{0\}$ in L. If there are several atoms $\{i\}$ for $i = 1$ to k, then we can use the construction for one atom to produce disjoint Π^0_1 classes $Q_1, \dots, Q_k$ such that $\mathscr{L}(Q_i)^*$ is isomorphic to the lattice $L_i = \{\emptyset\} \cup \{A \in L : i \in A\}$. It is then easy to see that for $Q = \cup_i Q_i$, $\mathscr{L}(Q)^*$ is isomorphic to L.

Suppose therefore that the generating tree S has a single atom $\{0\}$ and is a family of subsets of $\{0,1,\dots,m\}$. We will construct the class Q with corresponding subclasses Q_A for each $A \in L$ such that every subclass of Q differs from one of the Q_A by a finite set. The classes are constructed so that $A \subset B \iff Q_A \subset Q_B$. It is immediate that $Q_{\{0\}}$ is a minimal Π^0_1 class.

Our goal is to define a Π_1^0 class Q with natural subclasses Q_A for each $A \in L$ so that for each Π_1^0 class $P_e \subset Q$, there is some A such that the difference between P_e and Q_A is finite.

The class Q will have a single limit element x, which will also be the only element of Q containing infinitely many "1"s. If we express x in the form $0^{n_0} * 1 * 0^{n_1} * 1 * \dots$, let $\sigma_0 = 0$ and let $\sigma_k = 0^{n_0} * 1 * \dots * 0^{n_k}$, then the class $Q_{\{0\}}$ will have additional elements $x_{0,k} = \sigma_k * 0 * 1 * 0^\omega$ for each k.

For each $i \leq m$ with $i > 0$, we will have a corresponding *label* 1^{i+1} such that for $A \in L$ and $i \in A$, the elements of Q_A will all contain $0 * 1^{i+1} * 0$ as a substring. In fact, we will characterize Q_A as those elements of Q which have no labels of the form $0 * 1^{m+1} * 0$ for any $m \notin A$. Note that this will make Q_A a Π_1^0 subclass of Q. For each $B = A \cup \{i\} \in S$, we will define a sequence of elements $x_{i,k}$ which have labels for all $i \in B$ and no other labels. This will be done so that for each i, $x_{i,k}$ is an extension of σ_k but not an extension of σ_{k+1}.

A sketch of the class Q for the simple case of $S = \{\emptyset, \{0\}, \{0, 1\}\}$ is given below in Figure 2.

It follows from the above discussion that the map taking $A \in L$ to Q_A is a lattice homomorphism, that is, $A \subset B \iff Q_A \subset Q_B$.

The key to making the subclasses of Q, modulo finite difference, isomorphic to $\mathscr{L}$, is the following condition:

(*) For any $b \leq m$, any e and any A and B in S with $B = A \cup \{b\}$, if $P_e \cap (Q_B - Q_A)$ is infinite, then $Q_B - P_e$ is finite.

Given this condition, we now show that for every Π_1^0 subclass P_e of Q, there exists $C \in L$ such that $P_e = Q_C$ modulo finite difference. Just let

$$C = \bigcup \{A : Q_A - P_e \text{ is finite}\}.$$

Clearly $Q_C - P_e$ is finite. Now suppose by way of contradiction that $P_e - Q_C$ is infinite. Then there must be some $B \in S$ with $P_e \cap (Q_B - Q_C)$ infinite. Let B have minimal cardinality among the set of D such that $P_e \cap (Q_D - Q_C)$ is infinite and let A be the predecessor of B. Then there is a b such that $B = A \cup \{b\}$ and $P_e \cap (Q_B - Q_A)$ is infinite. It now follows from (*) that $Q_B - P_e$ is finite. But the definition of C now requires that $Q_B \subset Q_C$, contradicting the assumption that $P_e \cap (Q_B - Q_C)$ is infinite. Thus P_e and Q_C have a finite difference, as desired.

Now let us see how to obtain this condition in the construction. Recall that we are defining x as the limit of strings σ_k and also defining $x_{b,k}$ for each k and for $b \leq m$ as the limit of, say, $\mu_{b,k}$.

The requirements used in the construction to obtain condition (*) are the following, for each $b \leq m$ and each pair of natural numbers $e \leq j$.

Requirement $R_{b,j,e}$:

(i) if $b = 0$ and $x \in P_e$, then $x_{0,j} \in P_e$;
(ii) if $b > 0$, $a \leq_* b$, and $x_{b,j} \in P_e$, then $x_{a,k} \in P_e$ for all $k \geq j$.

(Recall that P_e is the e-th Π_1^0 class.) Let us demonstrate that these requirements imply the condition (*) given above.

Suppose therefore that $B = A \cup \{b\}$ and that $P_e \cap (Q_B - Q_A)$ is infinite and let $a \in B$. This means that $x_{b,j} \in P_e$ for infinitely many j and thus for some $j \geq e$.

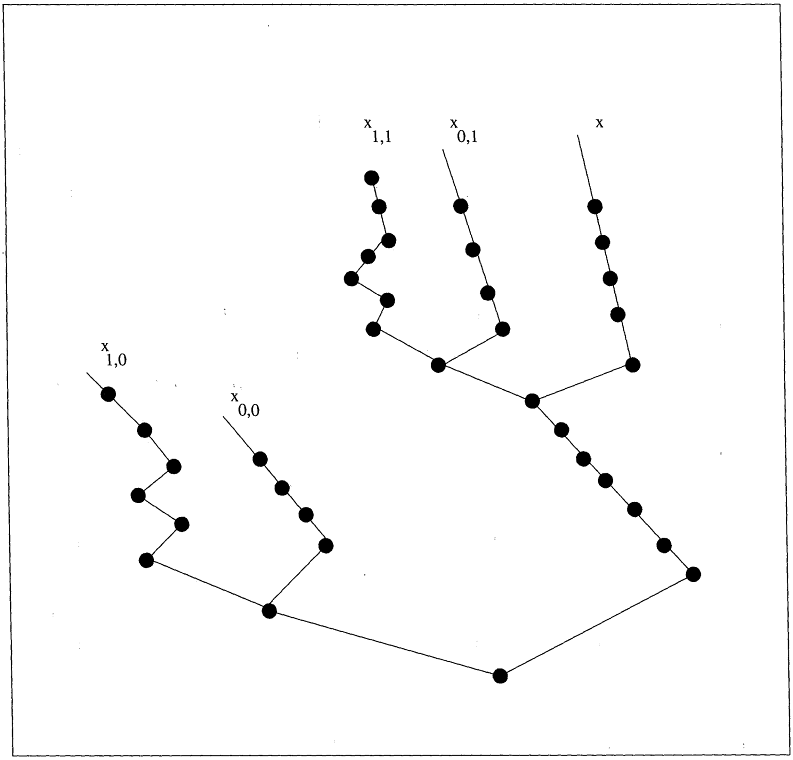

FIGURE 2. Q(S).

Then the requirement $R_{b,j,e}$ implies that $x_{a,k} \in P_e$ for all but finitely many k. Thus $Q_B - P_e$ is finite as desired.

We will show below that these requirements also imply that x is the unique limit point of Q and that x is not computable.

Priority is assigned to the requirements as follows. $R_{a,i,d}$ has higher priority than $R_{b,j,e}$ if either $i < j$, or $i = j$ and $d < e$, or $i = j$ and $d = e$ and $a < b$.

It remains to construct the set Q by a finite injury argument. The construction will proceed in stages. At stage s we will have, for $e \leq s$, strings $\sigma_e^s \prec \mu_{0,e}^s$, containing at least e 1's, such that, for all $e < s$, $\sigma_e^{s\frown}1 \prec \sigma_{e+1}^s$, together with strings $\mu_{b,k}^s$ for $1 \leq b \leq m$ and $k < s$ such that $\mu_{b,k}$ extends σ_k^s but does not extend σ_{k+1}^s. The construction will ensure the existence of the limits $\sigma_e = \lim_s \sigma_e^s$ for each e. The unique limit point x of Q will the union of $\{\sigma_e : e \in \omega\}$. For each b and k, the element $x_{b,k}$ of Q will be the limit of the strings $\mu_{b,k}^s$ in the sense that $x_{b,k}(i) = lim_s \mu_{b,k}^s(i)$ for each i. At the same time we will be defining a sequence $n(0) < n(1) < \ldots$ so that $s \leq n(s)$ and constructing a computable tree T in stages T^s. At stage s, we will have decided whether each finite sequence of length $n(s)$ is in T. This will ensure that T is computable.

We first give an outline of the construction for the case when L is a chain with 3 nodes $0, \{0\}$ and $\{0, 1\}$. We will build a computable tree T with

$$Q = [T] = \{x\} \cup \{x_{0,j} : j < \omega\} \cup \{x_{1,j} : j < \omega\},$$

where x is the unique limit path of Q, also called the *main path*. Thus $Q_0 = \emptyset$, $Q_{\{0\}} = \{x\} \cup \{x_{0,j} : j < \omega\}$ and of course $Q_{\{0,1\}} = Q$. The main path will have the form $0^{n_0} * 1 * 0^{n_1} * \dots$, while the isolated paths will each end in 0^ω. The paths $x_{1,j}$ will each have as a *label* the substring (11), while the other paths will not have this label. The isolated paths $x_{a,j}$ will agree with x at least as far as $0^{n_0} * 1 * \dots * 1 * 0^{n_j}$.

We achieve the requirements $R_{b,j,e}$ by working on the converses. That is, if it looks like $x_{0,j} \notin P_e$ but $x \in P_e$, then we move x to $x_{0,j}$ (by making $\mu^s_{0,j} \prec \sigma^{s+1}_e$) to ensure that $x \notin P_e$. Similarly, if for some $j < k$ it looks like $x_{0,j} \in P_e$ but $x_{0,k} \notin P_e$, then we move $x_{0,j}$ to $x_{0,k}$ to ensure that $x_{0,j} \notin P_e$. The other cases move $x_{1,j}$ to $x_{1,k}$ or move $x_{1,k}$ to $x_{1,j}$ for $j \le k$. The restriction that $e \le j$ will ensure that the construction converges.

To see that these requirements lead to the desired conclusion, we suppose now that some $P_e \subset Q$ and show that Q is equal (modulo finite difference) to one of the three sets Q_A defined above. If P_e is finite, then clearly $P_e = Q_0$ (modulo finite). If P_e is infinite, then it has a limit point, so that $x \in P_e$ and therefore, by part (i) of the Requirement, $x_{0,j} \in P_e$ for all $j \ge e$, so that $Q_{\{0\}} \subset P_e$ (modulo finite). In this case, if P_e contains just finitely many $x_{1,j}$, then $P_e = Q_{\{0\}}$ (modulo finite). If P_e contains infinitely many $x_{1,j}$, then, by part (ii), it must contain all points $x_{1,j}$ and $x_{0,j}$ for $j \ge e$, so that $P_e = Q$ (modulo finite).

We begin the construction by setting $n(0) = 0$ and letting σ^0_0 be the null string.

Now suppose we have completed the construction as far as stage s. Thus we have defined $n(s) > s$ and decided whether $\sigma \in T$ for all strings σ of length $\le n(s)$. We have also defined σ^s_e for all $e \le s$ and also $\mu^s_{a,k}$ for all $a \le m$ and $k < s$ as described above.

At stage $s + 1$, the triple (b, j, e) with $j \ge e$ and $b > 0$ requires action if we have $\mu^s_{b,j} \in T_e$ and we have some $a \le_* b$ and some $k > j$ such that $\mu^s_{a,k} \notin T_e$.

The triple $(0, j, e)$ with $j \ge e$ requires action if $\sigma^s_j \in T_e$ and there is some $k > j$ such that $\mu^s_{0,k} \notin T_e$.

If no triple requires action at stage $s+1$, then we simply extend the tree as follows.

For each $a \le m$, let $0 = a_0 <_* a_1 <_* \dots <_* a_n = a$ list the nodes below or equal to a in S, let

$$\ell(a) = a + 1 + a_0 + 2 + \dots + a_n + 2$$

and let $\ell = \max\{\ell(a) : a \le m\}$ and $n(s + 1) = n(s) + \ell$.

For all $a \le m$ and all $k < s$, let $\mu^{s+1}_{a,k} = \mu^s_{a,k} * 0^\ell$. Let $\sigma^{s+1}_k = \sigma^s_k$ and let $\sigma^{s+1}_{s+1} = \sigma^s_s * 1 * 0^{\ell-1}$. Finally, let

$$\mu^{s+1}_{a,s} = \sigma^s_s * 0^{a+1} * 1^{a_0+1} * 0 * 1^{a_1+1} * 0 * \dots * 0 * 1^{a_n+1} * 0^{\ell-\ell(a)}.$$

Otherwise, let (b, j, e) be the triple with highest priority which requires action at stage $s + 1$ and do the following.

Case I: $b = 0$. Then we have $\sigma^s_j \in T_e$ and $k > j$ such that $\mu^s_{0,k} \notin T_e$. Now the idea is to move σ_j to $\mu^s_{0,k}$, to abandon the part of the tree which branches off between σ_j^s and $\mu_{0,k}^s$ and restart the construction above the new σ_j^{s+1}. The details follow.

Define ℓ as above and let $n(s+1) = n(s) + (s+1-j)(\ell)$. For $i \leq s+1-j$, let

$$\sigma_{j+i}^{s+1} = \mu_{0,k}^s * (1 * 0^{\ell-1})^i.$$

For $a \leq m$ as above and for $i \leq s-j$, let

$$\mu_{a,j+i}^{s+1} = \sigma_{j+i}^{s+1} * 0^{a+1} * 1^{a_0+1} * 0 * 1^{a_1+1} * 0 * \cdots * 0 * 1^{a_n+1} * 0^{(s+1-j-i)\ell-\ell(a)}.$$

For $i < j$, let $\sigma_i^{s+1} = \sigma_i^s$ and for each a, let $\mu_{a,i}^{s+1} = \mu_{a,i}^s * 0^{(s+1-j)\ell}$.

Case II: $b > 0$. Then we have $\mu_{b,j}^s \in T_e$ and we have some $c \leq_* b$ and $k > j$ such that $\mu_{c,k}^s \notin T_e$. Now the idea is to move $\mu_{b,j}$ to $\mu_{c,k}^s$, move σ_{j+1} to σ_{k+1} and to abandon the part of the tree which branches off between σ_j^s and σ_k^s, except for the $\mu_{a,j}$ with $a \neq b$. The tree above σ_{k+1}^s is relabeled and the construction is restarted above σ_s^s. The details follow.

Define ℓ as above and let $n(s+1) = n(s) + (k-j)\ell$. Let $c = c_0 <_* c_1 <_* \cdots <_* c_r = b$ list the nodes of T between c and b and let

$$\mu_{b,j}^{s+1} = \mu_{c,k}^s * 1^{c_1+1} * 0 * \cdots * 0 * 1^{c_r+1} * 0^q,$$

where q is chosen so that $|\mu_{b,j}^{s+1}$ has length $n(s+1)$. For $a \neq b$, let

$$\mu_{a,j}^{s+1} = \mu_{a,j}^s * 0^{(k-j)\ell}.$$

Let $\sigma_j^{s+1} = \sigma_j^s$. For $0 < i \leq s-k$, let

$$\sigma_{j+i}^{s+1} = \sigma_{k+i}^s$$

and for i with $0 < i < k-j$ and for any $a \leq m$, let

$$\mu_{a,j+i}^{s+1} = \mu_{a,k+i}^s * 0^{(k-j)\ell}.$$

For $i \leq k-j+1$, let

$$\sigma_{s+j-k+i}^{s+1} = \sigma_s^s * (1 * 0^{\ell-1})^i$$

and for $a \leq m$ and $0 < i < k-j+1$, let

$$\mu_{a,s+j-k+i}^{s+1} = \sigma_{s+j-k+i}^{s+1} * 0^{a+1} * 1^{a_0+1} * 0 * 1^{a_1+1} * 0 * \cdots * 0 * 1^{a_n+1} * 0^{(k-j-i)\ell-\ell(a)}.$$

Finally, for $i < j$, let $\sigma_i^{s+1} = \sigma_i^s$ and for each a, let $\mu_{a,i}^{s+1} = \mu_{a,i}^s * 0^{(k-j)\ell}$.

In each case, a string σ of length $\leq n(s+1)$ is in T if and only if either $\sigma \prec \sigma_k^{s+1}$ for some k, or $\sigma \prec \mu_{a,k}^{s+1} * 0^t$ for some a, k, t.

CLAIM 2.5. *For every k, the sequence σ_k^s converges to some limit σ_k and for every $a \leq m$ and every e, there is a stage s such that, for all $t \geq s$, $\mu_{a,k}^{t+1}$ is an extension of $\mu_{a,k}^t$ by a string of 0's.*

PROOF. It follows from the construction that we only have $\sigma_k^{s+1} \neq \sigma_k^s$ when we take action on a requirement $R_{b,j,e}$ with $j \leq k$ and similarly we only move $\mu_{a,k}^s$ when we take action on $R_{b,j,e}$ with $j \leq k$.

Thus it suffices to show that for each k, there is a stage after which we never again take action on any requirement $R_{b,j,e}$ with $j \leq k$. We proceed by induction on k. For $k = 0$, the only possible requirements have the form $R_{b,0,0}$. For $b = 0$, one action at stage s will put $\sigma_0^s \notin T_0$ and no later action can injure this requirement. Now suppose that we have reached a stage s_0 such that we never act on requirement

$R_{0,0,0}$ after stage s_0. Then for each $b, c \leq m$ with $b, c \neq 0$, observe that action taken on requirement $R_{b,0,0}$ does not move $\mu_{c,0}$, so that one action taken on requirement $R_{c,0,0}$ will move $\mu_{c,0}$ out of T_0 and no further action will be required.

Now suppose that we have reached a stage s_{k-1} such that no action is ever taken on any requirement $R_{b,j,e}$ with $j < k$ after stage s_{k-1}. Then we see as in the $k = 0$ case above that there will be a stage after which we never act on requirement $R_{0,k,0}$ and then a stage $s_{k,0}$ after which we never act on requirement $R_{b,k,0}$ for any b. Since we always have $e \leq k$ in requirement $R_{b,k,e}$, we can show by induction on $e \leq k$ that there are stages $s_{k,e}$ after which we never act on requirement $R_{b,k,e}$ for any b. Thus after stage $s_k = s_{k,k}$, we never act on any requirement $R_{b,j,e}$ with $j \leq k$. ⊣

Since $\sigma_e^s \prec \sigma_{e+1}^s$ for all s and e, it follows that $\sigma_e \prec \sigma_{e+1}$ for all e. Thus we can define the limit point x of Q to be $x = \cup_e \sigma_e$.

For each $a \leq m$ and each k, the sequence $\mu_{a,k}^s$ likewise converges to a path $\mu_{a,k} \in Q$. Since all other paths are eventually terminated, Q consists of precisely the elements $x_{a,k}$ and the elements x_e. For each $k \geq e$, $x_{0,k}$ is an extension of σ_e, so that x is a limit point of Q. It is clear that each $x_{a,k}$ is isolated in Q since $\mu_{a,k}$ is eventually only extended by 0's in T. Thus x is the unique limit point of Q.

It remains to verify the requirements $R_{b,j,e}$ given above. Note that for each $A \in L$, $Q_A = \{x\} \cup \{x_{a,k} : a \in A \,\&\, k < \omega\}$.

Claim 2.6. *Let $e \leq j$.*

(*i*) *If $b = 0$ and if $x \in P_e$, then $x_{0,j} \in P_e$.*
(*ii*) *If $b > 0$, and $B = A \cup \{b\}$ for some $A, B \in S$ and $x_{b,j} \in P_e$, then $x_{a,k} \in P_e$ for all $a \in B$ and all $k \geq j$.*

Proof. For the first part, suppose that $x \in P_e$ and let s be a stage such that no action is taken on any requirement of priority less than or equal to $R_{0,e,j}$ after stage s. Then the condition must never require action at any stage $t + 1 > s$. It follows that $\sigma_j = \sigma_j^s$. Since $x \in P_e$, it follows that $\sigma_j \in T_e$, so that for all $t \geq s$, $\mu_{0,k}^t \in T_e$ for all $k \geq j$. It follows that $x_{0,j} \in P_e$.

For the second part, assume the hypothesis of part (ii) and let s be a stage such that no action is taken on any requirement of priority less than or equal to $R_{b,e,j}$ after stage s. Then the condition must never require action at any stage $t + 1 > s$. It follows that $\mu_{a,j} = \mu_{a,j}^s$ for all $a \in B$. Since $x_{b,j} \in P_e$, it follows that $\mu_{b,j} \in T_e$, so that for all $a \in B$ and all $t \geq s$, $\mu_{a,k}^t \in T_e$ for all $k \geq j$. It follows that $x_{a,k} \in P_e$, as desired. ⊣

It is important to note that these requirements, now verified, imply that the limit point x is not computable. If it were, then $\{x\}$ would be a Π_1^0 class, say P_e. But then we would have $x_{0,j} \in P_e$ for all $j \geq e$, which is a contradiction.

Finally, we consider the furthermore clause of the theorem, that is, that the theory of $\mathscr{L}(Q)$ is decidable. By a theorem of Lachlan [10], if a lattice $L \subset \mathscr{P}(\mathbb{N})$ is closed under finite differences, then the theory of L is many-one reducible to the theory of L^*.

Lemma 2.7. *Suppose that P is a countable Π_1^0 class such that every computable member of P is isolated. Then the lattice $\mathscr{L}(P)$ of Π_1^0 subclasses of P is isomorphic to a sublattice L of $\mathscr{P}(\mathbb{N})$ which is closed under finite differences.*

PROOF. Let $A = \{\alpha_n : n < \omega\}$ be a list of the isolated points in P. It is sufficient to show that a Π_1^0 subclass of P is determined by its intersection with A. To see this, suppose that Q_1 and Q_2 are two Π_1^0 classes having the same intersection with A. We first show by induction on the rank of $x \in P$ that if $x \in Q_i$ (where $i = 0, 1$), then for any open neighborhood U of x, there is an element of A which belongs to $Q_i \cap U$. The hypothesis covers the case of rank zero. Now suppose that $x \in Q_i$ and that x has rank α in P. Let U be an open set such that $x \in U$ and such that U contains no points of rank $\geq \alpha$ in P other than x. Since x is not computable, $Q_i \cap U$ must contain some point $y \neq x$ and necessarily y has rank $< \alpha$. It follows by induction that $Q_i \cap U$ contains an element of A. Now if $x \in Q_1$, then every neighborhood of x contains an element of $A \cap Q_1$ and therefore, by assumption, an element of $A \cap Q_2$. Since Q_2 is closed, it follows that $x \in Q_2$. Similarly, $x \in Q_2 \rightarrow x \in Q_1$. ⊣

The Π_1^0 class constructed above is certainly countable and the finitely many limit points are each non-computable. Thus by Lemma 2.7 the theory of $\mathscr{L}(Q)$ is many-one reducible to the theory of $\mathscr{L}(Q)^*$. But the latter is the theory of a finite structure and is therefore decidable.

This completes the proof of Theorem 2.3. ⊣

For the remainder of the section, we will show that the construction of Theorem 2.3 may not, in general, be achieved with a decidable class P. In Section 3, we will present a stronger result, that the theory of $\mathscr{L}(P)$ is undecidable, and in fact interprets the theory of arithmetic, whenever P is decidable and $\mathscr{L}(P)$ is not a Boolean algebra. We include the next result since it gives a more direct proof that no decidable Π_1^0 class P can have $\mathscr{L}(P)^*$ isomorphic to a finite lattice, such as the three-point lattice $\{0, 1, 2\}$, which is not a Boolean algebra.

THEOREM 2.8. *If P is a countably infinite, decidable Π_1^0 class, and $\mathscr{L}(P)^*$ is not a Boolean algebra, then $\mathscr{L}(P)^*$ is infinite.*

PROOF. First suppose that P has infinitely many limit points. This condition alone implies that $\mathscr{L}(P)^*$ is infinite; by the countability of P, there must be infinitely many $\{x_0, x_1, \dots\}$ which have rank one. This means that for each n, there is an interval U_n such that $P \cap U_n$ contains x_n and contains no other limit point of P. We claim that the sets $P \cap U_n$ are distinct modulo finite difference. Suppose by way of contradiction that $P \cap U_m$ and $P \cap U_n$ had a finite difference. Since x_m is a limit point of P, there is a sequence $y_0, y_1, \dots$ of (isolated) elements of $P \cap U_m$ which converges to x_m. Then all but finitely many of these y_k would belong to $P \cap U_n$ and therefore x_m would be in $P \cap U_n$, contradicting the assumptions above.

Now suppose that P has only finitely many limit points $\{x_0, \dots, x_k\}$. As above, we can separate them by intervals U_n so that $P \cap U_n$ contains x_n and no other limit point. Since $P - (U_0 \cup \cdots \cup U_n)$ contains no limit points and is therefore finite, we may assume that the sets $P \cap U_n$ partition P. The assumption that $\mathscr{L}(P)^*$ is not a Boolean algebra thus implies that $\mathscr{L}(P \cap U_n)^*$ is not a Boolean algebra for some n. Thus we may assume without loss of generality that P has a unique limit point.

Since $\mathscr{L}(P)^*$ is not a Boolean algebra, there must be some infinite subset P_0 of P such that $P - P_0$ is also infinite. Assuming that $\mathscr{L}(P)^*$ is finite, we may take P_0 to be minimal and P to be a minimal extension of P_0. That is, we may assume, without loss of generality, that $\mathscr{L}(P)^*$ has exactly 3 nodes, corresponding to $\emptyset$, P_0 and P. Now let $P = [T]$ where T has no dead ends, let $P_0 = [T_0]$, and let x be the unique limit point of P. Of course $x \in P_0$ since P_0 is infinite and x is the only limit point of P. Observe that for any $\sigma \in T - T_0$, σ has only finitely many extensions in P, since otherwise $P - P_0$ would contain a limit point of P.

Then we can recursively define a sequence $\sigma_0, \sigma_1, \dots$ of pairwise incompatible nodes in $T - T_0$, as follows. Let σ_0 be the least element of $T - T_0$. Given $\sigma_0, \dots, \sigma_n \in T - T_0$, there exists an element $y \in P - P_0$ which does not extend any of $\sigma_0, \dots, \sigma_n$ since $P - P_0$ is infinite and each σ_i has only finitely many extensions in P. Thus there exists some initial segment $\sigma \in T - T_0$ of y which is incompatible with each of $\sigma_0, \dots, \sigma_n$. Just take σ_{n+1} to be the least such σ (first under length and then lexicographically). The key conclusion now is that since T has no dead ends, each interval $I(\sigma_n)$ must contain a point x_n of $P - P_0$. Now consider the Π^0_1 class $P_1 = \{x \in P : (\forall n)\neg(\sigma_{2n} \prec x)\}$. Since each $\sigma_k \notin T_0$, we have $P_0 \subset P_1$ and in addition $x_{2n-1} \in P_1$ for each n. Thus P_1 is distinct modulo finite difference from the three subclasses which make up $\mathscr{L}(P)^*$. This contradiction demonstrates the result. ⊣

Note that we are not assuming in the previous theorem that $\mathscr{L}(P)$ is closed under finite differences. In particular, we are not assuming that every limit point of P is non-computable.

For the special case of a single node, there does exist a computable tree T with no dead ends such that $P = [T]$ is a minimal Π^0_1 class.

In the next section, we consider the general problem of a decidable Π^0_1 class where $\mathscr{L}(P)$ is not a Boolean algebra.

§3. **Decidable Π^0_1 classes.** In this section, we consider in more detail the theory of the lattice $\mathscr{L}(P)$ of Π^0_1 subclasses of a decidable Π^0_1 class when $\mathscr{L}(P)$ is not a Boolean algebra. By Theorem 2.8 this means that either P is uncountable or $\mathscr{L}(P)^*$ is infinite. We prove the following theorem.

THEOREM 3.1. *Suppose that P is a decidable Π^0_1-class such that $\mathscr{L}(P)$ is not a Boolean algebra. Then* $\mathrm{Th}(\mathscr{L}(P))$ *interprets* $Th(\mathbb{N}, +, \times)$.

Let $\mathscr{D}$ be the computable dense Boolean algebra. For ease of notation and also to conform with Nies [12], we will use the language of c.e. ideals of $\mathscr{D}$ under inclusion instead of Π^0_1-classes under inclusion. If H is a c.e. ideal of $\mathscr{D}$, let $\mathscr{L}(H)$ be the lattice of c.e ideals of $\mathscr{D}$ *containing* H.

THEOREM 3.2. *Suppose H is a computable ideal of $\mathscr{D}$ and $\mathscr{L}(H)$ is not a Boolean algebra. Then* $\mathrm{Th}(\mathscr{L}(H))$ *interprets* $Th(\mathbb{N}, +, \times)$.

The equivalence of Theorem 3.1 and the preceding theorem can be obtained using effective Stone duality. See Cenzer and Remmel [5] for details. It will be clear from the proof how the decidability of H is used: this enables one to see that a requirement is satisfied permanently, when it depends on the fact that a certain element of $\mathscr{D}$ which has been enumerated into an ideal is not in H.

We first need some terminology and notation. A c.e. Boolean algebra is given by a model $(\mathbb{N}, \preceq, \vee, \wedge)$ such that $\preceq$ is a c.e. relation which is a pre-ordering, $\vee, \wedge$ are total computable binary functions, and the quotient structure $\mathscr{B} = (\mathbb{N}, \preceq, \vee, \wedge)/\approx$ is a Boolean algebra (where $n \approx m \Leftrightarrow n \preceq m \;\&\; m \preceq n$). We can suppose that $0 \in \mathbb{N}$ names the least and $1 \in \mathbb{N}$ the greatest element of $\mathscr{B}$. For Σ^0_k-Boolean algebras, one requires that $\preceq$ be Σ^0_k and that $\wedge, \vee$ be computable in $\emptyset^{(k-1)}$. For a Σ^0_k

Boolean algebra $\mathscr{B}$, let

$$\mathscr{I}(\mathscr{B}) := \text{the lattice of } \Sigma^0_k\text{–ideals of } \mathscr{B}.$$

Clearly c.e. Boolean algebras correspond to c.e. ideals of $\mathscr{D}$, and similarly for computable. In this language, Theorem 3.2 can be restated a further time as follows: for a computable Boolean algebra $\mathscr{C}$, if $\mathscr{I}(\mathscr{C})$ is not a Boolean algebra, then $\mathrm{Th}(\mathscr{I}(\mathscr{C}))$ interprets $\mathrm{Th}(\mathbb{N}, +, \times)$.

PROOF. We will first prove the weaker result that $\mathrm{Th}(\mathscr{L}(H))$ is undecidable, and then obtain the full result by an extra argument. We use a result from Nies [12]. A c.e. Boolean algebra $\mathscr{B}$ is called *effectively dense* [12] if there is a computable F such that $\forall x\ [F(x) \preceq x]$ and

$$\forall x \not\approx 0\ [0 \prec F(x) \prec x]. \tag{1}$$

More generally, a Σ^0_k Boolean algebra $\mathscr{B}$ is effectively dense if the above holds with some $F \leq_T \emptyset^{(k-1)}$. In [12], it is proved that, for any effectively dense Σ^0_k Boolean algebra $\mathscr{C}$, $\mathrm{Th}(\mathscr{I}(\mathscr{C}))$ is hereditarily undecidable (i.e., all subtheories containing the valid sentences are undecidable). By the standard methods to transfer hereditary undecidability (see e.g., [1]), it suffices to give a coding in $\mathscr{L}(H)$ with parameters of $\mathscr{I}(\mathscr{C})$, for an effectively dense Σ^0_3 Boolean algebra $\mathscr{C}$.

In the following we describe how to determine $\mathscr{C}$ and how to do the coding. We first need some more notation.

DEFINITION 3.3. 1. For $S \subseteq \mathscr{D}$, let $[S]_{id}$ be the ideal of $\mathscr{D}$ generated by $S \cup H$.
2. A enumeration of an ideal X of $\mathscr{B} = \mathscr{D}/_H$ is given by a c.e. subset $\widetilde{X} = \bigcup_s \widetilde{X}_s$ of $\mathscr{D}$ such that $X = [\widetilde{X}]_{id}$. We let $X_s = [\widetilde{X}_s]_{id}$ (thereby slightly deviating from the notation in [13], where H is usually not decidable). We let (V_e) be a uniform enumeration of all c.e. ideals containing H.
3. For a c.e. ideal X, we let

$$x_0 = 0,\ x_s = \sup_{\mathscr{D}} X_s - \sup_{\mathscr{D}} X_{s-1} (s > 0). \tag{2}$$

Thus, $(x_n)_{n\in\mathbb{N}}$ is an effective "partition" generating X.
4. Capital letters $A, \dots, E, X, Y, V, W$ range over c.e. ideals of $\mathscr{D}$ containing H.
5. An element b of $\mathscr{B}$ is identified with the corresponding principal ideal $[\{b\}]_{id}$.
6. (*Splittings of ideals*) We write $B \sqcup C = A$ if $B \cap C = H$ and $B \vee C = A$. In this case we denote C by $\mathrm{Cpl}_A(B)$. We write $B \sqsubset A$ if $\exists C\ B \sqcup C = A$.

Fix $A \in \mathscr{I}(\mathscr{B})$ and choose a $\emptyset''$-listing (X_i) of $\mathscr{B}(A)$, where

$$\mathscr{B}(A) = \{X : X \sqsubset A\}.$$

Since $\mathscr{I}(\mathscr{B})$ is a distributive lattice, $(\mathscr{B}(A), \cap, \vee, \mathrm{Cpl}_A, H, A)$ is a Σ^0_3-Boolean algebra (with the presentation determined by that listing). We consider ideals of $\mathscr{B}(A)$. To avoid confusion, we will write "IDEAL" when we mean such a level 2 ideal. For certain A, E such that $A \subseteq E$, we will view

$$\mathscr{R}_E(A) = \{X \sqsubset E : X \subseteq A\}$$

as the IDEAL of negligible splittings of A. Note that $\{e : X_e \in \mathscr{R}_E(A)\}$ is a Σ_3^0-set. Let

$$\mathscr{C} = \mathscr{B}(A/_E) = \mathscr{B}(A)/_{\mathscr{R}_E(A)}. \tag{3}$$

We first give an outline of the coding. Under certain conditions on A and E (for instance, if E is nonprincipal), we will be able to show that $\mathscr{C}$ is effectively dense as a Σ_3^0 Boolean algebra. Then, to give the coding of $I(\mathscr{C})$ in $\mathscr{L}(H)$, we represent a Σ_3^0-IDEAL $I \in \mathscr{I}(\mathscr{C})$ by (any) $C \in \mathscr{L}(H)$ such that, for $X \sqsubset A$, $X \in \mathscr{I}(\mathscr{C})$ if and only if $X \cap C$ is negligible, that is, $X \cap C \subset R$ for some $R \in \mathscr{R}_E(A)$. Clearly, any subset of $\mathscr{B}(A)$ represented in this way is a Σ_3^0-IDEAL containing $\mathscr{R}_E(A)$. The main technical result, proved in [13], is that also each such IDEAL I can be represented.

Then with the listing $(X_i/_E)_{i\in\omega}$, $\mathscr{B}(A/_E)$ becomes a Σ_3^0 Boolean algebra. To obtain the desired Σ_3^0 Boolean algebra, we require that E is a nonprincipal ideal, $A \subseteq E$ is not a split of E and A also satisfies the following property.

DEFINITION 3.4. We say that A is locally principal (l.p.) in E if $A \subseteq E$ and

$$\forall e \in E[e \cap A \text{ is principal }].$$

Locally principal ideals were introduced by Nies in [13]. Note that this property of A, E can be expressed in $\mathscr{I}(\mathscr{B})$ in a first-order way, since the principal ideals are just the complemented elements of $\mathscr{I}(\mathscr{B})$. The motivation is that the situation $A \subseteq E$ is in a sense similar to an inclusion of sets: whenever $e \in E$, the intersection $A \cap e$ has only a finite amount of information.

Since $\mathscr{L}(H)$ is not a Boolean algebra, a nonprincipal E exists. We first supply the fact that an $A \subseteq E$ as required also exists.

LEMMA 3.5. *For any $E \not\trianglelefteq 1$, there is $A \subseteq E$, $A \not\trianglelefteq E$ such that A is l.p. in E.*

PROOF. Since $E \not\trianglelefteq 1$, we can fix an enumeration $E_s = [\{e_n : n < s\}]_{id}$, where (e_n) is a u.c.e. sequence of elements of $D - H$ which have pairwise meet H. It suffices to meet the requirements

$$R_n : \neg(A \sqcup V_n = E).$$

To do so, we reserve e_n for R_n. At stage s, for each $n < s$, if now $e_n \in V_{n,s}$, we put e_n into A (precisely speaking, into $\widetilde{A}$).

Clearly A is l.p. in E. Moreover, each requirement is met: If $A \vee V_n = E$, then, since we threaten to keep e_n out of A, $e_n \in V_{n,s}$ for some s. Then the construction ensures $V_n \cap A \neq H$. ⊣

In the following we fix A, E with the properties as above. We prove that the Σ_3^0 Boolean algebra $\mathscr{B}(A/_E)$ is effectively dense. Clearly if A is l.p. in E and $Y \sqsubset A$, then so is Y. So the following is sufficient.

LEMMA 3.6. *Suppose Y is l.p. in E. Then one can effectively obtain a splitting $Y = Y_0 \sqcup Y_1$ such that $Y \not\trianglelefteq E$ implies $Y_i \not\trianglelefteq E$ $(i = 0, 1)$.*

PROOF. Let $E = [\{e_n : n \in \mathbb{N}\}]_{id}$ as above. We call $S \subseteq E$ *small* if

$$\exists n S \subseteq e_0 \vee \ldots \vee e_n.$$

For c.e. ideals C, D let $C \searrow D$ be the ideal X given by enumerating (into a set $\widetilde{X}$) at stage s those x such that

$$x \in C_{s-1} \ \&\ x \notin D_{s-1} \ \&\ x \in D_s,$$

(and, as always, letting X_s be the ideal of D_s generated by $\widetilde{X}_s \cup H$). Similarly to the proof of the Friedberg Splitting Theorem [14], we meet the requirements

$$P_{e,i} : V_e \searrow Y \text{ not small } \Rightarrow\ V_e \searrow Y_i \nsubseteq H,$$

while ensuring that $Y = Y_0 \sqcup Y_1$.

We first verify that this is sufficient. Suppose $Y_i \sqsubset E$. Choose k such that $Y_i \sqcup V_k = E \ \&\ Y_i \cap V_k = H$. Then $V_k \searrow Y$ is not small: assume

$$V_k \searrow Y \subseteq \widehat{e}_n := e_0 \vee \ldots \vee e_n,$$

and let $V = [\{y \leq \mathrm{Cpl}_E(\widehat{e}_n) : \exists s\ (y \in V_{k,s} \ \&\ y \notin Y_s)\}]_{id}$. Then

$$(\widehat{e}_n \vee Y) \vee V = E \ \&\ (\widehat{e}_n \vee Y) \cap V = H.$$

Thus $(\widehat{e}_n \vee Y) \sqsubset E$, and since Y is l.p. in E, $Y \sqsubset E$.

Since $V_k \searrow Y$ is not small, $V_k \searrow Y_i \nsubseteq H$, contrary to $V_k \cap Y_i = H$. So it suffices to meet the requirements $P_{e,i}$.

Construction of Y_0, Y_1. At stage s determine the least $\langle e, i\rangle < s$ such that $P_{e,i}$ has not been met (i.e., $V_e \searrow Y_i[s] \subseteq H$ and $y_s \cap V_{e,s-1} \nsubseteq H$). Enumerate y_s into Y_i. If $\langle e, i\rangle$ fails to exist, put y_s into Y_0.

Clearly, $Y = Y_0 \sqcup Y_1$. To prove that $P_{e,i}$ is met, suppose that by stage t, P_k has been met for each $k < \langle e, i\rangle$. Since $V_e \searrow Y$ is not small, there is $s > t$ such that $y_s \notin H$, $y_s \wedge \widehat{e}_t \in H$ and $y_s \cap V_{e,s-1} \nsubseteq H$. Then the requirement is satisfied from stage $s + 1$ on. ⊣

Since $\mathscr{B}(A/_E)$ is an effectively dense Σ_3^0 Boolean algebra, by Nies [12], the lattice $\mathscr{I}(\mathscr{B}(A/_E))$ has a hereditarily undecidable theory. Therefore it is sufficient to give a coding with parameters of $\mathscr{I}(\mathscr{B}(A/_E))$ in $\mathscr{L}(H)$. We rely on the proof of Nies [13, Lemma 6.3], where it is shown that, if A is l.p. in E, then, for each Σ_3^0 IDEAL I of $\mathscr{B}(A)$ containing $\mathscr{R}_E(A)$, there is a C_I such that

$$I = \{X \in \mathscr{B}(A) : (\exists R \in \mathscr{R}_E(A))(C_I \cap X \subseteq R)\}. \tag{4}$$

(In [13] the assumption is made in the proof of Lemma 6.3 that the base Boolean algebra $\mathscr{D}/_H$ is effectively dense, but this is not needed.) Note that, conversely, each subset of $\mathscr{B}(A)$ determined by (4) is a Σ_3^0 IDEAL containing $\mathscr{R}_E(A)$. Since the set of these IDEALS corresponds to $\mathscr{I}(\mathscr{B}(A/_E))$, we obtain the desired coding: represent I by any C_I, and give a first-order formula $\varphi_{\subseteq}(C_1, C_2; A, E)$ expressing inclusion of the represented ideals in the obvious way.

This settles undecidability. We now give the extra argument needed to obtain an interpretation of $\mathrm{Th}(\mathbb{N}, +, \times)$ in $\mathrm{Th}(\mathscr{L}(H))$. First we prove a uniqueness property of $\mathscr{B}(A/_E)$.

PROPOSITION 3.7. *Suppose that*

$$E \not\sqsubseteq 1,\ A \subseteq E,\ A \not\sqsubseteq E,\ A \textit{ is l.p. in } E, \tag{5}$$

and the same properties also hold for $\widetilde{E}, \widetilde{A}$. *Then* $\mathscr{B}(A/_E) \cong \mathscr{B}(\widetilde{A}/_{\widetilde{E}})$ *via an isomorphism which is computable in* $\emptyset''$.

PROOF. A c.e. Boolean algebra $\mathscr{C}$ is called *effectively inseparable* (e.i.) if the sets $\{n \in \mathbb{N} : n \approx 0\}$, $\{n \in \mathbb{N} : n \approx 1\}$ (i.e., the sets of names for $0^{\mathscr{C}}, 1^{\mathscr{C}}$) are effectively inseparable. By the methods of Kripke and Pour-El [9], any two e.i. Boolean algebras are effectively isomorphic. We apply their result, relativized to $\emptyset''$. It suffices to show that under the given hypotheses $\mathscr{B}(A/_E)$ (with the presentation given at (3)) is $\emptyset''$-e.i.. Recall that (X_i) is an $\emptyset''$-listing of $\mathscr{B}(A)$. We prove that

$$(6) \qquad S = \{i : X_i \in \mathscr{R}_E(A)\},\ T = \{i : A - X_i \in \mathscr{R}_E(A)\}$$

are $\emptyset''$-e.i. sets. Fix a pair of Σ^0_3-sets $\widetilde{S}, \widetilde{T}$ which is e.i. relative to $\emptyset''$. It suffices to find a total $f \leq \emptyset''$ such that

$$(7) \qquad f(\widetilde{S}) \subseteq S, f(\widetilde{T}) \subseteq T.$$

Fix a u.c.e. double sequence Z^i_n of initial segments of $\mathbb{N}$ such that

$$i \in \widetilde{S} \Leftrightarrow \exists n Z^i_{2n} = \mathbb{N} \text{ and } i \in \widetilde{T} \Leftrightarrow \exists n Z^i_{2n+1} = \mathbb{N}.$$

For each i we will effectively obtain a splitting $A = A_0 \sqcup A_1$ such that $i \in \widetilde{S} \Rightarrow A_0 \sqsubset E$ and $i \in \widetilde{T} \Rightarrow A_1 \sqsubset E$. Then f, given by $f(i) =$ the first j such that $X_j = A_0$, is a function computable in $\emptyset''$ as desired. We employ a simple fact from Nies [13, Fact 6.1]. Recall that we are identifying elements of $\mathscr{B}$ and principal ideals.

FACT 3.8. *Suppose $B \subseteq E$ is a c.e. ideal such that $\forall k\ B \cap e_k = b_k$, where b_k is obtained effectively from k. Then $B \sqsubset E$.*

PROOF. Let $C = [\{e_k - b_k\}_{k \in \mathbb{N}}]_{id}$. Then $B \sqcup C = E$. ⊣

At stage s, we decide whether to put a_s into A_0 or into A_1, as follows: Compute the maximal k such that $a_s \wedge e_k \notin H$. Let m be minimal such that $|Z^i_{m,s}| > k$. If m is even or fails to exist put a_s into A_1, else into A_0.

To verify (7), first suppose $i \in \widetilde{S}$ and let m be least such that $Z^i_{2m} = \mathbb{N}$. Then Fact 3.8 implies $B = A_0 \sqsubset E$ as follows. Given k, since B is l.p. in A we can assume that $k > \max_{r<2m} |Z^i_r|$ (because finitely many b_i can be fixed in advance). Compute s such that $|Z^i_{2m,s}| > k$. Then $A_0 \cap e_k = A_{0,s} \cap e_k$, so let $b_k = \sup(A_{0,s} \cap e_k)$. If $i \in \widetilde{T}$, one proves $A_1 \sqsubset E$ in a similar fashion. This completes the proof of Proposition 3.7.

By the uniqueness up to $\emptyset''$ isomorphism of $\mathscr{B}(A/E)$, all possible structures $\mathscr{I}^* = \mathscr{I}(\mathscr{B}(A/E))$, where E, A satisfy (5), are isomorphic. By Nies [13] and the effective density of $\mathscr{B}(A/E)$, Th$(\mathscr{I}^*)$ interprets Th$(\mathbb{N}, +, \times)$. But

$$\mathscr{I}^* \models \varphi \Leftrightarrow \mathscr{L}(H) \models \exists E \exists A[(5)\ \&\ ''\mathscr{I}(\mathscr{B}(A/_E)) \models \varphi''],$$

so Th$(\mathscr{I}^*)$ can be interpreted in Th$(\mathscr{L}(H))$.

This demonstrates fact 3.8 and completes the proof of Proposition 3.7, Theorems 3.1 and 3.2. ⊣ ⊣

Open problem: Characterize those P such that Th$(\mathscr{L}(P))$ is decidable.

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