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RANDOMNESS NOTIONS AND PARTIAL RELATIVIZATION

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ABSTRACT

We study the computational complexity of an oracle set using a number of notions of randomness that lie between Martin–Löf randomness and 2-randomness in terms of strength. These notions are weak 2-randomness, weak randomness relative to $\emptyset'$, Demuth randomness and Schnorr randomness relative to $\emptyset'$. We characterize the oracles A such that $\mathsf{ML}[A] \subseteq \mathcal{C}$, where $\mathcal{C}$ is such a randomness notion and $\mathsf{ML}[A]$ denotes the Martin–Löf random reals relative to A, using a new meta-concept called partial relativization. We study the reducibility associated with weak 2-randomness and relate it with LR-reducibility.

1. Introduction

Studying the computational complexity of a set A of natural numbers is a fundamental goal of computability theory. As a tool, one often uses relativization, an important operation in both set theory and computability. All concepts in computability are ultimately defined in terms of computations; to relativize such a concept to an oracle set A means to enhance the underlying computational device (such as a Turing machine) by allowing it to ask whether certain numbers obtained during the computation are in A. Broadly speaking, one wants to understand the complexity of A by its effect on specific concepts when relativized to A.

There is a rich history of successful research in the case that the concept is a randomness notion. A central such notion is the one of Martin–Löf. Let μ denote the usual product measure on Cantor space 2^ω. A Martin–Löf test is a uniformly computably enumerable sequence (V_i) of open sets in Cantor space such that $\mu(V_i) \leq 2^{-i}$. A set is called **Martin–Löf random** if it passes all Martin–Löf tests in the sense that $Z \notin \bigcap_i V_i$.

Following [KT99], we say that an oracle A is **low for Martin–Löf** if each Martin–Löf random set is already Martin–Löf random relative to A. This class has multiple characterizations; for instance by [Nie05] it coincides with the K-trivial sets introduced by Chaitin [Cha76].

Martin–Löf randomness has been criticized for not being strong enough to appropriately formalize our intuition of a random set. For instance, relatively easily definable setscan be Martin–Löf random, such as the halting probability

Ω of a universal prefix-free machine, and some superlow sets (i.e., sets that are computable from the halting problem by means of a truth-table reduction). On the other hand, Martin–Löf randomness interacts very well with computability-theoretic concepts. Many examples of such an interaction are given in [Nie09a, Chapter 4]; see the beginning of that chapter for an overview.

We say that Z is 2-random if it is Martin–Löf random relative to the halting problem $\emptyset'$. In this paper we study the computational complexity of sets via relativization of randomness notions between Martin–Löf randomness and 2-randomness. In this way, we also find new interactions of these randomness notions with computability theoretic concepts. The notions include weak 2-randomness, where in the definition of tests the condition $\mu(V_i) \leq 2^{-i}$ is replaced by the weaker condition that $\lim_i \mu(V_i) = 0$; Demuth randomness, where passing the test means to be out of almost all V_i, but the components V_i can each be "replaced" a computably bounded number of times; and Schnorr randomness relative to $\emptyset'$, where tests are taken relative to $\emptyset'$, and in addition $\mu(V_i)$ is uniformly computable relative to $\emptyset'$.

There are two measures of computational complexity of a set A: absolute and relative. For the absolute complexity, one places A in classes, such as being low ($A' \leq_T \emptyset'$), or of hyperimmune-free Turing degree (each function computed by A is dominated by a computable function). For the relative complexity, one compares A with other sets via a reducibility such as Turing $\leq_T$.

We will study both aspects of complexity via relativization of randomness notions. For the **absolute complexity** aspect, consider randomness notions $\mathcal{C}$ and $\mathcal{D}$ where $\mathcal{D}$ is stronger than $\mathcal{C}$, i.e., we have the containment $\mathcal{D} \subset \mathcal{C}$. We ask:

which oracles A are computationally strong enough to ensure that $\mathcal{C}^A \subseteq \mathcal{D}$?

If $\mathcal{C}$ is Martin–Löf randomness and $\mathcal{D}$ is 2-randomness, then certainly any set Turing above $\emptyset'$ will have sufficient strength. However, there are others. Dobrinen and Simpson [DS04] called a set A **uniformly a.e. dominating** (u.a.e.d.) if A computes a function f such that for each Turing functional Ψ, for almost every Z, we have that Ψ^Z is total $\rightarrow \Psi^Z$ is dominated by f. Kjos-Hanssen, Miller and Solomon [KHMS10] showed that the sets A such that each Martin–Löf random in A is already 2-random coincide with the uniformly a.e. dominating sets.

In Section 3 we will answer the same question when $\mathcal{C}$ is Martin–Löf randomness, and $\mathcal{D}$ is any of the randomness notions mentioned above. For Demuth randomness, this answers a question left open in [FHM+10]. Each of the answers involves an important idea called **partial relativization**: a concept combining several computational notions is given, and only some of these components are relativized. This idea was introduced implicitly in papers of Simpson such as [CS07], and in more explicit form in Nies' 2009 talk [Nie09b]. See Subsection 2.1 for more detail.

For the **relative complexity** aspect, we consider reducibilities weaker than Turing. The first one was introduced in [Nie05] and has been widely studied since: A is LR reducible to a set B (denoted by $A \leq_{LR} B$) if every B-random set is A-random. (In particular, $\emptyset' \leq_{LR} B$ iff each B-random set is 2-random, which is equivalent to being u.a.e.d. as mentioned above.)

Generalizing the scheme that led to $\leq_{LR}$, for each randomness notion $\mathcal{C}$ we have an associated reducibility $\leq_{\mathcal{C}}$ given by

$$A \leq_{\mathcal{C}} B \Leftrightarrow \mathcal{C}^A \supseteq \mathcal{C}^B.$$

Namely, if A can find "regularities" in a set in the sense of $\mathcal{C}$, then so can B. In Section 4 we study for the first time such a reducibility other than $\leq_{LR}$: we consider the reducibility associated with weak 2-randomness, denoted by $\leq_{W2R}$. We show that it is unexpectedly close to $\leq_{LR}$. Firstly, $\leq_{W2R}$ implies $\leq_{LR}$. While this implication is shown to be strict, we also show that the degree equivalence classes corresponding to both weak reducibilities coincide. This extends the result that lowness for Martin–Löf coincides with lowness for weak 2-randomness (see [Nie09a, Thm. 5.5.17] and the references given there).

The degree equivalence classes are known to be countable by [KHMS10] combined with [Nie05]. Yet, we also show that there are continuum many Z such that $Z \leq_{W2R} \emptyset''$.

Every Δ^0_2 set is a Π^0_2 singleton. Hence, there is no weakly 2-random Δ^0_2 set. In the final Section 5 we study the LR-interaction of Δ^0_2 sets with weakly 2-random sets. We show that a Δ^0_2 set LR-below a weakly 2-random must be K-trivial. Further, there is a weakly 2-random set Z such that $Z \leq_{LR} \emptyset'$, and in fact Z is K-trivial relative to $\emptyset'$. Thus a weakly 2-random set can be very close to $\emptyset'$.

2. Background

2.1. Partial relativization. Recall that partial relativization of a computational concept to an oracle A means that we only relativize parts of its definition. In effect, we study what happens under restricted access to the oracle. In [CS07, Sim07] some properties obtained by partial relativization were shown to play an important role in the study of mass problems and the degrees of difficulty. Although we are not going to study these in the present paper, we mention the notions of bounded limit recursiveness and jump traceability as examples.

Given a class $\mathcal{C}$, we denote its full relativization to A by $\mathcal{C}[A]$. While full relativization to an oracle A is indicated with the phrase "relative to A" or "in A", partial relativization is indicated with the phrase "by A". In this subsection we give two examples of partial relativization which will be needed in Section 3.

First example: Let $J^X(e) = \Phi_e^X(e)$ where Φ_e is the e-th Turing functional; $J(e)$ is short for $J^\emptyset(e)$. Recall that a set Y is called diagonally non-computable (d.n.c.) if there is a function $f \leq_T Y$ such that $f(e) \neq J(e)$ whenever $J(e)$ is defined. Then Y is d.n.c. relative to an oracle A if there is a function $f \leq_T Y \oplus A$ such that $f(e) \neq J^A(e)$ whenever $J^A(e)$ is defined. We say that Y is d.n.c. *by* A if we can in fact choose $f \leq_T Y$; we do not relativize that component of the definition.

Recall that a set A is **generalized low** (GL_1) if $A' \leq_T \emptyset' \oplus A$. Equivalently, $\emptyset'$ is Turing complete relative to A, so this is an example of a full relativization. By the Arslanov completeness criterion relative to A, $\emptyset' \oplus A$ is d.n.c. relative to A iff $\emptyset'$ is Turing complete relative to A. If $\emptyset'$ is d.n.c. by A then it is d.n.c. relative to A, and hence GL_1.

Second example: We say that a sequence of sets $(T_n)_{n\in\mathbb{N}}$ is a trace for a function f if $f(n) \in T_n$ for all $n \in \mathbb{N}$. Also, a function h is a bound for (T_n) if $|T_n| < h(n)$ for all $n \in \mathbb{N}$. Recall that Y is c.e. traceable if there is a computable function h such that each function $f \leq_T Y$ has a uniformly c.e. trace with bound h. Since we trace only total functions, by a method of Terwijn and Zambella ([TZ01], or see [Nie09a, Thm. 8.2.3]), if Y is c.e. traceable then the bound on the required trace can be any non-decreasing unbounded computable function.

We say that Y is c.e. traceable *by* A if there is a computable function h such that each function $f \leq_T Y$ has a uniformly A-c.e. trace with bound h.

Thus, we only have to trace functions $f \leq_T Y$ (not $f \leq_T Y \oplus A$ as in full relativization). On the other hand, the bound on the size of the trace sets needs to be computable.

2.2. Randomness notions between Martin–Löf and 2-randomness. Recall from the introduction that a Martin–Löf test is a uniformly computably enumerable sequence $(V_i)_{i \in \mathbb{N}}$ of open sets in Cantor space such that $\mu(V_i) \leq 2^{-i}$, and a set is called Martin–Löf random if it passes all Martin–Löf tests in the sense that $Z \notin \bigcap_i V_i$.

Randomness notions other than Martin–Löf's are often obtained by varying the highly malleable concept of a Martin–Löf test, and sometimes also the passing condition.

- **Weak randomness** (or Kurtz randomness) is defined by asking that the tests have the special property that V_i is a clopen set generated by a uniformly computable finite set of strings.
- **Schnorr randomness** is defined by asking that the tests have the special property that their members have uniformly computable measure.
- **Weak 2-randomness** is the notion obtained when the condition $\mu(V_i) \leq 2^{-i}$ is replaced by the weaker condition that $\lim_i \mu(V_i) = 0$.

Note that Z is weakly random iff it is not a member of any null Π^0_1 class; Z is weakly 2-random if it is not a member of any null Π^0_2 class.

Weak 2-randomness is a natural notion of randomness which has a very simple definition. Its exact relation with Martin–Löf randomness was clarified by a result of Hirschfeldt/Miller (see [Nie09a, Section 5.3]): a set is weakly 2-random iff it is Martin–Löf random and it forms a minimal pair with $\emptyset'$.

Recall that a set is 2-random if it is Martin–Löf random relative to $\emptyset'$. We will only consider Schnorr and weak randomness relative to $\emptyset'$. Table 1 summarizes the relevant notations. For more background on algorithmic randomness we refer to [Nie09a, Chapter 3].

A Martin–Löf test $(U_i)_{i \in \mathbb{N}}$ is called **universal** if this single test is sufficient: $\bigcap_i U_i$ contains $\bigcap_k V_k$ for any Martin–Löf test $(V_k)_{k \in \mathbb{N}}$. It is well known that there is a universal Martin–Löf test. In contrast, with the exception of 2-randomness, the notions introduced above lack a universal test.

The following implications hold:

$$\mathsf{ML}[\emptyset'] \Rightarrow \mathsf{SR}[\emptyset'] \Rightarrow \mathsf{W2R} \Rightarrow \mathsf{Kurtz}[\emptyset'] \cap \mathsf{ML} \Rightarrow \mathsf{ML}. \tag{2.1}$$

Martin–Löf randomness	ML
weak randomness relative to $\emptyset'$	$\mathsf{Kurtz}[\emptyset']$
weak 2-randomness	$\mathsf{W2R}$
Schnorr random relative to $\emptyset'$	$\mathsf{SR}[\emptyset']$
2-randomness	$\mathsf{ML}[\emptyset']$

Table 1. Randomness notions and the symbols used to denote them.

The first implication follows from the definitions. The second follows from the observation that every null Π^0_2 class is contained in a Schnorr test. The third follows in a similar way (every $\Pi^0_1[\emptyset']$ class is a Π^0_2 class) and the fourth is trivial.

In the following we indicate why none of the implications can be reversed. The strictness of the first implication follows by relativizing the well known fact that some Schnorr random set is not Martin–Löf random.

The strictness of the second implication can be derived from the following, together with the result of [LMN07] that some weakly 2-random set is not GL_1.

Proposition 2.1: *Each set in* $\mathsf{SR}[\emptyset']$ *is* GL_1.

Proof. Uniformly in e, the set $\emptyset'$ can compute a stage s so large that e goes into A' after stage s for at most measure 2^{-e-1} oracles A. Let f be the $\emptyset'$-computable function that computes s from e. Given e and $s = f(e)$, the oracles A such that e goes into A' after stage s form a Σ^0_1 class V_e. Since $\mu V_e < 2^{-e}$, $\emptyset'$ can uniformly form a $\Sigma^0_1[\emptyset']$ class U_e that contains V_e and has measure exactly 2^{-e}. Then $S_i = \bigcup_{e>i} U_e$ determines a Schnorr test relative to $\emptyset'$. If $A \notin \bigcap_i S_i$, then, except for finitely many e, we have $e \in A'$ iff $e \in A'_{f(e)}$. Thus A is GL_1. ∎

The strictness of the third implication in (2.1) is shown in Theorem 2.3 below. Finally, for the strictness of the fourth implication, notice that some Martin–Löf random set is computable from $\emptyset'$, and hence a $\Pi^0_1[\emptyset']$ singleton.

Another strengthing of Martin–Löf randomness is Demuth randomness, introduced by Demuth [Dem88] to study differentiability of constructive functions defined on the unit interval. A **Demuth test** is a sequence of c.e. open sets $(V_i)_{i\in\mathbb{N}}$ such that $\forall i\, \mu V_i \le 2^{-i}$, and there is a function $f \le_{\mathrm{wtt}} \emptyset'$ such that V_i is the open set generated by the strings in $W_{f(i)}$ (viewed as a subset of $2^{<\omega}$).

A set Z **passes** the test if $Z \notin V_i$ for almost every i. We say that Z is **Demuth random** if Z passes each Demuth test. See [Nie09a, 3.6.24] for more detail, and a proof that Demuth randomness is incomparable with weak 2-randomness. Note that $\mathsf{SR}[\emptyset']$ is contained in the Demuth random sets. A more complex argument than in the previous result shows that each Demuth random set is in GL_1 (see [Nie09a, 3.6.26]).

2.3. The weakness of $\mathsf{Kurtz}[A]$ randomness. In the present subsection, which is independent of the rest of the paper, we investigate $\mathsf{Kurtz}[A]$ randomness for an arbitrary oracle A. It is easy to see that there is no A such that every Kurtz random relative to A is Martin–Löf random (in other words, there is no set in $\mathrm{High}(\mathsf{Kurtz}, \mathsf{ML})$ as defined at the beginning of Section 3 below). This follows from purely topological considerations. Since each component of a universal Martin–Löf test is dense in Cantor space, the non-ML random reals form a comeager class. On the other hand, for any A, the union of all measure zero $\Pi^0_1[A]$ classes is meager. Hence by the Baire category theorem, there is a Kurtz random relative to A that is not Martin–Löf random. One must work harder to answer the following question: what does Kurtz randomness relative to A imply if Z is already Martin–Löf random? We show that there is no oracle A such that Martin–Löf randomness and $\mathsf{Kurtz}[A]$ randomness together is enough to imply weak 2-randomness. First, we need the following lemma.

Lemma 2.2: *Let $P \subseteq 2^\omega$ be a nowhere dense Π^0_1 class. There is a null Π^0_2 class Q such that $Q \cap P$ is dense in P.*[1]

Proof. We will define Q to cover the left endpoints of maximal open intervals in $\overline{P} = 2^\omega - P$. Since P is nowhere dense, these points are dense in P. It will be helpful to use the euclidean metric on 2^ω; that is, for $X, Y \in 2^\omega$ we take $|X - Y|$ to be distance between the reals numbers in $[0, 1]$ whose binary expansions are given by X and Y.[2] We also use the natural order on 2^ω and

1 Liang Yu independently proved this lemma, and even without the assumption that P is nowhere dense. To show that this assumption is superfluous, let P be any Π^0_1 class and consider the null Π^0_2 class $Q \cup \bigcap_{i\in\omega} U_i$, where Q is the null Π^0_2 class constructed in our proof and (U_i) is a universal Martin–Löf test. Since Q covers the left endpoints of maximal open intervals in $2^\omega - P$ and $\bigcap_{i\in\omega} U_i$ is dense in 2^ω, the union is dense in P.

2 Strictly speaking, this is not a metric on 2^ω since the two distinct sequences representing a dyadic rational have distance zero from each other.

let $\mathcal{F} \subseteq 2^\omega$ represent the sequences with finitely many ones. For $s \in \omega$, let

$$V_s = \{X\colon (\exists t \geq s)(\exists A, B \in \mathcal{F})\ X \in P_s \text{ and } X < A < B \text{ and } [A, B] \cap P_t = \emptyset \text{ and } |A - X| < |B - A|/s\}.$$

It should be clear that V_s is a Σ^0_1 class. It is also easy to see that if X is the left endpoints of a maximal open interval in $\overline{P}$, then $X \in V_s$. Hence, letting $Q = \bigcap_{s\in\omega} V_s$, we have $X \in Q$. All that remains to prove is that $\mu(Q) = 0$, for which it is sufficient to show that $\lim_s \mu(V_s) = 0$.

Fix $s \in \omega$. Let (Y, Z) be a maximal interval in $\overline{P}$ and let $\ell = |Z - Y|$ be its length. Say that X is added to V_s with witnesses $A, B \in (Y, Z)$. If $X \notin (Y, Z)$, then it must be the case that $X < Y$ and $|Y - X| < |A - X| < \ell/s$. Thus we have $\mu(V_s) \leq (1 + 1/s)\mu(\overline{P})$. On the other hand, this estimate includes the measure of all the sequences in $\overline{P_s}$, but these have been excluded in the definition of V_s. So in fact, we have $\mu(V_s) \leq (1 + 1/s)\mu(\overline{P}) - \mu(\overline{P_s})$. But both $(1 + 1/s)\mu(\overline{P})$ and $\mu(\overline{P_s})$ approach $\mu(\overline{P})$ as s goes to infinity. Therefore, $\lim_s \mu(V_s) = 0$. ∎

Now we are ready to separate $\mathsf{Kurtz}[A]$ from the weakly 2-randoms within the class of Martin–Löf randoms.

Theorem 2.3: *For any oracle set A, there is a Martin–Löf random Z in $\mathsf{Kurtz}[A]$ that is not weakly 2-random.*

Proof. Let P be a Π^0_1 class containing only Martin–Löf random reals. Let Q be the measure zero Π^0_2 class from the lemma. We will, as in the remarks before Lemma 2.2, use the Baire category theorem, but this time with respect to the compact subspace P. Note that $Q \cap P$ is a G_δ set relative to P and it is dense in P, hence it is comeager in P. Next, consider a measure one $\Sigma^0_1[A]$ class V. Let $\sigma \in 2^{<\omega}$. If $[\sigma] \cap P \neq \emptyset$, then it is a non-empty Π^0_1 class containing a Martin–Löf random. So $\mu([\sigma] \cap P) > 0$ because otherwise we would obtain a Martin–Löf test which contains a Martin–Löf random set. Hence $V \cap [\sigma] \cap P$ is non-empty. Therefore, $V \cap P$ is dense in P. Since it is an open set relative to P, it is also comeager in P. By the Baire category theorem relative to P, there is a $Z \in P$ in the intersection of Q with (the countable collection of) all measure one $\Sigma^0_1[A]$ classes. Clearly Z is Kurtz random relative to A. Since $Z \in P$, it is Martin–Löf random. Finally, $Z \in Q$ implies that it is not weakly 2-random. ∎

3. Characterizing highness notions

For two classes $\mathcal{C}$ and $\mathcal{D}$ where usually $\mathcal{C} \supset \mathcal{D}$, we define High$(\mathcal{C}, \mathcal{D})$ to be the class containing all oracles A such that $\mathcal{C}^A \subseteq \mathcal{D}$. In this section we characterize highness notions when $\mathcal{C}$ is Martin–Löf randomness and $\mathcal{D}$ is a stronger randomness notion. The results are summarized in Table 2. We prove the characterizations in (a)–(d). As already mentioned, the equivalence (e) is due to Kjos-Hanssen, Miller and Solomon [KHMS10] (also see [Sim07] for a proof). In Corollary 4.4 we show that High$(\mathsf{W2R}, \mathsf{ML}[\emptyset'])$ also coincides with the uniformly a.e. dominating sets.

(a)	$A \in$ High$(\mathsf{ML}, \mathsf{Kurtz}[\emptyset'])$	$\emptyset'$ is non-d.n.c. by A
(b)	$A \in$ High$(\mathsf{ML}, \mathsf{W2R})$	
(c)	$A \in$ High$(\mathsf{ML}, \mathsf{Demuth})$	A is ω-c.e. tracing
(d)	$A \in$ High$(\mathsf{ML}, \mathsf{SR}[\emptyset'])$	$\emptyset'$ is c.e. traceable by A
(e)	$A \in$ High$(\mathsf{ML}, \mathsf{ML}[\emptyset'])$	A is u.a.e. dominating

Table 2. Highness classes with respect to randomness notions, and their computability-theoretic characterizations.

3.1. The class High$(\mathsf{ML}, \mathsf{W2R})$. We give a characterization of the highness property High$(\mathsf{ML}, \mathsf{W2R})$ in computability theoretic terms. Despite the fact that $\mathsf{W2R}$ is a stronger randomness notion than $\mathsf{ML} \cap \mathsf{Kurtz}[\emptyset']$, the computational strength that is required for an oracle A to turn $\mathsf{ML}[A]$ into a subclass of $\mathsf{Kurtz}[\emptyset']$ is the same as the strength required to turn it into a subclass of $\mathsf{W2R}$.

We start with the following lemma, which is a partial relativization of a result from [GM09]; the proof is due to the second author. Let $\mathsf{DNC}[A]$ denote the set of functions f such that $f(e) \neq J^A(e)$ whenever $J^A(e)$ is defined. Thus, Y is d.n.c. by A iff Y computes such a function.

Lemma 3.1: *If $A \in$ High$(\mathsf{ML}, \mathsf{Kurtz}[Y])$, then Y does not compute a $\mathsf{DNC}[A]$ function.*

Proof. Assume that $f \leq_T Y$ is a $\mathsf{DNC}[A]$ function. We show that Y computes an infinite subset D of a set that is ML-random in A. This shows that there is a set that is Martin–Löf random in A but is in a null $\Pi^0_1[Y]$ class, thus not in

$\mathsf{Kurtz}[Y]$. Let Q be a non-empty $\Pi^0_1[A]$ class of ML[A]-random sets. By a well known lemma of Kučera [Kuč85], we may assume that if $P \subseteq Q$ is a non-empty $\Pi^0_1[A]$ class, then we can compute, uniformly from an index for P, a k such that $2^{-k} < \mu P$.

Using f we compute a sequence $d_0 < d_1 < \cdots$ such that, for each n, the $\Pi^0_1[A]$ class $\{Z \in Q\colon\ d_0, \dots, d_{n-1} \in Z\}$ is non-empty. Let $D = \{d_0, d_1, d_2, \dots\}$. By compactness $\{Z \in Q\colon\ D \subseteq Z\}$ is non-empty. Suppose we have determined $d_0 < \cdots < d_{n-1}$ such that the $\Pi^0_1[A]$ class

$$P_n = \{Z \in Q\colon\ d_0, \dots, d_{n-1} \in Z\}$$

is non-empty. The set $G = \{m\colon\ \forall Z \in P_n\,[Z(m) = 0]\}$ is c.e. in A uniformly in an index for P_n. We will determine $d_n \notin G$. Since $P_n \subseteq Q$ is non-empty, compute k such that $2^{-k} < \mu P_n$ and hence $|G| \le k$.

Let $\omega^{<\omega}$ be the set of finite sequences of natural numbers. We denote concatenation of strings by $*$. Let $(S_\sigma)_{\sigma\in\omega^{<\omega}}$ be a uniformly computable sequence of sets such that $S_\varnothing = \mathbb{N}$ and, for each σ, $(S_{\sigma * i})_{i\in\mathbb{N}}$ is an infinite partition of S_σ into non-empty sets. Define a Turing functional Ψ as follows. Let $\Psi^A(\sigma) = i$ if i is the first number such that some element of $S_{\sigma * i}$ is enumerated in G. The relation '$\simeq$' means that if the left-hand side is defined then it is equal to the right-hand side. Let α be a computable function such that $J^A(\alpha(\sigma)) \simeq \Psi^A(\sigma)$ for all $\sigma \in \omega^\omega$, where J is the jump functional (i.e., $J^A(e) \simeq \Phi^A_e(e)$, where (Φ_e) is an effective list of all Turing functionals). Since f is d.n.c. relative to A, we have $f(\alpha(\sigma)) \neq \Psi^A(\sigma)$ for each σ.

Now let $\sigma_0 = \varnothing$ and $\sigma_{i+1} = \sigma_i * f(\alpha(\sigma_i))$ for $i < k$. Clearly $G \cap S_{\sigma_k} = \emptyset$ since for each $i < k$ some element of G is in some $S_{\sigma_i * r}$ for $r \neq f(\alpha(\sigma_i))$ (unless already $G \cap S_{\sigma_i} = \emptyset$). Choose $d_n > d_{n-1}$ in S_{σ_k}. Then d_n is as desired, and the sequence (d_i) is computable in Y. So $\bigcap_i P_i$ is a non-empty $\Pi^0_1[Y]$ class of measure 0 and it is contained in Q. Therefore there is a Martin–Löf random set relative to A that is not in $\mathsf{Kurtz}[Y]$. ∎

THEOREM 3.2: *For $A \in 2^\omega$, the following are equivalent:*

(i) $A \in \mathrm{High}(\mathsf{ML}, \mathsf{W2R})$,
(ii) $A \in \mathrm{High}(\mathsf{ML}, \mathsf{Kurtz}[\emptyset'])$,
(iii) *$\emptyset'$ does not compute a $\mathsf{DNC}[A]$ function.*

Proof. (iii)⇒(i) Assume that $\{V_n\}_{n\in\omega}$ is an effective sequence of Σ^0_1 classes such that $\mu(V_n) \to 0$. It suffices to show that $\bigcap_n V_n$ is contained in a Martin–Löf test relative to A. Note that $\emptyset'$ computes a function f such that $\mu(V_{f(k)}) \leq 2^{-k}$, for all $k \in \omega$. For a Σ^0_1 class V and rational $\varepsilon > 0$, let $(V)_\varepsilon$ denote the Σ^0_1 class uniformly obtained by enumerating V as long as the measure does not exceed ε. Since $\emptyset'$ does not compute a $\mathsf{DNC}[A]$ function, there are infinitely many k such that $f(k) = J^A(k)$, where J denotes the jump functional. Therefore, $S_m = \bigcup_{k>m}(V_{J^A(k)})_{2^{-k}}$ covers $\bigcap_{n\in\omega} V_n$, for each m (where $V_{J^A(k)}$ is taken to be empty if $J^A(k)\uparrow$). By definition, $\mu(S_m) \leq 2^{-m}$, so $\{S_m\}_{m\in\omega}$ is a Martin–Löf test relative to A that covers $\bigcap_n V_n$. Hence $A \in \mathrm{High}(\mathsf{ML}, \mathsf{W2R})$.

Since every $\Pi^0_1[\emptyset']$ class is a Π^0_2 class, we have (i)⇒(ii). Finally, (ii)⇒(iii) follows by Lemma 3.1 for $Y = \emptyset'$. ∎

By (i)↔(iii) of Theorem 3.2 and the remarks in Subsection 2.1, if A is not GL_1, then A is in $\mathrm{High}(\mathsf{ML}, \mathsf{W2R})$. In particular we obtain the following.

Corollary 3.3: *Let A be a Δ^0_2 set. Then the following are equivalent:*

(i) $A \in \mathrm{High}(\mathsf{ML}, \mathsf{W2R})$,
(ii) $A \in \mathrm{High}(\mathsf{ML}, \mathsf{Kurtz}[\emptyset'])$,
(iii) *A is not low.*

3.2. The class $\mathrm{High}(\mathsf{ML}, \mathsf{SR}[\emptyset'])$. Recall from Subsection 2.1 that Y is called c.e. traceable by A if there is a computable function h such that for each $f \leq_T Y$ there is an A-c.e. trace for f with bound h. The next theorem with $Y = \emptyset'$ characterizes the condition that $A \in \mathrm{High}(\mathsf{ML}, \mathsf{SR}[\emptyset'])$, row (d) in Table 2. First, we need the following fact whose proof relies on the Lebesgue density theorem. For a string τ, let $\mu_\tau(S)$ be the measure of a class $S \subseteq 2^\omega$ relative to $[\tau] = \{X \mid \tau \prec X\}$. That is, $\mu_\tau(S) = \mu([\tau] \cap S)/2^{-|\tau|}$.

Lemma 3.4 ([Nie09a], Lemma 8.3.4): *Suppose that $\bigcap_n U_n \subseteq R$ for open sets U_n, R with $\mu(R) < q < 1$. Then there is a string τ and $d \in \mathbb{N}$ such that $\mu_\tau(R) < q$ and $\mu_\tau(U_d - R) = 0$.*

Theorem 3.5: *Let $A, Y \in 2^\omega$. Then*

$$\mathsf{ML}[A] \subseteq \mathsf{SR}[Y] \;\Leftrightarrow\; Y \text{ is c.e. traceable by } A.$$

Proof. ⇐: It suffices to show that every Schnorr test relative to Y is contained in a Martin–Löf test relative to A. Let (V_i) be a Schnorr test relative to Y, i.e., a Martin–Löf test relative to Y where the sequence $(\mu(V_i))$ is Y-computable. Without loss of generality we can assume that $\mu(V_n) = 2^{-n-1}$ for each $n \in \mathbb{N}$. Now let (D_i) be an effective sequence of all finite sets. There is a Y-computable function f such that $V_n = \bigcup_i D_{f(n,i)}$ and $\mu(D_{f(n,i)}) \leq 2^{-n-i}$ for all $n, i \in \mathbb{N}$. Now consider a trace of $f(n,i)$ which is computable in A with bound $n+i$. That is, an A-c.e. sequence $(T_{n,i})$ such that $|T_{n,i}| \leq n+i$ and $f(n,i) \in T_{n,i}$ for all $n, i \in \mathbb{N}$. Without loss of generality we can assume that $T_{n,i}$ only contains numbers j such that $\mu(D_j) \leq 2^{-n-i}$. Define $U_n = \bigcup_i \bigcup_{j \in T_{n,i}} D_j$. Clearly $V_n \subseteq U_n$ for all $n \in \mathbb{N}$. Also,

$$\mu(U_n) \leq \sum_i (n+i) \cdot 2^{-n-i},$$

which means that (U_n) is a Martin–Löf test relative to A (modulo a computable shift of the indices).

$\Rightarrow$: Suppose $f \leq_T Y$ and we wish to build an A-c.e. trace for f with bound 2^n. It suffices to build an A-c.e. trace for the function given by $g(n) := nf(n) + n$. Let $B_{k,n}$ be the set of reals that have n consecutive 0s starting at the k-th digit. Clearly, $\mu(B_{k,n}) = 2^{-n}$ for all $k, n \in \mathbb{N}$. It is easy to check that the sets $U_d = \bigcup_{n>d} B_{g(n),n}$ form a Schnorr test relative to Y. Let R be the second member of the universal Martin–Löf test relative to A, so that $\mu(R) < 2^{-2}$. Since $\mathsf{ML}[A] \subseteq \mathsf{SR}[Y]$ we have $\bigcap_d U_d \subseteq R$. By Lemma 3.4 there is a string τ and $d \in \mathbb{N}$ such that $\mu_\tau(R) < 2^{-2}$ and $\mu_\tau(B_{g(n),n} - R) = 0$ for all $n > d$. Now let $n\mathbb{N}$ denote the multiples of n and consider the following trace:

$$T_n = \{k \in n\mathbb{N} \mid \mu_\tau(B_{k,n} - R) < 2^{-k-3}\}. \tag{3.1}$$

Since $B_{k,n}$ clopen and R is $\Sigma^0_1[A]$, the sequence (T_n) is uniformly c.e. in A. On the other hand, $g(n) \in T_n$ for all $n > d$, by the choice of d, τ.

It remains to show that the sequence $|T_n|$ is computably bounded. By (3.1) we have $\mu_\tau(\bigcup_{k \in T_n} B_{k,n} - R) < 2^{-2}$, which implies that

$$\mu_\tau\left(2^\omega - \bigcup_{k \in T_n} B_{k,n}\right) + \mu_\tau(R) \geq 1 - 2^{-2}.$$

Since $\mu_\tau(R) < 2^{-2}$, this means that $\mu_\tau(2^\omega - \bigcup_{k \in T_n} B_{k,n}) > 2^{-1}$. On the other hand, $\mu_\tau(B_{k,n}) = 2^{-n}$ for $n > |\tau|$. Since T_n consists of multiples of n, the sets

$B_{k,n}$, $k \in T_n$ are independent and

$$\mu_\tau\left(2^\omega - \bigcup_{k\in T_n} B_{k,n}\right) = (1-2^{-n})^{|T_n|}$$

for $n > |\tau|$. Hence $(1-2^{-n})^{|T_n|} > 2^{-1}$ which shows that $|T_n| < 2^n$, for $n > |\tau|$.[3] ∎

We note that the proof of Theorem 3.5 is an adaptation of the proof of Theorem 8.3.3 in [Nie09a].

3.3. The class High($\mathsf{ML}, \mathsf{Demuth}$). Note that $\emptyset'$ is c.e. traceable by A if there is a computable function h such that, for each Δ^0_2 function, there is an A-c.e. trace for f with bound h. In the foregoing subsection, we showed that this computability theoretic condition characterizes the class High($\mathsf{ML}, \mathsf{SR}[\emptyset']$). In this subsection we characterize the larger class High($\mathsf{ML}, \mathsf{Demuth}$) by a weaker variant of this property, which was introduced in [FHM$^+$10, Definition 21 of the Journal version].

A set A is called **ω-c.e.-tracing** if there is a computable function h such that each function $f \le_{\mathrm{wtt}} \emptyset'$ has an A-c.e. trace $(T_x)_{x\in\mathbb{N}}$ such that $|T^A_x| \le h(x)$ for each x. Since we trace only total functions, by the method of Terwijn and Zambella already mentioned above, the bound h can be replaced by any non-decreasing unbounded computable function without changing the class. In [FHM$^+$10, Prop. 32] it is shown that each ω-c.e. tracing set is in High($\mathsf{ML}, \mathsf{Demuth}$). We provide the converse implication. This establishes row (c) in Table 2.

Theorem 3.6: *For $A \in 2^\omega$, we have*

$$\mathsf{ML}[A] \subseteq \mathsf{Demuth} \Leftrightarrow A \text{ is } \omega\text{-c.e.-tracing.}$$

Proof. $\Leftarrow$: This was proved in [FHM$^+$10].
$\Rightarrow$: The proof is a modification of the proof of the corresponding implication in Theorem 3.5. Firstly, we provide a variant of Lemma 3.4 suitable for Demuth randomness.

Lemma 3.7: *Suppose that $\{Z \mid \exists^\infty n\, Z \in U_n\} \subseteq R$ for open sets U_n, R with $\mu(R) < q < 1$. Then there is a string τ and $d \in \mathbb{N}$ such that*

[3] This follows from the fact that $(1-1/k)^k < e^{-1}$ for any $k \ge 1$.

$$\mu_\tau(R) < q \text{ and } \forall n > d\,[\mu_\tau(U_n - R) = 0].$$

Assume the hypothesis holds but the conclusion fails. We define inductively a sequence of strings $(\tau_d)_{d\in\mathbb{N}}$ such that $\tau_0 \prec \tau_1 \prec \cdots$ and $\forall d\, \mu(R \mid \tau_d) < q$. Let τ_0 be the empty string. Suppose τ_d has been defined and $\mu(R \mid \tau_d) < q$. Then, since the Lemma fails, there is $n > d$ such that $\mu((U_n - R) \mid \tau_d) > 0$. So we can choose y such that $[y] \subseteq U_n$ and $\mu([y] - R \mid \tau_d) > 0$; in particular, $y \succcurlyeq \tau_d$. By the Lebesgue density theorem we may choose $\tau_{d+1} \succ y$ such that $\mu(R \mid \tau_{d+1}) < q$.

Now let $Z = \bigcup_d \tau_d$; then $\exists^\infty n\, Z \in U_n$ and $Z \notin R$, contradiction. This establishes the lemma.

To conclude the proof of Theorem 3.6, suppose f is an ω-c.e. function and we wish to build an A-c.e. trace for f with bound 2^n. As before, it suffices to build an A-c.e. trace for the function given by $g(n) := nf(n) + n$. Let $U_n = B_{g(n),n}$. Since g is ω-c.e., the sequence $(U_n)_{n\in\mathbb{N}}$ forms a Demuth test. As before let R be the second member of the universal Martin–Löf test relative to A, so that $\mu(R) < 2^{-2}$. By the hypothesis that $\mathsf{ML}[A] \subseteq \mathsf{Demuth}$, we may pick τ, d according to Lemma 3.7 where $q = 2^{-2}$.

Define the A-c.e. trace $(T_n)_{n\in\mathbb{N}}$ by (3.1). By the Lemma we have $\mu_\tau(R) < q$ and $\forall n > d\,[\mu_\tau(U_n - R) = 0]$. Hence, as before, $|T_n| < 2^n$ and $(T_n)_{n\in\mathbb{N}}$ is a trace for g. ∎

4. The reducibility associated with weak 2-randomness

Recall from the introduction the weak reducibility associated with weak 2-randomness: $A \leq_{W2R} B$ if each weak 2-random relative to B is weakly 2-random relative to A. In this section we study $\leq_{W2R}$ and compare it to $\leq_{LR}$.

A set is called **low for** Ω if Ω is Martin–Löf random relative to it. We show that $\leq_{LR}$ and $\leq_{W2R}$ coincide on the Δ^0_2 sets, as well as the low for Ω sets. Given that the low for Ω sets are downward closed with respect to $\leq_{LR}$, it follows that the two reducibilities have interesting common initial segments. On the other hand, we show that they differ on the class of Δ^0_3 sets.

The two reducibilities $\leq_{LR}$ and $\leq_{W2R}$ induce equivalence relations $\equiv_{LR}$ and $\equiv_{W2R}$ respectively on $\mathcal{P}(\mathbb{N})$, and therefore degree structures. We show that $\equiv_{LR}$, $\equiv_{W2R}$ coincide on all sets. Hence, although the degree structures differ as partially ordered sets, the actual degrees as equivalence classes coincide.

Barmpalias, Lewis and Soskova [BLS08a] proved that there are continuum many sets $\leq_{LR} \emptyset'$. We conclude Section 4 with a similar result, proving that there are continuum many sets $\leq_{W2R} \emptyset''$.

We will frequently use a theorem of Kjos-Hanssen, Miller and Solomon (also see [Nie09a, Thm. 5.6.9]).

Theorem 4.1 ([KHMS10]): *If $A \leq_{LR} B$ and $A \leq_T B'$ then every $\Pi^0_2[A]$ class is contained in a $\Pi^0_2[B]$ class of the same measure.*

In fact, the converse implication also holds.

4.1. $\leq_{W2R}$ implies $\leq_{LR}$. We say that a class $\mathcal{C} \subseteq 2^\omega$ is *bounded* if $\mu\mathcal{C} < 1$. Let U be the second component of a fixed universal oracle Martin–Löf test (thus $\mu U^X \leq 1/2$ for each oracle X). Kjos-Hanssen [KH07] proved that the following are equivalent for $X, Y \in 2^\omega$:

(a) $X \leq_{LR} Y$;
(b) there exists a bounded $\Sigma^0_1[Y]$ class V such that $U^X \subseteq V$.

This shows that $\leq_{LR}$ is Σ^0_3. We show that instead of $U^X \subseteq V$, one can equivalently require the weaker condition that $U^X - V$ is null.

Lemma 4.2: *The following are equivalent for $X, Y \in 2^\omega$:*

(a) *$X \leq_{LR} Y$.*
(b) *There exists a bounded $\Sigma^0_1[Y]$ class V such that $\mu(U^X - V) = 0$.*

Proof. We have (a) $\Rightarrow$ (b) from the Theorem of Kjos-Hanssen, so it suffices to show that (b) $\Rightarrow$ (a). Choose a bounded $\Sigma^0_1[Y]$ class V such that $\mu(U^X - V) = 0$, and a rational $q < 1$ such that $\mu(V) < q$. We claim that $\mu_\sigma(V) = 1$ for all σ such that $[\sigma] \subseteq U^X$. Otherwise, there is a σ such that $\mu_\sigma(V) < 1$ and $[\sigma] \subseteq U^X$. This implies

$$\mu(U^X - V) \geq \mu([\sigma] - V) > 2^{-|\sigma|}(1 - \mu_\sigma(V)) > 0,$$

which contradicts the hypothesis (b). Let

$$F = \{\tau \mid \tau \text{ is minimal such that } \mu_\tau(V) > q\}.$$

In other words, F is the set of all strings τ such that $\mu_\tau(V) > q$ and $\mu_\rho(V) \leq q$ for all proper prefixes ρ of τ. Then we have $U^X \subseteq [F]$ and $[F]$ is a $\Sigma^0_1[Y]$ class.

If (ρ_i) is a list of the strings in F, then

$$q \cdot \mu([F]) \leq \sum_i 2^{-|\rho_i|}\mu_{\rho_i}(V) = \mu(V \cap [F]) \leq \mu(V) < q,$$

which implies that $\mu(F) < 1$, proving (a). ∎

We will now use the foregoing lemma to show that $X \leq_{LR} Y$ is equivalent to $\mathsf{W2R}[Y] \subseteq \mathsf{ML}[X]$:

Theorem 4.3: *The following are equivalent for $X, Y \in 2^\omega$:*

(a) $X \leq_{LR} Y$.
(b) *Every weakly 2-random relative to Y is Martin–Löf random relative to X.*

Hence, $\leq_{W2R}$ implies $\leq_{LR}$.

Proof. By definition of $\leq_{LR}$ we have (a) $\Rightarrow$ (b). For (b) $\Rightarrow$ (a) suppose that $X \not\leq_{LR} Y$. We construct a set Z that is weakly 2-random relative to Y but not Martin–Löf random relative to X.

Let (U_i) be a universal oracle Martin–Löf test. By Lemma 4.2 we know that for every $\tau \in 2^{<\omega}$, every $\Sigma^0_1[Y]$ class V^Y and every $i \in \mathbb{N}$, if $\mu([\tau] - V^Y) > 0$ then there exists $[\sigma] \subseteq U^X_i$ such that $\tau \subset \sigma$ and $\mu([\sigma] - V^Y) > 0$. Otherwise, $(2^\omega - [\tau]) \cup V^Y$ would satisfy part (b) of Lemma 4.2. Let (S^e_j) be a double sequence of $\Sigma^0_1[Y]$ classes such that $S^e_{j+1} \subseteq S^e_j$ and every $\Pi^0_2[Y]$ class is of the form $\bigcap_j S^e_j$ for some e. We build $Z = \bigcup_s \sigma_s$ and a sequence of open sets (R_s) in stages.

Let $\sigma_0 = \emptyset$ and R_0 be S^0_j for the least j such that $\mu(S^0_j) < 2^{-2}$ if there is such, and $\emptyset$ otherwise. Inductively assume that $\mu([\sigma_s] - R_s) > 0$ and at stage $s+1$ we choose some $\sigma \supset \sigma_s$ such that $\sigma \in U^X_s$ and $\mu([\sigma] - R_s) > 0$. Let $\sigma_{s+1} = \sigma$. Let $q > 0$ be a rational such that $\mu([\sigma_{s+1}] - R_s) > q$ and let $R_{s+1} = R_s \cup S$ where S is S^{s+1}_j for the least j such that $\mu(S^{s+1}_j) < q$ if there is such, and $\emptyset$ otherwise. Notice that $\mu([\sigma_{s+1}] - R_{s+1}) > 0$.

The construction is well defined since R_s is $\Sigma^0_1[Y]$ for all $s \in \mathbb{N}$, so the required string σ will be found at every stage $s+1$. Moreover, $[\sigma_s] \not\subseteq R_s$ for all $s \in \mathbb{N}$ and $R_t \subseteq R_s$ for all $t < s$. So $Z = \bigcup_s \sigma_s$ is not in any R_t, which shows that it is not in any null $\Pi^0_2[Y]$ class. On the other hand $Z \in \bigcap_i U^X_i$, so it is not 1-random relative to X. ∎

As mentioned above, $\emptyset' \leq_{LR} Y$ is equivalent to Y being uniformly a.e. dominating. Hence Theorem 4.3 with $X = \emptyset'$ yields a further classification of a highness property along the lines of Table 2.

Corollary 4.4: High($\mathsf{W2R}, \mathsf{ML}[\emptyset']$) *equals the class of u.a.e.d. sets.*

In contrast, by the remarks after Theorem 3.2 we have $\overline{GL_1} \subseteq \mathrm{High}(\mathsf{ML}, \mathsf{W2R})$. Since $\overline{GL_1}$ is a much larger class than the u.a.e.d. sets, this gives evidence that the class ML is closer to $\mathsf{W2R}$ than $\mathsf{W2R}$ is to $\mathsf{ML}[\emptyset']$. Indeed, the computational strength required for an oracle A to lift Martin–Löf randomness to weak 2-randomness is just the property '$\emptyset'$ non-d.n.c. by A' (which contains $\overline{GL_1}$ and coincides with non-lowness for Δ^0_2 sets). This is much less than u.a.e. domination, which is the strength required for A to lift weak 2-randomness to 2-randomness.

4.2. When $\leq_{W2R}$ coincides with $\leq_{LR}$. The sets that are low for Ω are, by definition, closed downward with respect to $\leq_{LR}$; in other words, they form an initial segment of the LR degrees.

Theorem 4.5: *The relations $\leq_{W2R}$ and $\leq_{LR}$ coincide on the LR initial segment of sets that are low for Ω.*

Proof. Let X, Y be low for Ω reals such that $X \leq_{LR} Y$. In view of Theorem 4.3 it suffices to show that $X \leq_{W2R} Y$. By a theorem in [Mil10] we have that $X \leq_T Y'$. Then by Theorem 4.1 due to [KHMS10], every weakly 2-random real relative to Y is also weakly 2-random relative to X, i.e., $X \leq_{W2R} Y$. ∎

By Theorem 4.1 we also obtain the following.

Corollary 4.6: *The relations $\leq_{LR}, \leq_{W2R}$ coincide on the class of Δ^0_2 sets.*

4.3. When $\leq_{W2R}$ does not coincide with $\leq_{LR}$. First we show that the relations $\leq_{W2R}$ and $\leq_{LR}$ differ on every LR-lower cone of a non-K-trivial Δ^0_2 set.

Theorem 4.7: *If Y is Δ^0_2 and Y is not K-trivial, then for all $Z \geq_T \emptyset'$ there exists $X \leq_{LR} Y$ such that $X \oplus \emptyset' \equiv_T Z$.*

Proof. By [Bar10] we know that there is a perfect Π^0_1 class P such that $A \leq_{LR} Y$ for all $A \in P$. We use P and a standard coding to define $X \in P$ in stages s by finite extensions σ_s. Let $\sigma_0 = \emptyset$ and, if σ_s is defined, find (with oracle $\emptyset'$)

the least node $\tau \supset \sigma_s$ such that both $\tau * 0$, $\tau * 1$ are extendible in P. Then define $\sigma_{s+1} = \tau * Z(s)$. Clearly $Z \equiv_T X \oplus \emptyset'$ and $X \leq_{LR} Y$ since X belongs to P. ∎

Corollary 4.8: *If Y is Δ^0_2 and not K-trivial, then there exists $X \leq_{LR} Y$ such that $X \not\leq_{W2R} Y$.*

Proof. Let X be as in Theorem 4.7 for $Z = \emptyset'''$. It suffices to find a set A that is not weakly 2-random relative to X but is weakly 2-random relative to Y. Let A be a 3-random that is recursive in $\emptyset'''$. Since $Y \leq_T \emptyset'$, the set A is (weakly) 2-random relative to Y (i.e., $A \in \mathsf{ML}[Y']$). However,

$$A \leq_T \emptyset''' \leq_T X \oplus \emptyset' \leq_T X',$$

so A belongs to a null $\Pi^0_2[X]$ class; in fact, $\{A\}$ is $\Pi^0_2[X]$. Hence, A is not weakly 2-random relative to X. ∎

Next, we obtain a result that contrasts with Corollary 4.6. It follows by using lowness in the proof of Corollary 4.8.

Corollary 4.9: *The relations $\leq_{LR}, \leq_{W2R}$ do not coincide on the class of Δ^0_3 sets.*

Proof. Notice that the set X separating $\leq_{LR}$, $\leq_{W2R}$ that was constructed in the proof of Theorem 4.7 is computable in $\emptyset' \oplus Z$. Now in the statement of Corollary 4.8, pick Y such that $Y' \equiv_T \emptyset'$ and $Y \not\leq_{LR} \emptyset$. We modify the proof so that we separate $\leq_{LR}$, $\leq_{W2R}$ within Δ^0_3. Consider the X given by Theorem 4.7 for $Z = \emptyset''$. Let A be 2-random and computable in $\emptyset''$. Since Y is low, the set A is 2-random relative to Y, i.e., $A \in \mathsf{ML}[Y']$. In particular, it is weakly 2-random relative to Y. However,

$$A \leq_T \emptyset'' \leq_T X \oplus \emptyset' \leq_T X',$$

so A belongs to a null $\Pi^0_2[X]$ class. Hence, it is not weakly 2-random relative to X. Finally, note that Y is Δ^0_2 and $X \leq_T \emptyset' \oplus Z \equiv_T \emptyset''$ is Δ^0_3. ∎

4.4. The equivalence relations $\equiv_{LR}$ and $\equiv_{W2R}$ coincide.

From [KHMS10] and [Nie05] we know that if $A \equiv_{LR} B$ then $A' \equiv_{tt} B'$ (see [Sim07] for more discussion). This, combined with Theorem 4.1, gives the following.

Corollary 4.10: *For all sets A, B we have $A \equiv_{LR} B$ if and only if $A \equiv_{W2R} B$.*

Hence the equivalence classes induced by $\leq_{W2R}$ coincide with those induced by $\leq_{LR}$, but the ordering of them differs as was shown in Corollary 4.8.

4.5. An uncountable initial segment of $\leq_{W2R}$. Despite the above results, we do not have a characterization of $\leq_{W2R}$ similar to the one of Kjos–Hanssen mentioned above. In particular, we do not know whether $\leq_{W2R}$ is arithmetical (note that its definition is merely Π^1_1).

For a first approach, note that if every $\Pi^0_2[A]$ null class is contained in some $\Pi^0_2[B]$ null class (i.e., if A is "test-wise" reducible to B), then $A \leq_{W2R} B$. However, the converse is open:

Question 4.11: Does $A \leq_{W2R} B$ imply that every $\Pi^0_2[A]$ null class is contained in some $\Pi^0_2[B]$ null class?

As noted in Theorem 4.1, Kjos-Hanssen, Miller and Solomon [KHMS10] studied a stronger condition, that every $\Pi^0_2[A]$ class is contained in a $\Pi^0_2[B]$ class of the same measure. They proved that this condition is equivalent to $A \leq_{LR} B$ and $A \leq_T B'$, and hence arithmetical. We can separate this stronger condition from $A \leq_{W2R} B$ by proving that $A \leq_{W2R} B$ does *not* imply $A \leq_T B'$. In [BLS08a] it was shown that there are uncountably many sets $\leq_{LR} \emptyset'$. Since every lower Turing cone is countable, $A \leq_{LR} B$ does not imply $A \leq_T B'$. We follow a similar approach for $\leq_{W2R}$.

Theorem 4.12: *The class of sets $\{X\colon\ X \leq_{W2R} \emptyset''\}$ is uncountable.*

Proof. It suffices to build a perfect tree T and a Martin–Löf test (U_i) relative to $\emptyset''$ with the following property: for all $X \in [T]$, every null $\Pi^0_2[X]$ is contained in $\bigcap_i U_i$. A perfect tree can be seen as a function from strings to strings that preserves the prefix and incompatibility relations. Level n of T is the set of strings $T(\sigma)$ such that $|\sigma| = n$. We build T level by level, computably in $\emptyset''$. At stage e we define level e and enumerate into the open sets S_i, $i \leq e$. We ensure that the total measure of S_i is at most 2^{-i}. Our Martin–Löf test relative to $\emptyset''$ will be $U_j := \bigcup_{i>j} S_i$.

Consider a double sequence $(V_{e,j})$ of oracle Σ^0_1 classes such that $V^X_{e,j+1} \subseteq V^X_{e,j}$ for all $e, j \in \mathbb{N}$ and all sets X. Notice that every $\Pi^0_2[X]$ class is of the form $\bigcap_j V^X_{e,j}$ for some $e \in \mathbb{N}$. We refer to the map $X \to \bigcap_j V^X_{e,j}$ as the oracle Π^0_2 class with index e (the eth oracle Π^0_2 class). Level e of T will be devoted to dealing with the eth oracle Π^0_2 class. For each string σ, let T_σ be the full subtree of T above node $T(\sigma)$.[4] We consider a countable set of requirements that are sufficient for the proof. For each $e \in \mathbb{N}$ and each T_σ for σ of length e, we require that one of the following holds:

- for all $X \in [T_\sigma]$ the e-th $\Pi^0_2[X]$ class is not null, or
- for some $j \in \mathbb{N}$ and all $X \in [T_\sigma]$ we have $V^X_{e,j} \subseteq S_e$.

To see that this is sufficient, suppose that $X \in [T]$ and let $F = \bigcap_j V^X_{e,j}$ be a null $\Pi^0_2[X]$ class. Then we can show that $F \subseteq S_k$ for infinitely many k. Indeed, let $k_0 \in \mathbb{N}$ be given and let $e > k_0$ be an index of F. Let σ be the string of length e such that $X \in [T_\sigma]$. Since F is null, the construction will ensure that $V^X_{e,j} \subseteq S_e$ for some $j \in \mathbb{N}$ and all $X \in [T_\sigma]$. In particular, $F \subseteq S_e$.

The requirements can be written as follows:

$$R_e: \ \forall\sigma\ \forall X \ \left[|\sigma| = e \ \wedge \ X \in [T_\sigma] \Rightarrow \left(\mu\left(\bigcap_j V^X_{e,j}\right) > 0 \ \vee \ \exists j\ V^X_{e,j} \subseteq S_e\right)\right].$$

At level/stage e we first define splittings of the strings in the previous level, in order to ensure that T is perfect. After this preliminary step, we make a decision about how to deal with the eth Π^0_2 class (above each string of this level). In particular, for each node $T(\rho)$ on the eth level of T, we check if we can force $\bigcap_j V^X_{e,j}$ to be non-null for all $X \in [T_\rho]$. That is, for an appropriately small value 2^{-t}, we check if for all $\tau \supseteq T(\rho)$ and all $i \in \mathbb{N}$ there exists $\gamma \supseteq \tau$ such that $\mu(V^\gamma_{e,i}) > 2^{-t}$. In that case we let $f(\rho) = 0$ to declare this fact. In later stages we define T above ρ to ensure that $\mu(\bigcap_j V^X_{e,j}) \geq 2^{-t}$ for all $X \in [T_\rho]$.

Otherwise, for some $n > 1$ and $\zeta \supseteq T(\rho)$, the oracle class $V_{e,n}$ has the uniform bound 2^{-t} on the measure of $V^X_{e,n}$ for all X extending ζ. To declare this fact, we let $f(\rho) = n$ and move $T(\rho)$ to ζ.[5] By choosing appropriate extensions in later stages, under this hypothesis we will be able to enumerate into S_e all $V^X_{e,n}$ for $X \in T_\rho$ while keeping the measure of S_e small.[6]

To sum up, at stage e the following actions determine level e:

- split the strings of the previous level,
- define extensions of the current paths according to the decisions that have been made in previous stages about R_i, for $i < e$, and
- make a decision about how to satisfy R_e above each node of level e.

[4] Our trees are 'growing' upward.

[5] The final value of $T(\rho)$ is only fixed at the *end* of stage e.

[6] The method in this case is the same as in the proof in [BLS08a] that the class of sets $\leq_{LR} \emptyset'$ is uncountable.

Construction. At stage 0 define $T(\emptyset) = \emptyset$ (where $\emptyset$ is the empty sequence here). At stage $e > 0$ we can assume that all previous levels of T have been defined. Given σ of length e we define $T(\sigma)$ in e substages, corresponding to the indices of the first e oracle Π^0_2 classes (starting from index 1). We define $\tau_0, \dots, \tau_{e-1}$ successively, and set $T(\sigma) \supseteq \tau_{e-1}$. Define τ_0 so that incompatibility is met: let i be the last digit of σ and define $\tau_0 := T(\sigma^-) * i$, where σ^- is the predecessor of σ. Now if τ_j, $j < k$ have been defined and $k < e$, let $\rho_k = \sigma \restriction k$. If $f(\rho_k) = 0$, let τ_k be an extension of τ_{k-1} such that $\mu(V^{\tau_k}_{k,e}) > 2^{-2k-1}$. Otherwise, let τ_k be an extension of τ_{k-1} such that

$$\mu(V^{\tau}_{k,f(\rho_k)} - V^{\tau_k}_{k,f(\rho_k)}) \leq 2^{-2(e+1)-1} \quad \text{for all } \tau \supseteq \tau_k. \tag{4.1}$$

When τ_{e-1} is defined, using $\emptyset''$ as an oracle determine if the following is true:

$$\forall i \forall \rho \supseteq \tau_{e-1} \exists \gamma \supseteq \rho \; [\mu(V^{\gamma}_{e,i}) > 2^{-2e-1}]. \tag{4.2}$$

If (4.2) holds, set $f(\sigma) = 0$ and $T(\sigma) = \tau_{e-1}$. Otherwise, choose an $n \in \mathbb{N}$ and $\rho \supseteq \tau_{e-1}$ such that $\mu(V^{\gamma}_{e,n}) \leq 2^{-2e-1}$ and

$$\mu(V^{\gamma}_{e,n} - V^{\rho}_{e,n}) \leq 2^{-2(e+1)-1}, \tag{4.3}$$

for all $\gamma \supseteq \rho$. Let $T(\sigma) = \rho$ and $f(\sigma) = n$.

Finally, for all $k < e$ such that $f(\rho_k) > 0$ enumerate $V^{T(\sigma)}_{k,f(\rho_k)}$ into S_k.

Verification. First we note that the construction is well defined. That is, when the construction defines a string according to (4.1) or (4.3), the search halts. Otherwise, we could inductively push up the measure of $V^{\tau}_{k,f(\rho_k)}$ (or $V^{\gamma}_{e,n}$) as high as we would like, which is impossible.

Second, we show that $\mu(S_e) \leq 2^{-e}$ for all $e \in \mathbb{N}$. Notice that the only 'strategies' that enumerate into S_e are the nodes $T(\rho)$ with $|\rho| = e$ and $f(\rho) > 0$. There are at most 2^e such nodes ρ, so fix one. Let $S_e(\rho)$ be the part of S_e that is enumerated by T_ρ. Consider the full subtree T_ρ of T above $T(\rho)$.

By the construction, $\mu(V^{\tau}_{e,f(\rho)}) \leq 2^{-2e-1}$ for all strings $\tau \in T_\rho$. In particular, $\mu\big(V^{T_\rho(\emptyset)}_{e,f(\rho)}\big) \leq 2^{-2e-1}$. Also, by the way we define T_ρ we have

$$\mu\left(V^{T_\rho(\eta)}_{e,f(\rho)} - V^{T_\rho(\eta^-)}_{e,f(\rho)}\right) \leq 2^{-2(e+|\eta|)-1} \quad \text{for all } \eta \in 2^{<\omega} \text{ with } |\eta| > 0.$$

Hence,

$$\mu(S_e(\rho)) \leq 2^{-2e-1} + \sum_{i>0} 2^i \cdot 2^{-2(e+i)-1} = 2^{-2e},$$

and so $\mu(S_e) \leq 2^e \cdot 2^{-2e} = 2^{-e}$.

Third, we argue for the satisfaction of R_e. At stage e the construction defines $f(\rho)$ for all strings ρ of length e. Fix such a string ρ. If $f(\rho) = 0$, the subtree T_ρ is defined such that $\mu(V^X_{e,i}) > 2^{-2(e+1)-3}$ for all $X \in [T_\rho]$ and all $i \in \mathbb{N}$. Therefore $\mu(\bigcap_i V^X_{e,i}) > 0$ for all $X \in [T_\rho]$. On the other hand, if $f(\rho) > 0$ the construction enumerates $V^X_{e,f(\rho)}$ into S_e, for all $X \in [T_\rho]$. ∎

Corollary 4.13: *$A \leq_{W2R} B$ does not imply $A \leq_T B'$.*

5. The LR-interaction of Δ^0_2 sets with weakly 2-random sets

Note that $\leq_{LR}$ is a Σ^0_3 relation implied by $\leq_T$. In many further ways $\leq_{LR}$ is similar to $\leq_T$ [BLS08a, BLS08b]. In this section we study the LR relations between a Δ^0_2 set A and a weakly 2-random set Z. Recall from Subsection 2.2 that $A \leq_T Z$ implies that A is computable.

5.1. The case $A \leq_{LR} Z$. Recall that by [Nie05] the class of K-trivial sets coincides with the class of low for Martin–Löf random sets.

Proposition 5.1: *If Z is weakly 2-random then every Δ^0_2 set $A \leq_{LR} Z$ is K-trivial.*

Proof. We prove the contrapositive. Suppose that $A \leq_{LR} Z$ and A is not K-trivial. By the theorem of Kjos-Hanssen mentioned in Subsection 4.1, there is a bounded oracle Σ^0_1 class V such that $U^A \subseteq V^Z$, where U is a component of a universal oracle Martin–Löf test. But then Z is a member of the class

$$\{X \mid U^A \subseteq V^X\} = \bigcap_{n,s_0} \bigcup_{s>s_0} \{X \mid U^{A\restriction n}[s] \subseteq V^X\},$$

which is Π^0_2. Since A is K-trivial, it is also low for Martin–Löf. By a theorem of Stephan (see [BLS08a]) all non-trivial LR upper cones are null. Therefore, the above class is null. This shows that Z is not weakly 2-random. ∎

5.2. The case $Z \leq_{LR} A$ when $A = \emptyset'$. Recently there has been some interest in understanding the class of sets $\leq_{LR} \emptyset'$; see for example [Nie09a, Section 5.6]. In [BLS08a] it was shown that this class is uncountable, and in [BLS08b] that it contains sets of hyperimmune-free Turing degree. In the following we show that it contains a weakly 2-random set. Notice that by definition of $\leq_{LR}$ it does not contain 2-random sets.

Theorem 5.2: *There is a weakly 2-random Z that is K-trivial relative to $\emptyset'$. Thus $Z \leq_{LR} \emptyset'$. Moreover, Z can be chosen of hyperimmune-free Turing degree.*

Proof. By Nies [Nie05], a set Z is K-trivial relative to $\emptyset'$ iff $Z \oplus \emptyset' \leq_{LR} \emptyset'$. In particular, this notion is closed downward with respect to $\leq_T$. Kučera and Nies [KNxx] have shown the following. Let P be a non-empty Π^0_1 class. Suppose that $B >_T \emptyset'$ is Σ^0_2. Then there is a set $Z \in P$ of hyperimmune-free Turing degree such that $Z' \leq_T B$.

Now let P be a non-empty Π^0_1 class of ML-randoms. The members of P that form a minimal pair with $\emptyset'$ are weakly 2-random (see [Nie09a, Section 5.3]). Let $B >_T \emptyset'$ be a Σ^0_2 set that is K-trivial relative to $\emptyset'$. This exists by a relativization of the well known construction of a non-computable c.e. K-trivial set. By applying the above theorem we get Z is as required. Indeed, since the degree of Z is hyperimmune-free, it forms a minimal pair with $\emptyset'$. Hence it is weakly 2-random. Moreover, it is computable from B, therefore it is K-trivial relative to $\emptyset'$. ∎

Theorem 5.2 does not hold if we replace 'weakly 2-random' with $\mathsf{SR}[\emptyset']$. Indeed, [Nie09a, Exercise 5.5.10] shows that no Schnorr random set is K-trivial; the relativization of this argument to $\emptyset'$ shows that no set in $\mathsf{SR}[\emptyset']$ is K-trivial relative to $\emptyset'$. Also, notice that any K-trivial relative to $\emptyset'$ is computable from $\emptyset''$. This follows by relativization of the fact from [Cha76] that every K-trivial is Δ^0_2.

5.3. The case $Z \leq_{LR} A$ when $A' \leq_T \emptyset'$. Intuitively, it is possible to have a weakly 2-random set LR-below $\emptyset'$ (Theorem 5.2) because the LR lower cone below $\emptyset'$ is only Σ^0_4. In contrast, we show that if the oracle A (the top of the lower cone) is low then its LR lower cone is Σ^0_3, which forbids the existence of a weakly 2-random Z in this cone.

Proposition 5.3: *If $A' \leq_T \emptyset'$ then there is no weakly 2-random Z (and in fact no Z in $\mathsf{Kurtz}[\emptyset']$) such that $Z \leq_{LR} A$.*

Proof. Again by the theorem of Kjos-Hanssen, $Z \leq_{LR} A$ iff Z belongs to

$$\{X \mid \forall n \exists s\ U^{X\upharpoonright n} \subseteq V^A[s]\} \tag{5.1}$$

for some bounded oracle Σ^0_1 class V (where, as before, U is a member of the universal oracle Martin–Löf test). For fixed V this is a $\Pi^0_1[A']$ class. If

$A' \leq_T \emptyset'$ then (5.1) is a $\Pi^0_1[\emptyset']$ class (and so a Π^0_2 class). All lower LR cones are null by [BLS08a], so Z cannot be Kurtz random relative to $\emptyset'$ (or weakly 2-random). ∎

We note that if for some A there is a weakly 2-random $Z \leq_{LR} A$, this does not necessarily mean that there is a weakly 2-random in the same LR degree as A. For example, [Nie09a, Exercise 5.6.22] shows that the only c.e. LR degree that contains a Martin–Löf random set is the LR degree of $\emptyset'$. Also notice that by Proposition 5.1 there is no weakly 2-random in the LR degree of $\emptyset'$. We do not know whether the property of LR bounding a weakly 2-random is an LR-completeness criterion for Δ^0_2 sets; in other words, if the condition that 'A is low' in Proposition 5.3 can be replaced with 'A is Δ^0_2 and not LR complete'.

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