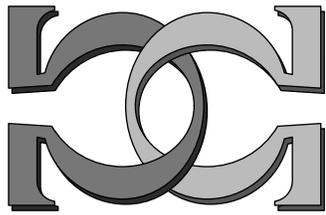
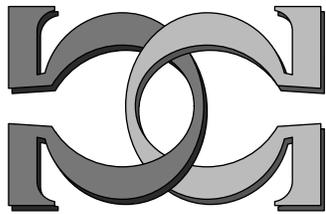


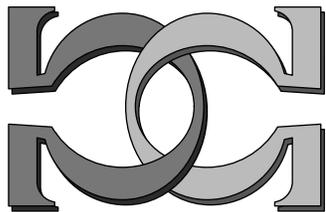
**CDMTCS  
Research  
Report  
Series**



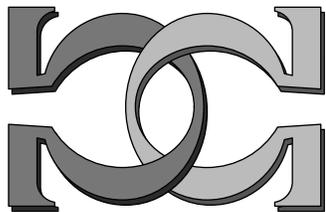
**Chaitin  $\Omega$  Numbers and  
Strong Reducibilities**



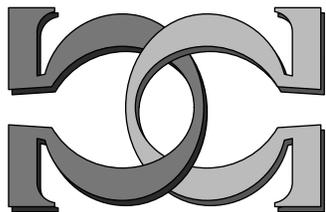
**Cristian S. Calude**  
Department of Computer Science  
University of Auckland  
Auckland, New Zealand



**André Nies**  
Department of Mathematics  
University of Chicago, IL U.S.A.



CDMTCS-062  
October 1997



Centre for Discrete Mathematics and  
Theoretical Computer Science

# Chaitin $\Omega$ Numbers and Strong Reducibilities\*

Cristian S. Calude<sup>†</sup> and André Nies<sup>‡</sup>

## Abstract

We prove that any Chaitin  $\Omega$  number (i.e., the halting probability of a universal self-delimiting Turing machine) is wtt-complete, but not tt-complete. In this way *we obtain a whole class of natural examples of wtt-complete but not tt-complete r.e. sets*. The proof is direct and elementary.

## 1 Introduction

Kučera [8] has used Arslanov's completeness criterion<sup>1</sup> to show that all random sets of r.e. T-degree are in fact T-complete. Hence, *every Chaitin  $\Omega$  number is T-complete*. In this paper we will strengthen this result by proving that *every Chaitin  $\Omega$  number is weak truth-table complete*. However, no Chaitin  $\Omega$  number can be tt-complete as, because of a result stated by Bennett [1] (see Juedes, Lathrop, and Lutz [9] for a proof), there is no random sequence  $\mathbf{x}$  such that  $K \leq_{tt} \mathbf{x}$ .<sup>2</sup> Notice that in this way *we obtain a whole class of natural examples of wtt-complete but not tt-complete r.e. sets* (a fairly complicated construction of such a set was given by Lachlan [10]).

---

\*The first has been partially supported by AURC A18/XXXXX/62090/F3414056, 1996. The second author was partially supported by NSF Grant DMS-9500983.

<sup>†</sup>Department of Computer Science, University of Auckland, Private Bag 92019, New Zealand. Email: [cristian@cs.auckland.ac.nz](mailto:cristian@cs.auckland.ac.nz).

<sup>‡</sup>Department of Mathematics, University of Chicago, Chicago IL., USA. Email: [nies@math.uchicago.edu](mailto:nies@math.uchicago.edu).

<sup>1</sup>An r.e.  $X$  is Turing equivalent to the halting problem iff there is a Turing computable in  $X$  function  $f$  without fixed-points, i.e.  $W_x \neq W_{f(x)}$ , for all  $x$ ; see Soare [12], p. 88.

<sup>2</sup>To keep the paper self-contained, a direct simple proof for Bennett result will be included.

We continue with a piece of notation. Let  $\mathbf{N}, \mathbf{Q}$  be the sets of non-negative integers and rationals. Let  $\Sigma = \{0, 1\}$  denote the binary alphabet,  $\Sigma^*$  is the set of (finite) binary strings,  $\Sigma^n$  is the set of binary strings of length  $n$ ; the length of a string  $x$  is denoted by  $|x|$ . By  $x|r$  we denote the prefix of length  $r$  of the string  $x$ . Let  $p(x)$  be the place of  $x$  in  $\Sigma^*$  ordered quasi-lexicographically. Let  $\Sigma^\omega$  the set of infinite binary sequences. The prefix of length  $n$  of the sequence  $\mathbf{x} \in \Sigma^\omega$  is denoted by  $\mathbf{x}|n$ . For every  $X \subset \Sigma^*$ ,  $X\Sigma^\omega$  stands for the cylinder generated by  $X$ , i.e., set of all sequences having a prefix in  $X$ .

Fix an acceptable gödelization  $(\varphi_x)_{x \in \Sigma^*}$  of all partial recursive (p.r.) functions from  $\Sigma^*$  to  $\Sigma^*$ , and let  $W_x = \text{dom}(\varphi_x)$  be the domain of  $(\varphi_x)$ . Denote by  $K$  the set  $\{x \in \Sigma^* \mid x \in W_x\}$ . A Chaitin computer (self-delimiting Turing machine) is a p.r. function  $C : \Sigma^* \xrightarrow{o} \Sigma^*$  with a prefix-free domain  $\text{dom}(C)$ . The program-size (Chaitin) complexity induced by Chaitin's computer  $C$  is defined by  $H_C(x) = \min\{|y| \mid y \in \Sigma^*, C(y) = x\}$  (with the convention  $\min \emptyset = \infty$ ).

A Chaitin computer  $U$  is *universal* if for every Chaitin computer  $C$ , there is a constant  $c > 0$  (depending upon  $U$  and  $C$ ) such that for every  $x$  there is  $x'$  such that  $U(x') = C(x)$  and  $|x'| \leq |x| + c$ ;<sup>3</sup>  $c$  is the “simulation” constant of  $C$  on  $U$ .

A Martin-Löf test is an r.e. sequence  $(V_i)_{i \geq 0}$  of subsets of  $\Sigma^*$  satisfying the following measure-theoretical condition:

$$\mu(V_i \Sigma^\omega) \leq 2^{-i},$$

for all  $i \in \mathbf{N}$ . Here  $\mu$  denotes the usual product measure on  $\Sigma^\omega$ , given by  $\mu(\{w\}\Sigma^\omega) = 2^{-|w|}$ , for  $w \in \Sigma^*$ .

An infinite sequence  $\mathbf{x}$  is *random* if for every Martin-Löf test  $(V_i)_{i \geq 0}$ ,  $\mathbf{x} \notin \bigcap_{i \geq 0} V_i \Sigma^\omega$ . A real  $\alpha \in (0, 1)$  is *random* in case its binary expansion is a random sequence.<sup>4</sup>

The halting probability of Chaitin's computer  $C$  is

$$\Omega_C = \mu(\text{dom}(C)\Sigma^\omega) = \sum_{x \in \text{dom}(C)} 2^{-|x|}.$$

Any real  $\Omega_C$  is recursively enumerable (r.e.) in the sense that the set  $\{q \in (0, 1) \cap \mathbf{Q} \mid q < \Omega_C\}$  is r.e. (see more about r.e. reals in [3]). Reals of the form  $\Omega_U$ , for some universal Chaitin computer  $U$ , are called *Chaitin ( $\Omega$ ) numbers* (see [4, 6, 2]). Chaitin [4] has proved that *every Chaitin number is random*. See Calude [2] for more details.

For a set  $A \subset \Sigma^*$  we denote by  $\chi_A$  the characteristic function of  $A$ . We say that  $A$  is Turing reducible to  $B$ , and we write  $A \leq_T B$ , if there is an oracle Turing machine  $\varphi_w^B$  such that  $\varphi_w^B(x) = \chi_A(x)$ . We say that  $A$  is weak truth-table reducible to  $B$ , and we write  $A \leq_{wtt} B$ , if  $A \leq_T B$  via a Turing reduction which on input  $x$  only queries strings of length less than  $g(x)$ , where  $g : \Sigma^* \rightarrow \mathbf{N}$  is a fixed recursive function. We

<sup>3</sup>In fact,  $c$  can be effectively obtained from  $U$  and  $C$ .

<sup>4</sup>Actually, the choice of base is irrelevant, cf. Theorem 6.111 in Calude [2].

say that  $A$  is truth-table reducible to  $B$ , and we write  $A \leq_{tt} B$ , if there is a recursive sequence of Boolean functions  $\{F_x\}_{x \in \Sigma^*}$ ,  $F_x : \Sigma^{r_x+1} \rightarrow \Sigma$ , such that for all  $x$ , we have  $\chi_A(x) = F_x(\chi_B(0)\chi_B(1)\cdots\chi_B(r_x))$ .<sup>5</sup> An r.e. set  $A$  is tt(wtt)-complete if  $K \leq_{tt} A$  ( $K \leq_{wtt} A$ ). See Odifreddi [11] for more details.

## 2 Main Results

In what follows we will fix a universal Chaitin computer  $U$  and write  $H = H_U$ ,  $\Omega = \Omega_U$ .

**Theorem 2.1** *The set  $\mathcal{H} = \{(x, n) \mid x \in \Sigma^*, n \in \mathbf{N}, H(x) \leq n\}$ <sup>6</sup> is wtt-complete.*

**Proof.** We will refine the proof by Arslanov and Calude in [7]. To this aim we will use Arslanov's Completeness Criterion (see Theorem III.8.17 in Odifreddi [11], p. 338) for wtt-reducibility

*an r.e. set  $A$  is wtt-complete iff there is a function  $f \leq_{wtt} A$  without fixed-points*

and the estimation due to Chaitin [4, 5] (see Theorem 5.4 in Calude [2], pp. 77):

$$\max_{x \in \Sigma^n} H(x) = n + O(\log n). \quad (1)$$

First we construct a positive integer  $c > 0$  and a p.r. function  $\psi : \Sigma^* \xrightarrow{o} \Sigma^*$  such that for every  $x \in \Sigma^*$  with  $W_x \neq \emptyset$ ,

$$U(\psi(x)) \in W_x, \quad (2)$$

and

$$|\psi(x)| \leq p(x) + c. \quad (3)$$

Consider now a Chaitin computer  $C$  such that  $C(0^{p(x)}1) \in W_x$  whenever  $W_x \neq \emptyset$ . Let  $c'$  be the simulation constant of  $C$  on  $U$ , and let  $\theta$  be a p.r. function satisfying the following condition: if  $C(u)$  is defined, then  $U(\theta)(u) = C(u)$  and  $|\theta(u)| \leq |u| + c'$ . Put

---

<sup>5</sup>Note that in contrast with tt-reductions, a wtt-reduction may diverge.

<sup>6</sup>This set is essential in deriving Chaitin's information-theoretical version of incompleteness, [4].

$c = c' + 1$  and notice that in case  $W_x \neq \emptyset$ ,  $C(0^{p(x)}1) \in W_x$ , so  $\theta(0^{p(x)}1)$  is defined and belongs to  $W_x$ . Finally, put  $\psi(x) = \theta(0^{p(x)}1)$  and notice that

$$|\psi(x)| = |\theta(0^{p(x)}1)| \leq |0^{p(x)}1| + c' = p(x) + c.$$

Next define the function

$$F(y) = \min\{x \in \Sigma^* \mid H(x) > p(y) + c\},$$

where the minimum is taken according to the quasi-lexicographical order and  $c$  comes from (3). In view of (1) it follows that

$$F(y) = \min\{x \in \Sigma^* \mid H(x) > p(y) + c, |x| \leq p(y) + c\}.$$

The function  $F$  is total,  $H$ -recursive and  $U(\psi(y)) \neq F(y)$  whenever  $W_y \neq \emptyset$ . Indeed, if  $W_y \neq \emptyset$  and  $U(\psi(y)) = F(y)$ , then  $\psi(y)$  is defined, so  $U(\psi(y)) \in W_y$  and  $|\psi(y)| \leq p(y) + c$ . But, in view of the construction of  $F$ ,  $H(F(y)) > p(y) + c$ , an inequality which contradicts (3):  $H(F(y)) \leq |\psi(y)| \leq p(y) + c$ .

Let  $f$  be an  $H$ -recursive function satisfying  $W_{f(y)} = \{F(y)\}$ . To compute  $f(y)$  in terms of  $F(y)$  we need to perform the test  $H(x) > p(y) + c$  only for those strings  $x$  satisfying the inequality  $|x| \leq p(y) + c$ , so the function  $f$  is wtt-reducible to  $\mathcal{H}$ .

We conclude by proving that for every  $y \in \Sigma^*$ ,  $W_{f(y)} \neq W_y$ . If  $W_{f(y)} = W_y$ , then  $W_y = \{F(y)\}$ , so by (3),  $U(\psi(y)) \in W_y$ , that is  $U(\psi(y)) = F(y)$ . Consequently, by (2)  $H(F(y)) \leq |\psi(y)| \leq p(y) + c$ , which contradicts the construction of  $F$ .  $\square$

**Theorem 2.2** *The set  $\mathcal{H}$  is wtt-reducible to  $\Omega$ .*

**Proof.** Let  $g : \mathbf{N} \rightarrow \Sigma^*$  be a recursive, one-to-one function which enumerates the domain of  $U$  and put  $\omega_m = \sum_{i=0}^m 2^{-|g(i)|}$ . Given  $x$  and  $n > 0$  we compute the smallest  $t \geq 0$  such that

$$\omega_t \geq 0.\Omega_0\Omega_1 \cdots \Omega_n.$$

From the relations

$$0.\Omega_0\Omega_1 \cdots \Omega_n \leq \omega_t < \omega_t + \sum_{s=t+1}^{\infty} 2^{-|g(s)|} = \Omega < 0.\Omega_0\Omega_1 \cdots \Omega_n + 2^{-n}$$

we deduce that  $|g(s)| > n$ , for every  $s \geq t + 1$ . Consequently, if  $x$  is not produced by an element in the set  $\{g(0), g(1), \dots, g(t)\}$ , then  $H(x) > n$  as  $H(x) = |g(s)|$ , for some  $s \geq t + 1$ ; conversely, if  $H(x) \leq n$ , then  $x$  must be produced via  $U$  by one of the elements of the set  $\{g(0), g(1), \dots, g(t)\}$ .  $\square$

Since the result in Juedes, Lathrop, and Lutz [9] is obtained in a rather indirect way, we conclude the paper by proving directly that  $K \not\leq_{tt} \mathbf{x}$ , for every random sequence  $\mathbf{x}$ .

**Theorem 2.3** *If  $K \leq_{tt} \mathbf{x}$ , then  $\mathbf{x}$  is not random.*

**Proof.** Assume  $\mathbf{x}$  is random and  $K \leq_{tt} \mathbf{x}$ , that is there exists a recursive sequence of Boolean functions  $\{F_u\}_{u \in \Sigma^*}$ ,  $F_u : \Sigma^{r_u+1} \rightarrow \Sigma$ , such that for all  $w \in \Sigma^*$ , we have  $\chi_A(w) = F_w(x_0 x_1 \cdots x_{r_w})$ . We will construct a Martin-Löf test  $V$  such that  $\mathbf{x} \in \bigcap_{n \geq 0} V_n \Sigma^\omega$ , which will contradict the randomness of  $\mathbf{x}$ .

For every string  $z$  let

$$M(z) = \{u \in \Sigma^{r_z+1} \mid F_z(u) = 0\}.$$

Consider the set

$$\{z \in \Sigma^* \mid \mu(M(z)\Sigma^\omega) \geq \frac{1}{2}\}$$

of inputs to the  $tt$ -reduction of  $K$  to  $\mathbf{x}$  where at least half of the possible oracle strings give the output 0. This set is r.e., so let  $W_{z_0}$  be a name for it. From the construction it follows that

$$z_0 \in K \Leftrightarrow F_{z_0}(x_0 x_1 \cdots x_{r_{z_0}}) = 1,$$

hence if we put  $r = r_{z_0} + 1$  and

$$V_0 = \{u \in \Sigma^r \mid \mu(M(z_0)\Sigma^\omega) \geq \frac{1}{2} \Leftrightarrow F_{z_0}(u) = 1\}$$

we ensure that  $V$  is r.e. and  $\mu(V_0 \Sigma^\omega) \leq \frac{1}{2}$ . Moreover  $\mathbf{x} \in V_0 \Sigma^\omega$ , because if  $u = \mathbf{x}|r$ , then

$$\mu(M(z_0)\Sigma^\omega) \geq \frac{1}{2} \Leftrightarrow z_0 \in K \Leftrightarrow F_{z_0}(u) = 1.$$

Assume now that  $z_n, V_n$  have been constructed such that  $\mathbf{x} \in V_n \Sigma^\omega$  and  $\mu(V_n \Sigma^\omega) \leq 2^{-n-1}$ . Let  $z_{n+1} \notin \{z_0, z_1, \dots, z_n\}$  be such that

$$W_{z_{n+1}} = \{u \in \Sigma^* \mid \mu(M(u)\Sigma^\omega \cap V_n \Sigma^\omega) \geq \frac{1}{2} \cdot \mu(V_n \Sigma^\omega)\}.$$

Then

$$z_{n+1} \in K \Leftrightarrow \mu(M(u)\Sigma^\omega \cap V_n\Sigma^\omega) \geq \frac{1}{2} \cdot \mu(V_n\Sigma^\omega).$$

Finally put  $r = r_{z_{n+1}+1}$  and

$$V_{n+1} = \{u \in \Sigma^r \mid u|_{r_{z_n}} \in V_n \wedge (\mu(M(z_{n+1})\Sigma^\omega \cap V_n\Sigma^\omega) \geq \frac{1}{2} \cdot \mu(V_n\Sigma^\omega) \Leftrightarrow F_{z_{n+1}}(u) = 1)\}$$

and note that  $V_{n+1}$  is r.e.,  $\mathbf{x} \in V_{n+1}$  and

$$\mu(V_{n+1}\Sigma^\omega) \leq \frac{1}{2} \cdot \mu(V_n\Sigma^\omega) \leq 2^{-n-2}.$$

Consequently,  $(V_n)_n$  is a Martin-Löf test with  $\mathbf{x} \in \bigcap_{n \geq 0} V_n\Sigma^\omega$ . □

## References

- [1] C. H. Bennett. Logical depth and physical complexity, in R. Herken (ed.). *The Universal Turing Machine. A Half-Century Survey*, Oxford University Press, Oxford, 1988, 227–258.
- [2] C. Calude. *Information and Randomness. An Algorithmic Perspective*. Springer-Verlag, Berlin, 1994.
- [3] C. S. Calude, P. H. Hertling, B. Khossainov, Y. Wang. Recursively enumerable reals and Chaitin  $\Omega$  numbers, *CDMTCS Research Report* 055, 1997, 22 pp.
- [4] G. J. Chaitin. *Information, Randomness and Incompleteness, Papers on Algorithmic Information Theory*, World Scientific, Singapore, 1987. (2nd ed., 1990)
- [5] G. J. Chaitin. On the number of  $N$ -bit strings with maximum complexity, *Applied Mathematics and Computation* 59(1993), 97-100.
- [6] G. J. Chaitin. *The Limits of Mathematics*, Springer-Verlag, Singapore, 1997.
- [7] G. J. Chaitin, A. Arslanov, C. Calude. Program-size complexity computes the halting problem, *EATCS Bull.* 57 (1995), 198-200.
- [8] A. Kučera. Measure,  $\Pi_1^0$ -classes and complete extensions of PA, In: H. D. Ebbinghaus and G. H. Müller (eds.) *Recursion Theory Week* (Oberwolfach, 1984), Lecture Notes in Math., 1141, Springer-Verlag, Berlin, 1985, 245-259.

- [9] D. Juedes, J. Lathrop, and J. Lutz. Computational depth and reducibility, *Theoret. Comput. Sci.* 132 (1994), 37-70.
- [10] A. H. Lachlan. wtt-complete sets are not necessarily tt-complete, *Proc. Amer. Math. Soc.* 48 (1975), 429-434.
- [11] P. Odifreddi. *Classical Recursion Theory*, North-Holland, Amsterdam, Vol.1, 1989.
- [12] R. I. Soare. *Recursively Enumerable Sets and Degrees*, Springer-Verlag, Berlin, 1987.