

Coding Methods in Computability
Theory and Complexity Theory

Habilitationsschrift

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Preface

A major part of computability theory focuses on the analysis of a few structures of central importance. As a tool, the method of coding with first-order formulas has been applied with great success. It was used to determine the complexity of the elementary theory, to provide restrictions on automorphisms, and even to obtain definability results. As an example, consider \mathcal{R}_T , the structure of computably enumerable (c.e.) Turing degrees. The analysis by coding methods began with the proof by Harrington and Shelah [22] that $\text{Th}(\mathcal{R}_T)$ is undecidable. Extending the coding methods used, Harrington and Slaman (unpublished) gave an interpretation of $\text{Th}(\mathbb{N}, +, \times)$, also called *true arithmetic*, in $\text{Th}(\mathcal{R}_T)$. Here an *interpretation* is a many-one-reduction of theories based on a computable map defined on sentences in some natural way. A different approach to the same problem, due to Slaman and Woodin, introduced a very versatile way of coding copies of $(\mathbb{N}, +, \times)$ into \mathcal{R}_T with parameters, which was a main ingredient for the investigations in Nies, Shore and Slaman [47]. In the latter work, the definability of some important classes, including Low_2 and High_1 , is proved. Moreover, it is shown that no automorphism of \mathcal{R}_T can change the second jump of a degree, and that a coding of \mathbb{N} in \mathcal{R}_T without parameters exists. In a different direction, Lempp, Nies and Slaman [34], combining the Harrington-Shelah type of coding with algebraic methods, proved that the $\forall\exists\forall$ -theory of \mathcal{R}_T (as a partial order) is undecidable.

We will describe how a similar program can be carried out for several other structures, including \mathcal{R}_m , the structure of c.e. many one degrees, \mathcal{R}_{wtt} , the structure of c.e. weak truth table degrees and \mathcal{E} , the lattice of c.e. sets under inclusion. In all cases we will obtain undecidability of, or even an interpretation of $\text{Th}(\mathbb{N})$ in the theory of the structure. For \mathcal{R}_m , we also obtain definability results and restrictions on automorphisms. Moreover, for both \mathcal{R}_m and \mathcal{R}_{wtt} a coding of \mathbb{N} without parameters can be given. On the other hand, for \mathcal{E} such stronger coding properties must fail: no infinite linear order can be coded without parameters. In connection with the study

of \mathcal{E} , we also consider the lattices $\mathcal{I}(\mathcal{B})$ of c.e. ideals for certain c.e. boolean algebras \mathcal{B} and prove that their theories are undecidable. These lattices, besides being of intrinsic interest in effective algebra, can be coded into many important structures, like degree structures from complexity theory “low down”. Thereby they provide a tool to prove undecidability for theories from very different contexts.

While so far most structures from computability theory (and complexity theory) were studied in isolation, our approach has a unifying aspect, since first general tools and concepts of a model theoretic flavor are developed, which then re-emerge again and again. For instance, for \mathcal{R}_m , \mathcal{E} and to some extent the lattices $\mathcal{I}(\mathcal{B})$, we will prove definability lemmas which give a way to pass from arithmetical definability of subsets of a structure to definability with parameters in the structure. These definability lemmas constitute the main tool for our analyses by coding methods of the structures in question.

The first chapter and to some extent the second chapter are of an introductory nature. The methods in Section 2.2 and Section 2.3 appeared first in Nies [42] and Nies [44], respectively. The first three sections from Chapter 3 are also from [42]. Chapter 4 is based on Harrington and Nies [21], but contains substantial improvements in Section 4.4 which lead to new results about fragments of $\text{Th}(\mathcal{E}^*)$ in Section 4.6. Chapter 5 appeared in Nies [45], as did Section 6.1. Section 6.2 is based on Downey and Nies [14], while Chapter 7 contains very recent results of the author. An extended version of this work containing also material about \mathcal{R}_T will appear as a book in the *Oxford Logic Guides*.

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Chapter 1

The objects of investigation

We introduce the structures we will study and discuss some of their basic properties.

1.1 Structures based on computably enumerable sets

A central notion in logic is the notion of a computably enumerable (c.e., or r.e.) set of natural numbers. In this section we review structures based on c.e. sets. The study of global and local properties of these structures is regarded as a central topic in computability theory.

1.1.1 Degree structures

The relative computational complexity of c.e. sets is investigated through the study of the uppersemilattices \mathcal{R}_m and \mathcal{R}_T of enumerable many-one degrees and of enumerable Turing (T -)degrees, and also of the degree structures \mathcal{R}_{wtt} and \mathcal{R}_{tt} which arise from reducibilities between \leq_m and \leq_T . These reducibilities are obtained from Turing-reducibility by more and more restricting the underlying concept of oracle computation: for subsets X, Y of \mathbb{N} , $X \leq_{wtt} Y$ if $X \leq_T Y$ via an oracle computation procedure where the largest oracle question asked is recursively bounded in the input; $X \leq_{tt} Y$ if such an oracle computation procedure is total for every oracle. Finally, $X \leq_m Y$ if there is a computable function f such that $n \in X \Leftrightarrow f(n) \in Y$ for all n . To avoid trivialities, we actually allow TRUE and FALSE as values of f . Since each reducibility \leq_r is a preordering, we obtain a degree structure \mathcal{R}_r of r -degrees of c.e. sets, which is an upper semilattice with a least

element, denoted by $\mathbf{0}$, and a largest element, denoted by $\mathbf{1}$. The degree $\mathbf{0}$ consists of the computable sets, and $\mathbf{1}$ is the r -degree of the halting problem. The r -degree of a set $X \subseteq \mathbb{N}$ is denoted by $\text{deg}_r(X)$.

Ever since Post's problem was formulated [50] which asks whether $\mathbf{0}, \mathbf{1}$ are the only c.e. T -degrees, the study of \mathcal{R}_T has been a mainstay of computability theory. A wide range of facts about \mathcal{R}_T , all formalizable within first-order logic, were found. The structure is dense (Sacks [51]), has pairs with infimum $\mathbf{0}$ (called *minimal pairs*; Yates [58]) but also nonzero degrees which don't bound any minimal pairs (Lachlan; see [56]). Such properties seem to reflect pathological rather than orderly behavior of \mathcal{R}_T . The structure \mathcal{R}_m , on the other hand, is much more homogeneous and well-behaved, and in fact is the only c.e. degree structure which permits a characterization (Denisov [13], see also Section 3.4). While \mathcal{R}_{tt} exhibits quite a pathological behavior as well, \mathcal{R}_{wtt} is at the borderline. For instance, the same theorems about minimal pairs hold as for \mathcal{R}_T , but it shares with \mathcal{R}_m the property of being *distributive* as an upper semilattice, namely

$$(1.1) \quad \forall x \forall a \forall b [x \leq a \vee b \Rightarrow \exists a_0 \leq a \exists b_0 \leq b \ x = a_0 \vee b_0]$$

(see Lachlan [30] for a proof).

For the study of enumerable sets, the reducibilities refining T -reducibility are interesting partially because they are more closely related to structural properties of an enumerable set than T -reducibility is. For instance, a maximal enumerable set must have minimal many-one degree, and a hypersimple set is necessarily *wtt*-incomplete, but not always T -incomplete (see Odifreddi [49, p. 338]).

1.1.2 C.e. sets under inclusion and ideal lattices

A more algebraic aspect of computably enumerable sets is captured by the lattice \mathcal{E} of computably enumerable sets under inclusion. This view of c.e. sets is the most elementary one, because no further concepts are required to relate them. Clearly \mathcal{E} is a distributive lattice with least and greatest elements. Moreover, \mathcal{E} satisfies the reduction principle:

$$(1.2) \quad \forall A \forall B \exists \tilde{A} \subseteq A \exists \tilde{B} \subseteq B [\tilde{A} \cap \tilde{B} = \emptyset \ \& \ \tilde{A} \cup \tilde{B} = A \cup B].$$

Despite of the conceptually simple way \mathcal{E} is introduced, it is a structure of great algebraic complexity. Several interrelated directions in the study of \mathcal{E}

have been pursued: one is the investigation of automorphisms (Soare [55]), a further one is the relationship between the behavior of an enumerable set as an element of \mathcal{E} and its computational complexity (see e.g. Martin [37] and Harrington and Soare [23]). Here we follow another approach, initiated by the undecidability proofs for $\text{Th}(\mathcal{E})$ due to Herrmann [25] and Harrington: the approach of studying coding and definability.

The next type of structures we consider is actually based not on c.e. sets but on c.e. ideals. A boolean algebra \mathcal{B} is *computably enumerable* if $\mathcal{B} = \mathcal{D}/H$ for a c.e. ideal H of the computable dense boolean algebra \mathcal{D} . Let $\mathcal{I}(\mathcal{B})$ be the lattice of c.e. ideals of a c.e. boolean algebra \mathcal{B} (thus, if $\mathcal{B} = \mathcal{D}/H$, $\mathcal{I}(\mathcal{B})$ is the lattice of c.e. ideals of \mathcal{D} containing H). We list some properties of $\mathcal{I}(\mathcal{B})$ which show that, in a sense, $\mathcal{I}(\mathcal{B})$ is similar to \mathcal{E} . First, $\mathcal{I}(\mathcal{B})$ is a distributive lattice with least and greatest elements. It is easy to prove that $\mathcal{I}(\mathcal{B})$ also satisfies the reduction principle. All principal ideals $[0, b]_{\mathcal{B}}$ of \mathcal{B} are in $\mathcal{I}(\mathcal{B})$. The class of principal ideals is definable in $\mathcal{I}(\mathcal{B})$: an ideal is principal iff it is complemented in $\mathcal{I}(\mathcal{B})$.

It is possible that $\mathcal{I}(\mathcal{B}) \cong \mathcal{B}$, even for a dense c.e. \mathcal{B} : one can construct a dense \mathcal{B} such that every c.e. ideal is principal (Martin and Pour-El [38]). However, the type of c.e. boolean algebras we consider here have a very complex lattice of c.e. ideals. We call a c.e. boolean algebra \mathcal{B} *effectively dense* if, for each element x of \mathcal{B} , we can effectively find an element $y \leq x$ such that $x \neq 0$ implies $0 < y < x$. Thus e.g. the recursive dense boolean algebra is effectively dense, but in fact many other c.e. presentations of the countable dense boolean algebra are as well. For instance, consider the Lindenbaum algebra of sentences over Peano arithmetic. This c.e. boolean algebra is effectively dense by Rosser's theorem, a refinement of Gödel's second incompleteness theorem (Example 5.1.1 below).

1.2 Structures from Complexity Theory

In complexity theory, one considers sets of strings, mostly from $\{0, 1\}^{<\omega}$, instead of sets of numbers. Polynomial time bounded analogs of the recursion theoretic reducibilities were introduced. For instance, for $X, Y \subseteq \{0, 1\}^{<\omega}$, polynomial time many-one reducibility is defined by

$$X \leq_m^p Y \Leftrightarrow (\exists f \in \text{PTIME}[X = f^{-1}(Y)]),$$

(where f , as before, may have TRUE and FALSE as values). Polynomial time Turing reducibility is defined by $X \leq_T^p Y \Leftrightarrow$ there is a polynomial time bounded deterministic oracle Turing machine taking inputs in $\{0, 1\}^{<\omega}$

which computes X if the oracle is Y . Analogs of some other reducibilities, like truth-table reducibility, can be defined in a similar way. We let (Rec_r^p, \leq_r^p) be the p.o. of polynomial time r -degrees of computable sets, where \leq_r^p is a polynomial time reducibility in between (and including) \leq_m^p and \leq_T^p . As before, Rec_r^p is an u.s.l. which has a least element \mathbf{o} , the degree consisting of sets in PTIME. But Rec_r^p has no greatest element.

The fact that the base sets are computable allows for a method radically different from the methods used in computability theory: the delay diagonalization (or looking-back) method introduced in Landweber, Lipton and Robertson [32]. They used the technique to reprove Ladner's result [31] that Rec_r^p is dense (see also Balcazar e.a. [9]). The idea is as follows: in a construction of a computable set A , at stage s $A^=s = A \cap \Sigma^s$ is determined. If s is large enough, one can in time polynomial in s see if a requirement was satisfied at a much earlier stage (which may involve checking if some short strings are in given computable sets). Then at stage s one can react accordingly, e.g. by starting to work on a different requirement.

Slaman and Shinoda [52] gave an interpretation of $\text{Th}(\mathbb{N})$ in $\text{Th}(Rec_T^p)$, but left open the case of polynomial time many-one degrees. Three years later, Ambos Spies and Nies [4] proved that $\text{Th}(Rec_m^p)$ is undecidable. However, the two latter results use the so-called "speed-up technique" introduced by Ambos-Spies, a method which leads to computable sets of very high complexity (usually nonelementary sets). From a complexity theorist's point of view, such sets are not very relevant because they are only computable in an ideal sense. Therefore here we will consider degree structures based on sets of low complexity. Let

$$\text{DTIME}(h) := \{X \subseteq \{0, 1\}^{<\omega} : X \text{ can be computed in time } O(h)\}.$$

A function $h : \mathbb{N} \mapsto \mathbb{N}$ is *time constructible* if $h(n)$ can be computed in time $O(h(n))$ (here we identify \mathbb{N} with $\{0\}^{<\omega}$). We will prove that, for each time constructible h which dominates the polynomial $n \mapsto n^k$ for each k , $(\text{DTIME}(h), \leq_r^p)$ has an undecidable theory (Downey and Nies [14]). Thus, for instance the polynomial time T -degrees of sets in exponential time have an undecidable theory.

A set A is *tally* if $A \subseteq \{0\}^{<\omega}$. For the result of Downey and Nies mentioned above, we will in fact prove that each initial interval $[\mathbf{o}, \mathbf{a}]$ has an undecidable theory, where $\mathbf{a} \neq \mathbf{o}$ is the degree of very particular type of a tally set, called a super sparse set. This notion was introduced by Ambos-Spies [2]. One requires that $A \subseteq \{0^{f(k)} : k \in \mathbb{N}\}$, for a time constructible f which increases so fast that " $A(0^{f(k)} = 1 ?)$ " can be determined in time $O(f(k+1))$. These

sets allow us some of the advantages of the speed-up technique, while still existing in each class $\text{DTIME}(h)$, h as above.

Chapter 2

Theories and coding

A *theory* is a consistent set of first-order sentences in some language closed under logical inference. Given a theory T in an effective first-order language, an important first question is whether the theory is decidable. Such investigations were initiated by Gödel (implicit in [19]) and Tarski [57] and have played an important role ever since. If \mathbf{A} is a structure whose theory is known to be undecidable, an interesting further problem is to determine the computational complexity of $\text{Th}(\mathbf{A})$. If \mathbf{A} can be coded in $(\mathbb{N}, +, \times)$, an upper bound for its computational complexity is the degree of $\text{Th}(\mathbb{N}, +, \times)$ (this theory is also called *true arithmetic*), because there is an interpretation of $\text{Th}(\mathbf{A})$ in true arithmetic. For most of the structures introduced in the previous chapter, we will give an interpretation in the other direction. So $\text{Th}(\mathbf{A})$ has the same computational complexity as true arithmetic.

A further question we will consider is which fragments of an undecidable theory T are undecidable.

2.1 Coding

We explain coding with first-order formulas and introduce the central concept of a coding scheme. Consider first-order languages L_0, L_1 over finite symbols sets, and suppose that L_0 is relational. We intend to code L_0 -structures \mathbf{C} into L_1 -structures \mathbf{A} , by using an appropriate collection of L_1 -formulas. We represent elements of \mathbf{C} by elements in an \mathbf{A} -definable set D , modulo an \mathbf{A} -definable equivalence relation \equiv (\mathbf{A} -definable means definable in \mathbf{A} with parameters). Then the relations of \mathbf{C} give rise to corresponding relations on D/\equiv , which we also require to be \mathbf{A} -definable. The uniformity is embodied in the fact that all the definability requirements are satisfied via

a fixed collection of formulas, called a scheme. Thus, a *scheme for coding L_0 -structures into L_1 -structures* is given by a collection of L_1 -formulas

(2.1)

$$S = \varphi_{dom}(x; \tilde{p}), \varphi_{\equiv}(x, y; \tilde{p}), (\varphi_R(x_1, \dots, x_n; \tilde{p}))_R \text{ relation symbol of } L_0,$$

together with a *correctness condition* $\alpha(\tilde{p})$ which expresses to the least that actually an L_1 structure is coded by \tilde{p} . The correctness condition states that

- $D = \{x : \varphi_{dom}(x; \tilde{p})\}$ is nonempty,
- $\equiv = \{x, y : \varphi_{\equiv}(x, y; \tilde{p})\}$ is an equivalence relation when restricted to D , and
- the relations on D defined by the formulas φ_R are compatible with \equiv .

We say that \mathcal{C} is *coded* in \mathcal{A} via S and a list of parameters \tilde{a} in \mathcal{A} if the structure defined by S with these parameters on D/\equiv equals \mathcal{C} .

Coding of this kind was introduced to prove in an indirect way that the theory of a class of L_1 -structures is undecidable. For *uniform coding* (up to isomorphism) of a class \mathcal{C} of L_0 -structures in a class \mathcal{A} of L_1 -structures one requires that via a fixed scheme of formulas with parameters a copy of each structure from \mathcal{C} in some structure \mathcal{A} from \mathcal{A} can be coded if appropriate values in \mathcal{A} are substituted for the parameters.

Given a first-order language L , L -*valid* is the set of valid L -sentences. A theory $T \subseteq L$ is *hereditarily undecidable* (h.u.) if, for each X ,

$$L\text{-valid} \subseteq X \subseteq T \Rightarrow X \text{ undecidable.}$$

The following well-known fact (see for instance Burris and Sankappanavar [10]) is used to transfer hereditary undecidability of theories of classes.

Fact 2.1.1 *If $\text{Th}(\mathcal{C})$ is h.u. and \mathcal{C} can be uniformly coded in \mathcal{A} , then $\text{Th}(\mathcal{A})$ is h.u.*

For instance, to show that the theory of the structure of r.e. m -degrees is undecidable, one can use for \mathcal{C} the class of finite distributive lattices, viewed as partial orders: each such lattice is isomorphic to an initial interval $[\mathbf{o}, \mathbf{a}]$ of the r.e. m -degrees (Lachlan [29]). So \mathcal{C} is uniformly coded in the class $\{\mathcal{R}_m\}$. Since $\text{Th}(\mathcal{C})$ is known to be h.u., $\text{Th}(\mathcal{R}_m)$ is undecidable.

Clearly, in $(\mathbb{N}, +, \times)$ (or in fact in any model of Peano arithmetic) one can uniformly code, say, the class of finite undirected graphs. By Theorem 2.3.1 below, this class has a h.u. theory, so by Fact 2.1.1 $\text{Th}(\mathbb{N}, +, \times)$ is h.u. So, as a special case of Fact 2.1.1, we obtain

Corollary 2.1.2 ([10]) *If $(\mathbb{N}, +, \times)$ can be coded in a structure \mathbf{A} with parameters, then $\text{Th}(\mathbf{A})$ is h.u.*

We will give more details on this method when we discuss undecidable fragments in Section 2.3 below. Next we consider interpreting $\text{Th}(\mathbb{N})$ in $\text{Th}(\mathbf{A})$. Here a central notion is the following.

Example 2.1.3 *A scheme S_M for coding models of some finitely axiomatized fragment PA^- of Peano arithmetic (in the language $L(+, \times)$) is given by the formulas*

$$(2.2) \quad \varphi_{num}(x, \bar{p}), \varphi_{\equiv}(x, y, \bar{p}), \varphi_+(x, y, z; \bar{p}), \varphi_{\times}(x, y, z; \bar{p})$$

and a correctness condition $\alpha_0(\bar{p})$ which says that

- $\varphi_{\equiv}(x, y, \bar{p})$ defines an equivalence relation \equiv on $\{x : \varphi_{num}(x; \bar{p})\}$
- φ_+ and φ_{\times} define binary operations on the set $\{x : \varphi_{num}(x; \bar{p})\}$ which are compatible with \equiv
- $\{x : \varphi_{num}(x; \bar{p})\} / \equiv$ with the corresponding operations satisfies the finitely many axioms of PA^- .

(Formally, we view $L(+, \times)$ as a language with two ternary relation symbols.) In our applications, the axioms ensure that \mathbf{M} has a standard part. For instance think of PA^- as Robinson arithmetic Q . In some applications it is necessary to represent numbers by equivalence classes of tuples of a fixed length (as opposed to elements). Thus the coding is similar to the coding of \mathbb{Q} in \mathbb{Z} given by the quotient field construction, where a rational is represented by a pair of integers (but we may also use parameters). To adapt the definitions, in the above x, y, z have to be interpreted as tuples of variables.

Notice that we are now interested in the collection of coded structures as they are, not only in structures up to isomorphism. In fact we will code more general objects than structures into \mathbf{A} : we drop the condition that there be a domain formula and a formula φ_{\equiv} . Thus an *object scheme* for coding in an L_1 -structure is given by a list of L_1 -formulas

$$\varphi_1, \dots, \varphi_n$$

with a shared parameter list \bar{p} , together with a correctness condition $\alpha(\bar{p})$.

Example 2.1.4 A scheme S_g for defining a function g is given by a formula $\varphi_1(x, y; \bar{p})$ defining the relation between inputs and outputs; and a correctness condition $\alpha(x, y; \bar{p})$ which says that a function is defined: $\forall x \exists^{\leq 1} y \varphi_1(x, y; \bar{p})$

Example 2.1.5 We will often consider object schemes for classes of n -ary relations on \mathbf{A} . Such a scheme is given by a formula $\varphi(x_1, \dots, x_n; \bar{p})$ and a correctness condition $\alpha(\bar{p})$.

For instance, if \mathbf{A} is a linear order and \mathcal{C} is the set of closed intervals, then \mathcal{C} is uniformly definable via the scheme consisting of $\varphi_1(x; a, b) \Leftrightarrow a \leq x \leq b$ and the correctness condition $\alpha(a, b) \Leftrightarrow a \leq b$.

In general, an object scheme S_X introduces a new type of object. The parameters \bar{p} satisfying $\alpha(\bar{p})$ code an object, and S_X acts as a decoding key. Using this coding, it becomes possible to quantify over objects of the new type (a form of second order quantification) in the first-order language of \mathbf{A} . Thus one can quantify over uniformly definable classes in the sense of the following definition.

Definition 2.1.6 (i) A class \mathcal{C} of objects of a common type is uniformly definable in \mathbf{A} if, for some scheme S , \mathcal{C} is the class of objects coded via S as the parameters range over tuples in \mathbf{A} which satisfy the correctness condition.

(ii) \mathcal{C} is weakly uniformly definable if \mathcal{C} is contained in a uniformly definable class.

We can perform basic mathematical operations on objects of two possibly different types and obtain a uniform way of coding objects of a yet different type. For example, we can define a scheme S for the compositions $g \circ h$ of maps g, h defined by schemes S_g, S_h . Furthermore, we can express basic relationships between coded objects by first order conditions on codes; for instance we can express the relationship “ g is a partial map from \mathbf{M}_0 to \mathbf{M}_1 ”, where $\mathbf{M}_0, \mathbf{M}_1$ are coded via S_M and g is coded via S_g , by formulas of \mathcal{R} .

Notation 2.1.7 We use the following convention throughout: If a scheme S_X is given, variables X, X_0 , etc. denote objects coded by this scheme for a particular parameter list \bar{p} satisfying the correctness condition. If it is necessary to mention the parameters explicitly, we write $X(\bar{p})$ (or $X_{\bar{p}}$, $X_0(\bar{p})$, etc. We say that \bar{p} codes $X(\bar{p})$ via S_X .

We will use the term “scheme” to refer to either a coding scheme or an object scheme. It will be clear from the context which notion is meant.

2.2 Interpreting true arithmetic

In the following we assume that \mathbf{A} is a structure which can be coded in $(\mathbb{N}, +, \times)$. Note that there is an onto map $\gamma : \mathbb{N} \rightarrow \mathbf{A}$ such that the preimages of the relations and functions of \mathbf{A} are arithmetical. For instance, if \mathbf{A} is \mathcal{R}_m , let $\gamma(i) = \text{deg}_m(W_i)$. We call the preimage of a relation R under γ the corresponding *index relation* (index set if R is unary) and sometimes write ΘR for this preimage. In fact we will often identify R and ΘR .

To interpret $\text{Th}(\mathbb{N})$ in $\text{Th}(\mathbf{A})$, for \mathcal{R}_m and \mathcal{E} we carry out the following two steps:

(2.3) Specify a scheme S_M as in Example 2.1.3 and a special list \bar{p}
such that $M_{\bar{p}}$ is standard

(2.4) Find an additional correctness condition $\alpha_{st}(\bar{p})$
which holds iff \bar{p} code a copy of $(\mathbb{N}, +, \times)$

The point is that the condition $\alpha_0(\bar{p})$ from Example 2.1.3 only gives an approximation to standardness. While (2.3) can be seen as a “local” coding, relying on very special parameters, recognizing standardness of an *arbitrary* $M_{\bar{p}}$ in a first-order way depends on how $M_{\bar{p}}$ relates to its context, namely the whole structure \mathbf{A} . Thus for (2.4) we will use particular properties of \mathbf{A} .

If β is a sentence in the language of arithmetic, let $\tilde{\beta}(\bar{p})$ be the translation of β , namely the formula obtained by replacing $=, +, \times$ by their definitions via $\varphi_{\equiv}, \varphi_+, \varphi_{\times}$ and relativizing the quantifiers to those x satisfying $\varphi_{num}(x, \bar{p})$. Then

(2.5) $(\mathbb{N}, +, \times) \models \beta \Leftrightarrow \mathbf{A} \models \exists \bar{p}[\alpha_{st}(\bar{p}) \ \& \ \tilde{\beta}(\bar{p})]$.

Since $\tilde{\beta}(\bar{p})$ is obtained in an effective way, $\text{Th}(\mathbb{N}) \leq_m \text{Th}(\mathbf{A})$.

In order to carry out (2.3), it is often useful if one just has to code a computable directed graph (V, E) into \mathbf{A} with parameters (where in fact $V = \mathbb{N}$). Here we provide a parameterless coding of $(\mathbb{N}, +, \times)$ in a particular such graph

$$(2.6) \quad (V_{\mathbb{N}}, E_{\mathbb{N}}),$$

which is a recursive irreflexive partial order. To construct this partial order, one starts with a countable antichain of minimal elements p_n which will represent the numbers n . Then, for each $n, m \in \mathbb{N}$ one adds an element $c_{n,m}$ to P_A which represents the pair (p_n, p_m) . Next, one adds ascending chains of lengths 2 and 3, respectively, from p_n to $c_{n,m}$ and from p_m to $c_{n,m}$. Finally, to code addition, add a chain of length 4 from p_{n+m} to $c_{n,m}$ and for multiplication, add a chain of length 5 from $p_{n \times m}$ to $c_{n,m}$.

We now discuss how to carry out (2.4), assuming that some scheme S_M as in (2.3) has been specified. We first assume that $\varphi_{\equiv}(x, y; \bar{p})$ defines the trivial equivalence relation $x = y$ (thus, numbers are represented by certain elements of \mathbf{A}). If we were allowed to quantify over subsets of $M_{\bar{p}}$, for any $M_{\bar{p}}$, then we could simply use Dedekind's second-order axiomatization of $(\mathbb{N}, +, \times)$: we would require that each subset of $M_{\bar{p}}$ which contains $0^{M_{\bar{p}}}$ and is closed under successor equals $M_{\bar{p}}$. Of course we cannot quantify over all such subsets in the first-order language of \mathbf{A} , but we can try to quantify over sufficiently many, by using some uniform definability result. The following two facts specify which subsets must be included.

Fact 2.2.1 *Suppose \mathbf{A} is coded in $(\mathbb{N}, +, \times)$. Then, for some fixed k , the standard part S of each coded model M has a Σ_k^0 index set.*

Proof. Note that $\gamma(i) \in S \Leftrightarrow \exists n \in \mathbb{N} \exists y_0, \dots, y_n \in \mathbf{A}$

$$[y_0 = 0^M \ \& \ y_n = \gamma(i) \ \& \ (\forall i < n) \mathbf{A} \models \varphi_{\oplus}(y_i, 1^M, y_{i+1})].$$

Since \mathbf{A} is coded in $(\mathbb{N}, +, \times)$, this is a Σ_k^0 property of i , for some fixed k depending only on \mathbf{A} and the scheme S_M . \diamond

We call a subset of M a Σ_k^0 -subset if its index set is Σ_k^0 . Make sure not to confuse Σ_k^0 -subsets of M with sets which can be defined by a Σ_k^0 -formula from the point of view of M .

Fact 2.2.2 *Suppose that the collection of Σ_k^0 subsets of any $M_{\bar{p}}$ is weakly uniformly definable via a formula $\varphi(x; \bar{q})$, where \bar{q} is a parameter list containing \bar{p} . Then $Th(\mathbb{N})$ can be interpreted in $Th(\mathbf{A})$.*

Proof. Let $\alpha_{st}(\bar{p})$ be the formula expressing

for all \bar{q} , if $\{x : \varphi(x; \bar{q})\}$ is a subset of $M_{\bar{p}}$ which contains $0^{M_{\bar{p}}}$ and is closed under successor then it equals $M_{\bar{p}}$.

This is certainly satisfied if $M_{\bar{p}}$ is standard. If $M_{\bar{p}}$ is not standard, then, the standard part is a Σ_k^0 -set which therefore can be defined via some \bar{q} . So the statement fails. \diamond

If φ_{\equiv} defines a nontrivial equivalence relation, with an adjustment of the terminology carried out in the following definition the previous considerations are still valid.

Definition 2.2.3 *A class $\widehat{S} \subseteq \{x : \varphi_{num}(x, \bar{p})\}$ represents a subset $S \subseteq M_{\bar{p}}$ if \widehat{S} is \equiv -closed and $\widehat{S}/_{\equiv} = S$. We call S a Σ_k^0 -subset of $M_{\bar{p}}$ if \widehat{S} has a Σ_k^0 index set.*

(Note that in the above we really mean the restriction of \equiv to $\{x : \varphi_{num}(x, \bar{p})\}$.) To show that the Σ_k^0 -subsets of $M_{\bar{p}}$ are weakly uniformly definable usually involves proving a sufficiently strong uniform definability result. Such a result can be derived for \mathcal{R}_m , as well as for \mathcal{E} . The definability lemma for \mathcal{R}_m states that

(2.7) for each $k \geq 3, N \geq 1$, the class of N -ary Σ_k^0 relations which are contained in some $[\mathbf{o}, \mathbf{c}]$, $\mathbf{c} < \mathbf{1}$, is uniformly definable.

It was first proved in Nies [42], by induction on k (see Section 3.2). Later, Harrington used the same general method to prove a similar result for \mathcal{E} , which we call the ideal definability lemma: for an r.e. set A , let $\mathcal{B}(A)$ be the Boolean algebra of components of c.e. splittings of A , and let $\mathcal{R}(A)$ be the ideal of $\mathcal{B}(A)$ consisting of the computable subsets of A . An ideal I of $\mathcal{B}(A)$ is called k -acceptable if $\mathcal{R}(A) \subseteq I$ and $\{e : W_e \in I\}$ is Σ_k^0 . Harrington's ideal definability lemma asserts that, for any odd $k \geq 3$,

(2.8) the class of k -acceptable ideals of $\mathcal{B}(A)$ is uniformly definable.

Again, it is proved by induction, here over odd $k \geq 3$ (see Section 4.2). The definability lemma for \mathcal{R}_m is in fact so strong that it can also be used to code copies of $(\mathbb{N}, +, \times)$ with parameters, i.e., to carry out (2.3). For \mathcal{E} , some extra work is required, but the ideal definability lemma is still a main

ingredient. In this way, an intermediate coding in cumbersome auxiliary structures, like the recursive boolean pairs in Herrmann [25], can be avoided. For the structure $\mathbf{D}_T(\leq \emptyset')$ of T -degrees of Δ_2^0 sets (Shore [53]), as well as for \mathcal{R}_{tt} (Nies and Shore [46]), one also satisfies (2.3) and (2.4), but no general definability lemmas are used to interpret $\text{Th}(\mathbb{N})$ in the theory. Instead, the coding of copies \mathbf{M} of $(\mathbb{N}, +, \times)$ is made more “effective” (in the sense of the arithmetical complexity of a function g such that $n^{\mathbf{M}} = \deg_r(W_{g(n)})$), so that, e.g. in the case of \mathcal{R}_{tt} , the standard part of any coded \mathbf{M} is actually Σ_3^0 . Now, a rather weak definability result suffices: each Σ_3^0 -ideal of T -incomplete c.e. tt -degrees has an *exact pair*, namely it has the form $[\mathbf{o}, \mathbf{a}] \cap [\mathbf{o}, \mathbf{b}]$ for appropriate $\mathbf{a}, \mathbf{b} \in \mathcal{R}_{tt}$.

In the case of \mathcal{R}_T , another approach yet has been carried out in order to satisfy (2.4): one considers not only coded copies of $(\mathbb{N}, +, \times)$, but also coded partial isomorphisms between them (called comparison maps). Extending work by Slaman and Woodin, in Nies e.a. [47], schemes S_M, S_g are determined such that for each coded copies $\mathbf{M}_0, \mathbf{M}_1$ of $(\mathbb{N}, +, \times)$ there is a map g which extends the isomorphism between the standard parts of the coded models. Then, $\mathbf{M}_{\overline{p}}$ is standard iff for each \mathbf{M} , some $g : \mathbf{M}_{\overline{p}} \rightarrow \mathbf{M}$ is total. The latter condition can be expressed in the first-order language of \mathcal{R}_T . Thus, standard models are singled out as the “shortest” coded models. The idea of using “comparison maps” is essential to obtain the definability results in Nies e.a. [47], for instance the definability without parameters of Low_2 and High_1 .

For \mathcal{R}_{wtt} we will develop in Chapter 7 a parameter free coding of a copy of $(\mathbb{N}, +, \times)$. We represent the number n by all sets of cardinality n in the uniformly definable class of “EN-sets”. A scheme is needed to code maps between EN-sets in order to express for instance that two EN-sets have the same cardinality. A similar result can be obtained for \mathcal{R}_m (Section 3.4). The first application of this variant was to the upper semilattice of c.e. equivalence relations modulo finite variants (Nies [41]).

2.3 Undecidable fragments of theories

In the context of fragments of theories we only consider coding *up to isomorphism*. A formula is Σ_k if it has the form

$$(\exists \dots \exists) (\forall \dots \forall) (\exists \dots \exists) \dots \psi,$$

with $k - 1$ quantifier alternations and ψ quantifier free, and Π_k if it has the form

$$(\forall \dots \forall) (\exists \dots \exists) (\forall \dots \forall) \dots \psi.$$

Given an undecidable theory T , an interesting further question to ask is which *fragments* $T \cap \Sigma_k$ and $T \cap \Pi_k$ are undecidable, for several reasons. Firstly, the sentences which occur in mathematical practice usually have a low number of quantifier alternations. So, even after undecidability of T is known, the question remains which feasible fragments are undecidable. Secondly, a sharp classification at which fragment an undecidable theory T becomes undecidable gives more precise information about T than a plain undecidability proof. (Ershov gave an example of an undecidable theory where all fragments are decidable. However, an undecidability result obtained indirectly via Fact 2.1.1 gives actually undecidability of some fragment.) Finally, if $T = \text{Th}(\mathcal{C})$ for some class of structures \mathcal{C} , sometimes one can interpret the sentences in a fragment algebraically. Then a decision procedure for that fragment gives algebraic information about \mathcal{C} . For instance, the Π_1 -theory of a variety is closely connected to the word problem of its finitely presented members. Moreover, Π_2 -sentences in the language of p.o. can be interpreted as statements about possible extensions of embeddings of finite partial orders.

In the following we will develop a version for fragments of the method to obtain undecidability of theories of classes in an indirect way which was outlined in Section 2.1. Given a first-order language L , a set of sentences $U \subseteq L$ is *hereditarily undecidable* (h.u.) if, for each X ,

$$L - \text{valid} \cap U \subseteq X \subseteq U \Rightarrow X \text{ undecidable.}$$

This extends the previous definition given before Fact 2.1.1 for *theories* U . Disjoint sets $A, B \subseteq \mathbb{N}$ are called *recursively inseparable* if there is no computable set R such that $A \subseteq R$ & $B \subseteq \mathbb{N} - R$. Then (provided we have chosen some Gödel numbering of the formulas in L),

$$U \subseteq L \text{ is h.u.} \Leftrightarrow L - \text{valid} \cap U, L - U \text{ are recursively inseparable.}$$

In order to obtain an undecidability result for a low-level fragment of $\text{Th}(\mathcal{A})$ from the method in Fact 2.1.1, one has to invent a coding of \mathcal{C} in \mathcal{A} of maximum economy. Therefore it is useful to consider a class \mathcal{C} in a language L_0 without equality. The following theorem was proved by Lavrov and is reproved in Nies [44].

Theorem 2.3.1 *The Σ_2 -theory of the class of finite undirected graphs in the language without equality is hereditarily undecidable.*

As in Section 2.1, consider first-order languages L_0, L_1 over finite symbols sets, and suppose that L_0 is relational and has no equality symbol. First we have to clarify when an L_0 -structure is said to be coded in an L_1 -structure. Let $\varphi_{eq}(x, y) \in L_0$ be the formula expressing that x, y behave in the same way with respect to all elements of the structure. Thus $\varphi_{eq}(x, y)$ is the conjunction of formulas of the type

$$\forall x[(Rxz \Leftrightarrow Ryz) \& (Rzx \Leftrightarrow Rzy)],$$

for each relation symbol R of L_0 . Given an L_0 -structure \mathbf{C} , let

$$eq(\mathbf{C}) = \{x, y : \mathbf{C} \models \varphi_{eq}(x, y)\},$$

and let $\mathbf{C}/_{eq(\mathbf{C})}$ be the structure on equivalence classes defined in the canonical way. It is easy to verify that

$$\mathbf{C} \models \psi \Leftrightarrow \mathbf{C}/_{eq(\mathbf{C})} \models \psi,$$

for each L_0 -sentence ψ , by an induction on $|\psi|$.

A Σ_k -scheme is given by a list of formulas

$$(2.9) \quad S = \varphi_{dom}(x; \tilde{p}), \varphi_R(x_1, \dots, x_n; \tilde{p}), \varphi_{\overline{R}}(x_1, \dots, x_n; \tilde{p})_{R \text{ relation symbol of } L_0},$$

together with a *correctness condition* which expresses that

$$D = \{x : \varphi_{dom}(x; \tilde{p})\}$$

is nonempty, and that the relations on D defined by the formulas $\varphi_R, \varphi_{\overline{R}}$ are complements of each other. These condition can be expressed by universally quantified boolean combinations of Σ_k -formulas, and therefore by a Π_{k+1} -formula $\alpha(\tilde{p})$. We say that the L_0 -structure \mathbf{C} is *coded* in \mathbf{A} via the Σ_k -scheme S and a list of parameters \tilde{a} in \mathbf{A} if $\mathbf{A} \models \alpha(\tilde{a})$ and

$$\mathbf{C}/_{eq(\mathbf{C})} \cong \mathbf{D}/_{eq(\mathbf{D})},$$

where $D = \{x : \varphi_{dom}(x; \tilde{p})\}$ and \mathbf{D} is the L_0 -structure on D induced by the formulas φ_R . For uniform Σ_k -coding of a class \mathcal{C} of L_0 -structures in a class \mathcal{A} of L_1 -structures one requires that a fixed Σ_k -scheme of formulas with parameters defines a copy of each structure from \mathcal{C} in some structure \mathbf{A} from \mathcal{A} if appropriate values in \mathbf{A} are substituted for the parameters. For example, the class of finite undirected graphs from Theorem 2.3.1 is

uniformly Σ_1 -coded in the class of finite p.o. as $L(\leq)$ -structures, via the following Σ_1 -scheme without parameters:

$$(2.10) \quad \begin{aligned} \varphi_{dom}(x) &\Leftrightarrow \exists u \exists v [u < x < v] \\ \varphi_E(x, y) &\Leftrightarrow x \not\leq y \not\leq x \ \& \ \exists z \ x, y \leq z \\ \varphi_{\overline{E}}(x, y) &\Leftrightarrow x \not\leq y \not\leq x \ \& \ \exists z \ z \leq x, y \end{aligned}$$

(here $x < y$ stands for $x \leq y \ \& \ y \not\leq x$).

We are now ready to obtain a version of Fact 2.1.1 for fragments.

Transfer Lemma 2.3.2 *Let $r \geq 2, k \geq 1$.*

(i) *If \mathcal{C} can be uniformly Σ_k^0 -coded in \mathcal{D} without parameters, then*

$$\Sigma_r - \text{Th}(\mathcal{C}) \text{ h.u.} \Rightarrow \Sigma_{r+k-1} - \text{Th}(\mathcal{D}) \text{ h.u.}$$

(ii) *If \mathcal{C} can be uniformly Σ_k^0 -coded in \mathcal{D} with parameters, then*

$$\Pi_{r+1} - \text{Th}(\mathcal{C}) \text{ h.u.} \Rightarrow \Pi_{r+k} - \text{Th}(\mathcal{D}) \text{ h.u.}$$

Thus, combining (2.10) with Theorem 2.3.1, we obtain from (i) that the Σ_2 -theory of the class of finite partial orders is h.u.

Proof. The idea is to define an effective map F from L_0 -sentences to L_1 sentences which maps L_0 -valid into L_1 -valid and sentences $\varphi \notin \text{Th}(\mathcal{C})$ to a sentence $F(\varphi) \notin \text{Th}(\mathcal{D})$.

Given an L_0 -sentence φ in normal form, the translation $\tilde{\varphi}(\overline{p})$ ($\tilde{\varphi}$ if no parameters are used in the Σ_k -scheme) is obtained by relativizing the quantifiers to $\{x : \varphi_{dom}(x; \overline{p})\}$ and replacing the atomic formulas $R\overline{x}$ and $\neg R x_1, \dots, x_n$ by $\varphi_R(\overline{x})$ and $\varphi_{\overline{R}}(x_1, \dots, x_n)$ ($\overline{x} = x_1, \dots, x_n$) in a way to minimize the number of quantifier alternations. If the innermost quantifier in φ is existential, replace $R\overline{x}$ by $\varphi_R(\overline{x})$ and replace $\neg R x_1, \dots, x_n$ by $\varphi_{\overline{R}}(x_1, \dots, x_n)$. Otherwise, replace $R\overline{x}$ by the Π_k -formula $\neg \varphi_R(\overline{x})$ and replace $\neg R x_1, \dots, x_n$ by the Π_k -formula $\neg \varphi_{\overline{R}}(\overline{x})$. For instance, if φ is $\exists x \forall y [Rxy \vee \neg R yx]$, then $\tilde{\varphi}(\overline{p})$ is

$$\exists x [\varphi_{dom}(x; \overline{p}) \ \& \ \forall y [\varphi_{dom}(y; \overline{p}) \Rightarrow \neg \varphi_{\overline{R}}(x, y; \overline{p}) \vee \neg \varphi_R(y, x; \overline{p})]].$$

Note that the translation of a Σ_r -sentence is a Σ_{r+k-1} -formula and that the translation of a Π_{r+1} -sentence is a Π_{r+k} -formula.

Let

$$F(\varphi) = \forall \bar{p}[\alpha(\bar{p}) \Rightarrow \tilde{\varphi}(\bar{p})]$$

(and $F(\varphi) = \alpha \Rightarrow \tilde{\varphi}$ if there are no parameters). Clearly,

$$\varphi \in L_0 - \text{valid} \Rightarrow F(\varphi) \in L_1 - \text{valid}.$$

Moreover,

$$\varphi \notin \text{Th}(\mathcal{C}) \Rightarrow F(\varphi) \notin \text{Th}(\mathcal{D}),$$

because, if φ fails in some structure $\mathbf{C} \in \mathcal{C}$, then $F(\varphi)$ fails in a structure $\mathbf{D} \in \mathcal{D}$ coding \mathbf{C} , the counterexample for $\forall \bar{p}[\dots]$ being provided by the list of parameters used for the coding.

For the proof of (i), note that, if φ is a Σ_r -sentence, then $F(\varphi)$ is logically equivalent to a Σ_{r+k-1} -sentence, since $r+k-1 \geq k+1$ and $\alpha(\bar{p})$ is a Π_{k+1} -formula.

For (ii) we argue in a similar way, using the fact that for a Π_{r+1} -sentence φ , $F(\varphi)$ is logically equivalent to a Π_{r+k} -sentence. \diamond

Chapter 3

C.e. many-one degrees

We first concentrate on the proof of the definability lemma for \mathcal{R}_m (2.7) explained in Section 2.2, which leads to an interpretation of $\text{Th}(\mathbb{N})$ in $\text{Th}(\mathcal{R}_m)$. Then we proceed to results about \mathcal{R}_m of a model theoretic nature. First we derive a local definability result for automorphisms. Next we strengthen the result that there is an interpretation of $\text{Th}(\mathbb{N})$ in $\text{Th}(\mathcal{R}_m)$ by developing a coding of $(\mathbb{N}, +, \times)$ in \mathcal{R}_m without parameters. Let $\mathcal{R}_m^- = \mathcal{R}_m - \{\mathbf{1}\}$. We show that there is an incomplete $\mathbf{e} \in \mathcal{R}_m$ such that $[\mathbf{o}, \mathbf{e}]$ is an elementary submodel of \mathcal{R}_m^- via the inclusion embedding. In particular, \mathcal{R}_m^- (and hence \mathcal{R}_m , since $[\mathbf{o}, \mathbf{e}] \cup \{\mathbf{1}\} \prec \mathcal{R}_m$) has a proper elementary submodel, i.e. is not a minimal model over the empty set. It follows from results of Slaman and Woodin [54] that $\mathbf{D}_T(\leq \emptyset')$ is both a minimal model and a prime model. For the c.e. degree structures except \mathcal{R}_m , both questions remain open.

3.1 Preliminaries

To prove the definability lemma we interpret an elementary non-extensional set theory in \mathcal{R}_m^- . A set $S \subseteq \mathcal{R}_m^-$ is called *uniformly computably enumerable (u.c.e.)* if $S = \{\mathbf{a}_n : n \in \mathbb{N}\}$ for some u.c.e. sequence (\mathbf{a}_n) . We will first show the uniform definability of the u.c.e. (and in particular of the nonempty finite) sets $S \subseteq \mathcal{R}_m^-$ from one parameter. This leads to a definable “element” relation on \mathcal{R}_m^- .

Theorem 3.1.1 *The u.c.e. sets of incomplete m -degrees are uniformly definable via a formula $\varphi_{\in}(x; a)$ with one parameter.*

Notice that the parameter a is not uniquely determined. Hence the elementary set theory we interpret in \mathcal{R}_m will not be extensional.

An *ideal* of an upper semilattice is a nonempty subset which is closed downward and under supremum. If I is an ideal in \mathcal{R}_m , we will say that \mathbf{b} is a *strong minimal cover of I* if $I = [\mathbf{o}, \mathbf{b}]$. A strong minimal cover is necessarily *join irreducible*, namely it is not the supremum of two smaller degrees. We first state a Lemma which is a special case of Theorem 3.1 in Ershov and Lavrov [15] (where a completely different notation is used). Inspection of their proof shows that the strong minimal cover is obtained in an effective way.

Lemma 3.1.2 ([15]) *Suppose that $I \subseteq \mathcal{R}_m^-$ is a Σ_3^0 -ideal and $D \subseteq \mathcal{R}_m^-$ is a u.c.e. set. Then one can effectively obtain a strong minimal cover \mathbf{b} of I such that $\mathbf{b} \not\leq \mathbf{d}$ for each $\mathbf{d} \in D$. \diamond*

Proof of Theorem 3.1.1 We have to determine a formula $\varphi_\in(x; a)$ with the following property: given a u.c.e. sequence (\mathbf{a}_n) of incomplete m -degrees, there is a parameter \mathbf{a} such that $\{\mathbf{a}_n : n \in \mathbb{N}\} = \{\mathbf{b} : \mathcal{R}_m \models \varphi_\in(\mathbf{b}; \mathbf{a})\}$. Applying Lemma 3.1.2 to each Σ_3^0 -ideal $[0, \mathbf{a}_n]$ and the u.c.e. set $D = \{\mathbf{a}_n : n \in \mathbb{N}\}$, we obtain a u.c.e. sequence (\mathbf{b}_n) such that, for each n , \mathbf{b}_n is a strong minimal cover of $[0, \mathbf{a}_n]$ which is not below \mathbf{a}_m for any m . For each n, m , $\mathbf{b}_n = \mathbf{b}_m$ or $\mathbf{b}_n, \mathbf{b}_m$ are incomparable.

Recall that, by a result of Lachlan (see [56, p. 45]), the complete c.e. m -degree is join irreducible in \mathcal{R}_m . So $I \subseteq \mathcal{R}_m^-$. Now, let \mathbf{a} be a strong minimal cover of the Σ_3^0 -ideal I generated by $\{\mathbf{b}_n : n \in \mathbb{N}\}$. The set $\{\mathbf{b}_n : n \in \mathbb{N}\}$ is definable in $[0, \mathbf{a}]$ as the set of maximally join irreducible elements: if $\mathbf{x} < \mathbf{a}$, then $\mathbf{x} \leq \mathbf{b}_0 \vee \dots \vee \mathbf{b}_n$ for some n . By the distributivity of \mathcal{R}_m (1.1), there exists $\mathbf{c}_i \leq \mathbf{b}_i$ such that $\mathbf{x} = \mathbf{c}_0 \vee \dots \vee \mathbf{c}_n$. If \mathbf{x} is join irreducible, this implies that $\mathbf{x} \leq \mathbf{b}_i$ for some i , and if \mathbf{x} is maximally join irreducible, then even $\mathbf{x} = \mathbf{b}_i$.

To show, conversely, that each \mathbf{b}_m is maximally join irreducible in $[0, \mathbf{a}]$, suppose that $\mathbf{b}_m \leq \mathbf{x}$ for some join irreducible $\mathbf{x} < \mathbf{a}$. In the same way as above, we obtain $\mathbf{x} \leq \mathbf{b}_i$ for some i . Hence $\mathbf{b}_m \leq \mathbf{b}_i$ and therefore $\mathbf{x} = \mathbf{b}_m$. From the definability of $\{\mathbf{b}_n : n \in \mathbb{N}\}$ in $[0, \mathbf{a}]$ we obtain the definability of $\{\mathbf{a}_n : n \in \mathbb{N}\}$. The formula $\varphi_\in(x; a)$ is given by

$$\exists b(b \text{ maximally join irreducible in } [0, a] \ \& \ [0, b] = [0, x]).$$

\diamond

Corollary 3.1.3 *(i) Every finite set $F \subseteq \mathcal{R}_m^-$ is uniformly definable from an incomplete m -degree \mathbf{a} , which can be obtained effectively in F .*

(ii) There exist definable projection functions pr_1, pr_2 so that

$$\forall \mathbf{x}_1, \mathbf{x}_2 < \mathbf{1} \exists \mathbf{y} < \mathbf{1} [pr_1(\mathbf{y}) = \mathbf{x}_1 \ \& \ pr_2(\mathbf{y}) = \mathbf{x}_2].$$

Moreover, an index for such a degree \mathbf{y} can be obtained effectively in indices for $\mathbf{x}_1, \mathbf{x}_2$.

Proof. (i) Suppose that $F = \{deg_m(W_i) : i \in D_z\}$, where D_z is a strong index for a finite set. If $D_z = \emptyset$, let $\mathbf{a} = 0$. Else in some effective way obtain a u.c.e. sequence (\mathbf{a}_n) such that $F = \{\mathbf{a}_n : n \in \mathbb{N}\}$ and apply Theorem 3.1.1. (ii) Recall that, in set theory, the ordered pair $\langle x_1, x_2 \rangle$ is represented by $\{\{x_1\}, \{x_1, x_2\}\}$. Regardless of extensionality, the analog in \mathcal{R}_m^- of such an object will determine both of its components. We let $pr_i(\mathbf{y}) = \mathbf{x}_i$ ($i = 1, 2$) in case \mathbf{y} is a set of this form, with respect to the element relation defined by the formula φ_\in , and $pr_i(\mathbf{y}) = \mathbf{o}$, otherwise. The maps pr_i are clearly definable in \mathcal{R}_m . Moreover, given $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{R}_m^-$, by repeated applications of (i) we can effectively in indices for $\mathbf{x}_1, \mathbf{x}_2$ determine a $\mathbf{y} < \mathbf{1}$ such that $pr_i(\mathbf{y}) = \mathbf{x}_i$ for $i = 1, 2$. \diamond

If $\mathbf{x}_1, \mathbf{x}_2$ are given by indices for c.e. sets, let $p(\mathbf{x}_1, \mathbf{x}_2)$ denote the degree \mathbf{y} obtained at the end of the previous proof. If $\mathbf{x}_1, \mathbf{x}_2 < \mathbf{1}$, then $\mathbf{x}_1 \vee \mathbf{x}_2 \leq p(\mathbf{x}_1, \mathbf{x}_2) < \mathbf{1}$. Keep in mind that $p(\mathbf{x}_1, \mathbf{x}_2)$ really depends on the indices used to represent $\mathbf{x}_1, \mathbf{x}_2$.

3.2 The definability lemma for \mathcal{R}_m

This section and the following do not use any particular property of \mathcal{R}_m beyond that the ordering is Σ_3^0 as a relation on indices and Theorem 3.1.1.

Definability Lemma 3.2.1 *For each $k \geq 3, N \geq 1$, the class of Σ_k^0 relations on intervals $[\mathbf{o}, \mathbf{c}]$ of \mathcal{R}_m such that $\mathbf{c} < \mathbf{1}$ is weakly uniformly definable. In fact, there exist formulas $\varphi_{k,N}(x_1, \dots, x_N; c, a)$ with the following property:*

if $Z \subseteq [\mathbf{o}, \mathbf{c}]^N$ is Σ_k^0 , then one can effectively in \mathbf{c} and a Σ_k^0 -representation of Z determine an $\mathbf{a} < \mathbf{1}$ such that $\varphi_{k,N}$ defines Z in \mathcal{R}_m with the parameters \mathbf{c}, \mathbf{a} .

Proof. We first make the extra assumption that we also possess a lower bound $\mathbf{d} > \mathbf{o}$ for Z , namely that $Z \subseteq [\mathbf{d}, \mathbf{c}]^N$. Thus we will construct

formulas $\varphi_{k,N}$ as in the statement of the lemma, but with an additional parameter d , and we will obtain \mathbf{a} effectively in \mathbf{c}, \mathbf{d} and a Σ_k^0 -representation of Z .

The formulas $\varphi_{k,N}$ are defined by recursion over k (N is fixed). For notational simplicity, we will carry out the recursion for $N = 2$. The fact that \mathbf{a} can be determined effectively is needed to make the recursion work.

First let $k = 3$. We will define u.c.e. sequences of incomplete m -degrees $\mathbf{a}_n, \mathbf{b}_n$ such that

$$(3.1) \quad Z = \{\langle \mathbf{a}_n, \mathbf{b}_n \rangle : n \in \mathbb{N}\} \cap [\mathbf{d}, \mathbf{1}]^2.$$

Recall that ΘZ is the index relation associated with Z . Since ΘZ is Σ_3^0 , there is a u.c.e. sequence of sets $(X_{\langle i,j,k \rangle})$ such that $\langle i,j \rangle \in \Theta Z \Rightarrow \exists k X_{\langle i,j,k \rangle} = \mathbb{N}$ and $\langle i,j \rangle \notin \Theta Z \Rightarrow \forall k X_{\langle i,j,k \rangle}$ finite. This follows e.g. from Soare [56, p. 68]. Now, for $n = \langle i,j,k \rangle$, let

$$A_n = W_i \cap X_{\langle i,j,k \rangle} \text{ and } B_n = W_j \cap X_{\langle i,j,k \rangle}.$$

If $\mathbf{a}_n = \deg_m(A_n)$ and $\mathbf{b}_n = \deg_m(B_n)$, then $\mathbf{d} \neq \mathbf{o}$ implies (3.1).

Since $\{\langle \mathbf{a}_n, \mathbf{b}_n \rangle : n \in \mathbb{N}\}$ is a u.c.e. set of incomplete degrees, by Theorem 3.1.1 this set can be defined from a parameter \mathbf{a} . Then Z is definable from the parameters \mathbf{d}, \mathbf{a} via a fixed formula $\varphi_{3,2}(x_1, x_2; d, \mathbf{a})$ (the upper bound was not needed yet):

$$\begin{aligned} \varphi_{3,2}(x_1, x_2; d, \mathbf{a}) \equiv & x_1 \geq d \ \& \ x_2 \geq d \ \& \\ & \exists y [\varphi_{\in}(y, \mathbf{a}) \ \& \ pr_1(y) = x_1 \ \& \ pr_2(y) = x_2]. \end{aligned}$$

Clearly \mathbf{a} was obtained effectively in \mathbf{d} and a Σ_3^0 -representation of ΘZ . Next suppose ΘZ is Σ_{k+1}^0 ($k \geq 3$). We will show the definability of $Y = [\mathbf{d}, \mathbf{c}] - Z$. Notice that ΘY is Π_{k+1}^0 . One might attempt the following:

1. write $\Theta Y = \bigcap_{n \in \mathbb{N}} R_n$, where each R_n is an N -ary relation on indices, closed under \equiv_m and Σ_k^0 uniformly in n
2. use the inductive hypothesis to define from parameters \mathbf{d}, \mathbf{c} and \mathbf{b}_n the relation on m -degrees given by R_n
3. finally apply Theorem 3.1.1 to the u.c.e. sequence (\mathbf{b}_n) in order to define Y from the new parameter \mathbf{b} obtained, as the intersection of these relations.

But there is a flaw in this approach, the careless handling of index sets. For instance, if $k = 4$ and $Y = [\mathbf{o}, \mathbf{v})$ for some $\mathbf{o} < \mathbf{v}$, then such a representation implies that $\Theta Y = R_n \cap \{e : \deg_m(W_e) \leq \mathbf{v}\}$, for some n and hence that Y is Σ_3^0 . This is not the case when $[\mathbf{o}, \mathbf{v}]$ is effectively isomorphic to \mathcal{R}_m . But the approach can be rescued if carried out separately for each pair of m -degrees. The relation Y is the effective union of Π_{k+1}^0 relations $Y_{\langle i, j \rangle}$ of cardinality ≤ 1 , where

$$Y_{\langle i, j \rangle} = Y \cap \{(\deg_m(W_i), \deg_m(W_j))\}.$$

We will find a u.c.e. sequence $(\mathbf{a}_{\langle i, j \rangle})$ so that $\mathbf{a}_{\langle i, j \rangle} < \mathbf{1}$ and for some fixed formula $\psi(x, y; a)$, each $Y_{\langle i, j \rangle}$ is definable from $\mathbf{a}_{\langle i, j \rangle}$ via ψ . Then, if $\mathbf{a} < \mathbf{1}$ defines the set $\{\mathbf{a}_{\langle i, j \rangle} : i, j \in \mathbb{N}\}$ via φ_\in ,

$$(3.2) \quad Y = \{\langle \mathbf{x}, \mathbf{y} \rangle : \mathcal{R}_m \models \exists a' [\varphi_\in(a', \mathbf{a}) \ \& \ \psi(\mathbf{x}, \mathbf{y}; a')]\}.$$

Hence the complement Z is definable from the parameters \mathbf{d}, \mathbf{c} and \mathbf{a} . Moreover \mathbf{a} was obtained effectively in \mathbf{d}, \mathbf{c} and a Σ_{k+1}^0 representation of ΘZ . To determine the sequence $(\mathbf{a}_{\langle i, j \rangle})$, let $A = W_i, B = W_j$ and $S = \Theta Y_{\langle i, j \rangle}$. We will construct $\mathbf{a}_{\langle i, j \rangle}$ uniformly in $i, j, \Theta Y$ and \mathbf{c}, \mathbf{d} . From a Π_{k+1}^0 -representation of ΘY one obtains a uniform sequence (R_n) of Σ_k^0 relations such that

$$(3.3) \quad S = \bigcap_n R_n.$$

Since $Y_{\langle i, j \rangle} \subseteq \{(\deg_m(A), \deg_m(B))\} \cap [\mathbf{d}, \mathbf{c}]$ and $k \geq 3$, we may suppose that

$$(3.4) \quad R_n xy \Rightarrow \mathbf{d} \leq \deg_m(W_x), \deg_m(W_y) \leq \mathbf{c} \ \& \ W_x \equiv_m A \ \& \ W_y \equiv_m B.$$

We will define modified Σ_k^0 -relations \tilde{R}_n such that (3.3) still holds, but also each \tilde{R}_n is compatible with \equiv_m . We cannot simply take the closure under \equiv_m , since it may be the case that $S = \emptyset$ because, for each n , $R_n xy$ holds with different indices x, y for A, B . Instead, we reduce the relations R_n , making use of the fact that S is \equiv_m -closed in order to maintain (3.3). This takes three steps.

1. First we find a uniform sequence (R'_n) of Σ_k^0 relations such that $R_n = R'_n$ if

$$(3.5) \quad \forall u, v [W_u \equiv_m A \ \& \ W_v \equiv_m B \Rightarrow R_n uv],$$

and otherwise R'_n is finite. Note that (3.5) is a $\Pi_1^{\emptyset^{(k)}} (= \Pi_2^{\emptyset^{(k-1)}})$ -statement about n , uniformly in indices for A, B . Thus we can effectively determine a set $X_n = W_u^{\emptyset^{(k-1)}}$ such that, if (3.5) holds, then $X_n = \mathbb{N}$, and X_n is finite otherwise. Then $R'_n = R_n \cap (X_n \times X_n)$ is a Σ_k^0 -relation as desired.

2. We view each relation R'_n as a relation c.e. in $\emptyset^{(k-1)}$. Thus we are effectively given an enumeration $R'_n = \bigcup_s R'_{n,s}$, where the sequence $(R'_{n,s})_{s \in \mathbb{N}}$ of strong indices for finite sets of pairs is recursive in $\emptyset^{(k-1)}$. We define $\emptyset^{(k-1)}$ -recursive relations $R''_n \subseteq R'_n$ as follows: at stage s , allow a pair $\langle x, y \rangle \in R'_{n,s}$ into $R''_{n,s}$ only if $|R'_{m,s}| > n$ for all $m < n$. If (3.5) holds for each R_n (and hence for each R'_n), then $\forall n R''_n = R'_n$, but otherwise $R''_n = \emptyset$ for almost all n .
3. Finally let \tilde{R}_n be the closure of R''_n under the equivalence relation \equiv_m .

To verify that $S = \bigcap_n \tilde{R}_n$, let $x, y \in \mathbb{N}$ be arbitrary. First suppose that Sxy . Then, because S is compatible with \equiv_m , (3.5) holds for each n . Hence $\forall n \tilde{R}_n xy$. Conversely, if $\forall n \tilde{R}_n xy$, then $R''_n \neq \emptyset$ for each n . Then R'_n is infinite, and therefore (3.5) holds. Since $S = \bigcap_n R_n$, this implies that Sxy . Since (3.4) holds for the sequence (\tilde{R}_n) and \tilde{R}_n is Σ_k^0 uniformly in n , it is possible to apply the inductive hypothesis to each relation on \mathcal{R}_m given by \tilde{R}_n . Thus we obtain a u.c.e. sequence (\mathbf{b}_n) of incomplete m -degrees such that

$$\langle \mathbf{x}, \mathbf{y} \rangle \in Y_{i,j} \Leftrightarrow \text{for each } n, \mathcal{R}_m \models \varphi_{k,2}(\mathbf{x}, \mathbf{y}; \mathbf{b}_n).$$

Define $\{\mathbf{b}_n : n \in \mathbb{N}\}$ from a parameter $\mathbf{a}_{i,j} < \mathbf{1}$. Then

$$Y_{i,j} = \{\langle \mathbf{x}, \mathbf{y} \rangle : \mathcal{R}_m \models \forall b [\varphi_{k,2}(b, \mathbf{a}_{i,j}) \rightarrow \varphi_{k,2}(\mathbf{x}, \mathbf{y}; b)]\}.$$

Since we have determined $\mathbf{a}_{i,j}$ effectively, this concludes the proof of the lemma for an interval $[\mathbf{d}, \mathbf{c}]$, $\mathbf{d} \neq \mathbf{o}$.

To reduce the general case to this, by Lemma 3.1.2, (effectively in \mathbf{c}) obtain a minimal m -degree \mathbf{d} such that $\mathbf{d} \not\leq \mathbf{c}$. Now we apply the above with

$\tilde{\mathbf{c}} = \mathbf{c} \vee \mathbf{d}$ instead of \mathbf{c} . Note that the intervals $[\mathbf{o}, \mathbf{c}]$ and $[\mathbf{d}, \tilde{\mathbf{c}}]$ are isomorphic: the isomorphism is $\mathbf{x} \rightarrow \mathbf{x} \vee \mathbf{d}$ and its inverse is $\mathbf{y} \rightarrow \mathbf{y} \wedge \mathbf{c}$. If $Z \subseteq [\mathbf{o}, \mathbf{c}]^N$ is Σ_k^0 ($k \geq 3$), then so is $\tilde{Z} = \{\mathbf{x} \vee \mathbf{d} : \mathbf{x} \in Z\}$. Hence \tilde{Z} is uniformly definable from the parameters $\mathbf{d}, \tilde{\mathbf{c}}$ and a parameter $\tilde{\mathbf{a}}$. Let $\mathbf{a} = p(\mathbf{d}, \tilde{\mathbf{a}})$. Then $\mathbf{a} < \mathbf{1}$ and $Z = \{\mathbf{x} \wedge \mathbf{c} : \mathbf{x} \in \tilde{Z}\}$ is uniformly definable from \mathbf{a} and \mathbf{c} . (Note that, also in the general case, \mathbf{a} is obtained effectively.) \diamond

The proof shows that in fact, for $k \geq 3$, every Σ_k^0 relation $Z \subseteq [\mathbf{o}, \mathbf{c}]$ can be defined from parameters via a Σ_{k+C} formula, for a fixed C . However, we only obtain weak uniform definability, namely some extra relations may be definable via our formulas. The following proposition shows that the upper bound $\mathbf{c} < \mathbf{1}$ in the definability lemma is necessary.

Proposition 3.2.2 *There is a Σ_4^0 -relation on \mathcal{R}_m which is not definable from parameters.*

Proof. We use the fact that \mathcal{R}_m has uncountably many automorphisms. First, by repeated applications of Lemma 3.1.1, for each $m > 0$ one can construct \mathbf{c} such that $||[\mathbf{o}, \mathbf{c}]|| = m$. Let

$$R = \{(\mathbf{a}, \mathbf{b}) : \exists n > 0 (||[\mathbf{o}, \mathbf{a}]|| \geq n \ \& \ \text{deg}_m(W_n) = \mathbf{b})\}.$$

Clearly R is Σ_4^0 . Assume that R is definable from a parameter list. Because there are uncountably many automorphisms, there must be a non-identity automorphism Φ which fixes the parameter list. Then Φ respects R . We show that $\Phi(\mathbf{b}) = \mathbf{b}$ for each \mathbf{b} , a contradiction. Given \mathbf{b} , let $n > 0$ be minimal such that $\text{deg}_m(W_n) = \mathbf{b}$. Then, for each \mathbf{a} , $R\mathbf{a}\mathbf{b} \Leftrightarrow ||[\mathbf{o}, \mathbf{a}]|| \geq n$. Hence for each \mathbf{c} , $R\mathbf{c}\Phi(\mathbf{b}) \Leftrightarrow ||[\mathbf{o}, \mathbf{c}]|| \geq n$. This implies that $\Phi(\mathbf{b}) = \text{deg}_m(W_n)$. \diamond

3.3 Interpreting true arithmetic in $\text{Th}(\mathcal{R}_m)$

Theorem 3.3.1 *$\text{Th}(\mathbb{N})$ can be interpreted in $\text{Th}(\mathcal{R}_m)$.*

Proof. We follow the framework of Section 2.2. Fix any $\mathbf{e} < \mathbf{1}$ such that $[\mathbf{o}, \mathbf{e}]$ is infinite. We carry out (2.3), representing numbers by the degrees in $[\mathbf{o}, \mathbf{e}]$.

Let $h \leq_T \emptyset^{(3)}$ be any map such that $\mathbf{a}_n = \text{deg}_m(W_{h(n)})$ is a non-repeating list of all the degrees in $[\mathbf{o}, \mathbf{e}]$. Then addition and multiplication on $\{\mathbf{a}_n : n \in \mathbb{N}\}$

(= $[\mathbf{o}, \mathbf{e}]$), viewed as ternary relations, are Σ_4^0 . By the definability lemma, these relation can be defined from a list of parameters $\bar{\mathbf{p}}$, which includes \mathbf{e} . To determine a $\alpha_{st}(\bar{\mathbf{p}})$ in (2.4), consider an arbitrary parameter list $\bar{\mathbf{p}}$. First, beyond the correctness condition $\alpha_0(\bar{\mathbf{p}})$ from Example 2.1.3 we require that $\mathbf{e} < \mathbf{1}$ and $M_{\bar{\mathbf{p}}} \subseteq [\mathbf{o}, \mathbf{e}]$. Let k be the least number such that, for each $M_{\bar{\mathbf{p}}}$, the standard part is Σ_k^0 . Using the definability lemma, we can now express in a first-order way that $M_{\bar{\mathbf{p}}}$ is standard: we require that

each subset of $M_{\bar{\mathbf{p}}}$ defined from any parameters via the formula $\varphi_{k,1}$ (and therefore, each Σ_k^0 -subset of $M_{\bar{\mathbf{p}}}$) which contains $0^{M_{\bar{\mathbf{p}}}}$ and is closed under taking the successor function of $M_{\bar{\mathbf{p}}}$ equals $M_{\bar{\mathbf{p}}}$. ◇

3.4 Model theoretic results on \mathcal{R}_m

We survey several results.

By the techniques of Denisov [13], \mathcal{R}_m possesses continuum many automorphisms. We apply the coding of copies of $(\mathbb{N}, +, \times)$ to derive a uniform definability result for the restrictions of automorphisms to proper initial intervals. In particular, there are only countably many such restrictions, and the abundance of automorphisms stems from the many possibilities to put them together.

Theorem 3.4.1 *The class of partial maps*

$$\{\Phi[[\mathbf{o}, \mathbf{e}]] : \mathbf{e} < \mathbf{1} \ \& \ \Phi \in \text{Aut}(\mathcal{R}_m)\}$$

is weakly uniformly definable.

Proof. Suppose that $\mathbf{e} < \mathbf{1}$ and $\Phi \in \text{Aut}(\mathcal{R}_m)$. Then $\mathbf{c} = \mathbf{e} \vee \Phi(\mathbf{e}) < \mathbf{o}$ by the aforementioned result of Lachlan (see [56, p. 45]). Let

$$U = \{\langle x, \Phi(x) \rangle : x \leq \mathbf{e}\}.$$

Then $U \subseteq [\mathbf{o}, \mathbf{c}] \times [\mathbf{o}, \mathbf{c}]$. By the definability lemma, it is therefore sufficient to show that, for some constant k not depending on U , U is Σ_k^0 .

If $[\mathbf{o}, \mathbf{e}]$ is finite then U is Σ_3^0 . Now suppose otherwise. As in the proof of Theorem 3.3.1, fix a parameter list $\bar{\mathbf{p}}$ coding a copy M of $(\mathbb{N}, +, \times)$ so that the domain of M equals $[\mathbf{o}, \mathbf{e}]$. If M' is the structure coded by $\bar{\mathbf{p}}' = \Phi(\bar{\mathbf{p}})$ via S_M , then the domain of M' is $[\mathbf{o}, \mathbf{e}']$ and M' is also a copy of $(\mathbb{N}, +, \times)$. Clearly,

$$\langle \mathbf{x}, \mathbf{y} \rangle \in U \Leftrightarrow \exists n \in \mathbb{N} [\mathbf{x} = n^{\mathbf{M}} \ \& \ \mathbf{y} = n^{\mathbf{M}'}].$$

This shows that U is Σ_k^0 for some sufficiently large fixed k . \diamond

Next we give a coding without parameters of a copy of $(\mathbb{N}, +, \times)$ and show that the set of tops of finite initial intervals is definable.

Theorem 3.4.2 *A copy of $(\mathbb{N}, +, \times)$ can be coded in \mathcal{R}_m without parameters.*

Proof. The formula φ_\in from Theorem 3.1.1 determines a scheme S_P to code subsets of \mathcal{R}_m (with a vacuous correctness condition). We plan to represent the number $n \in \mathbb{N}$ by all sets P such that $P = n$. Thus let

$$N = \{\mathbf{a} : |P_{\mathbf{a}}| < \infty\}.$$

To obtain a scheme as in (2.2) but with an empty parameter list, we have to give first-order definitions without parameters of N , $\{\langle \mathbf{a}, \mathbf{b} \rangle : \mathbf{a}, \mathbf{b} \in N \ \& \ |P_{\mathbf{a}}| = |P_{\mathbf{b}}|\}$ and the ternary relations on N corresponding to the arithmetical operations $+$, \times .

The following formula determines an object scheme S_C to code binary relations: $\varphi(x, \mathbf{y}; \mathbf{b}) \equiv$

$$\exists z < 1 [\varphi_\in(z; \mathbf{b}) \ \& \ \mathbf{x} = pr_1(z) \ \& \ \mathbf{y} = pr_2(z)].$$

(see Corollary 3.1.3 for a definition of pr_1, pr_2 .) Clearly we can express in a first-order way that C is a bijection between sets defined coded via some fixed schemes.

For a first-order definition of N , note that for $\mathbf{a} < \mathbf{1}$, $\mathbf{a} \in N$ iff there is a bijection between $P_{\mathbf{a}}$ and some initial segment of a copy of $(\mathbb{N}, +, \times)$ coded by the scheme of the preceding section. By the results in Section 3.1, such a bijection can be coded via S_C .

Using elementary set theory in \mathcal{R}_m^- , we can also define in a first-order way the other relations needed. Let $\varphi_{\equiv}(x, y)$ be a formula expressing

$$\exists C [C \text{ is bijection } P_x \mapsto P_y]$$

and $\varphi_+(x, y, z)$ be a formula expressing

$$\exists u \exists v [\varphi_{\equiv}(x, u) \ \& \ \varphi_{\equiv}(y, v) \ \& \ P_z = P_u \cup P_v \ \& \ P_u \cap P_v = \emptyset].$$

For $\varphi_{\times}(x, y, z)$ we express in terms of definable projection maps that P_z has the same size as the cartesian product $P_x \times P_y$. Thus $\varphi_{\times}(x, y, z)$ expresses

$$\exists C_1 \exists C_2 \quad C_1 : P_z \mapsto P_x \text{ onto } \& C_2 : P_z \mapsto P_y \text{ onto } \& \\ \forall a \in P_x \forall b \in P_y \exists ! q \in P_z [C_1(q) = a \& C_2(q) = b].$$

◇

Corollary 3.4.3 $\{b : [\mathbf{o}, b] \text{ is finite } \}$ is definable in \mathcal{R}_m .

Proof.

$$[\mathbf{o}, b] \text{ is finite } \Leftrightarrow \exists C \exists a \in N [C \text{ is bijection between } [\mathbf{o}, b] \text{ and } P_a]. \quad \diamond$$

One can in fact obtain a stronger result, using the proof of Theorem 3.3.1: if $\mathcal{C} \subseteq \mathcal{R}_m$ is a class such that “ $\mathbf{a} \in \mathcal{C}$ ” only depends on the isomorphism type of $[\mathbf{o}, \mathbf{a}]$, then \mathcal{C} is definable iff \mathcal{C} has an arithmetical index set. Thus a restricted maximum definability property holds (see Section 4.8 below for a definition). The full maximum definability property in \mathcal{R}_m , which would state that a relation is definable without parameters iff it is invariant under automorphisms and arithmetical, is unknown.

Next we show the existence of an incomplete $\mathbf{e} \in \mathcal{R}_m$ such that $[\mathbf{o}, \mathbf{e}]$ is an elementary submodel of \mathcal{R}_m^- via inclusion. We use a version of the elementary chain principle. Write $\mathbf{A} \prec_{\Sigma_k} \mathbf{B}$ if \mathbf{A} is a submodel of \mathbf{B} and the inclusion map is a Σ_k -elementary embedding.

Lemma 3.4.4 ([11]) *If $\mathbf{A}_0 \prec_{\Sigma_k} \mathbf{A}_1 \prec_{\Sigma_k} \dots$ is a Σ_k -elementary chain and $\mathbf{A}_\omega = \bigcup_{i \in \omega} \mathbf{A}_i$, then $\mathbf{A}_i \prec_{\Sigma_k} \mathbf{A}_\omega$ for each i . Moreover, if $\mathbf{A}_i \prec_{\Sigma_k} \mathbf{B}$ for each i , then $\mathbf{A}_\omega \prec_{\Sigma_k} \mathbf{B}$.* ◇

Theorem 3.4.5 $\forall \mathbf{a} < \mathbf{1} \exists \mathbf{e} < \mathbf{1} [\mathbf{a} \leq \mathbf{e} \& [\mathbf{o}, \mathbf{e}] \prec \mathcal{R}_m^-]$.

Proof. We use the terminology and techniques of Denisov [13] (see also Odifreddi [48]), which we review briefly. A main concept is the notion of an L -semilattice (called effective distributive upper semilattice in [48]), which is a type of distributive upper semilattice with $0, 1$. Lachlan [29] proved that up to isomorphism the L -semilattices are the initial intervals of \mathcal{R}_m . We also need the following main tool for the characterization of \mathcal{R}_m from [13]. Enumerated L -semilattices are L -semilattices with a presentation so that certain effectivity conditions are satisfied. By the proof of Lachlan’s characterization of initial intervals, each L -semilattice $[\mathbf{o}, \mathbf{x}]$ is equipped with such

an enumeration. Denisov's main technical result is the following saturation property of \mathcal{R}_m .

For enumerated L-semilattices U_0, U and effective embeddings $g : U_0 \mapsto \mathcal{R}_m^-, h : U_0 \mapsto U$ as initial intervals, there is an effective embedding as an initial interval $f : U \mapsto \mathcal{R}_m$ such that $g = f \circ h$.

Moreover, the proof in [13] shows that an index for f is obtained in an effective way. Now, for each \mathbf{x} we can effectively obtain \mathbf{y} such that

1. $\mathbf{x} < \mathbf{1} \Rightarrow \mathbf{x} < \mathbf{y} < \mathbf{1}$, and
2. $[\mathbf{o}, \mathbf{y}] \cong \mathcal{R}_m$ via an (effective) isomorphism which acts as the identity on $[\mathbf{o}, \mathbf{x}]$.

To see this, consider the (effective) inclusion embedding h of the enumerated L-semilattice $U_0 = [\mathbf{o}, \mathbf{x}]$ into the enumerated L-semilattice $L = \mathcal{R}_m \cup \{\mathbf{t}\}$, where \mathbf{t} is a new largest element. By the above, obtain an effective $f : L \rightarrow \mathcal{R}_m$ which is the identity on $[\mathbf{o}, \mathbf{x}]$, and obtain \mathbf{y} as the image of $\mathbf{1} \in U$. Let us write $\mathbf{y} = F_0(\mathbf{x})$. F_0 is an effective map on indices for c.e. m -degrees. Thus (like the function p introduced above) $F_0(\mathbf{x})$ really depends on the index via which \mathbf{x} is given. Iterating F_0 we obtain, by the effectivity of Denisov's construction, for any $\mathbf{x} < \mathbf{1}$ a u.c.e. chain

$$\mathbf{x} < F_0(\mathbf{x}) < F_0(F_0(\mathbf{x})) < \dots$$

In a sense we will obtain \mathbf{e} by iterating F_0 on \mathbf{a} ω^ω many times. The construction bears some resemblance to the reflection theorems from set theory.

Let $F_1(\mathbf{x})$ be a degree \mathbf{y} such that $[\mathbf{o}, \mathbf{y}] = \bigcup_i [\mathbf{o}, F_0^{(i)}(\mathbf{x})]$. We can obtain \mathbf{y} effectively in \mathbf{x} by applying Theorem 3.1.2. Moreover, $\mathbf{x} < F_1(\mathbf{x})$. More generally, if $F_k(\mathbf{x})$ has been defined for all \mathbf{x} , F_k is effective on indices and $F_k(\mathbf{x}) > \mathbf{x}$ for $\mathbf{x} < \mathbf{1}$, let $F_{k+1}(\mathbf{x})$ be a degree \mathbf{y} such that $[\mathbf{o}, \mathbf{y}] = \bigcup_i [\mathbf{o}, F_k^{(i)}(\mathbf{x})]$. Then F_{k+1} is a function on indices with the same properties.

Claim 3.4.6 For $\mathbf{x} < \mathbf{1}$, $k \geq 0$, $[0, F_k(\mathbf{x})] \prec_{\Sigma_k} \mathcal{R}_m^-$.

Proof of the Claim. By induction on k . For $k = 0$, we assert that $[0, F_0(\mathbf{x})]$ is embedded as an ordering into \mathcal{R}_m^- , which is correct. To prove the statement for $k + 1$, let $\mathbf{z} = F_{k+1}(\mathbf{x})$, $\mathbf{z}_j = F_k^{(j)}(\mathbf{x})$ ($j \geq 0$). By the inductive hypothesis, $[\mathbf{o}, \mathbf{z}_j] \prec_{\Sigma_k} \mathcal{R}_m^-$, so the elementary chain principle implies that

$$(3.6) \quad [\mathbf{o}, \mathbf{z}] \prec_{\Sigma_k} \mathcal{R}_m^- \text{ and } \forall j [\mathbf{o}, \mathbf{z}_j] \prec_{\Sigma_k} [\mathbf{o}, \mathbf{z}].$$

Suppose $\mathbf{b}_0, \dots, \mathbf{b}_{r-1} < \mathbf{z}$, and consider the formula

$$\varphi(\bar{\mathbf{b}}) = \exists \tilde{y} \psi(\bar{\mathbf{b}}, \tilde{y}),$$

where ψ is a boolean combination of Σ_k -formulas and \tilde{y} is a tuple of variables of a certain length. We have to show that

$$[\mathbf{o}, \mathbf{z}] \models \varphi(\bar{\mathbf{b}}) \Leftrightarrow \mathcal{R}_m^- \models \varphi(\bar{\mathbf{b}}).$$

1. First suppose that $[\mathbf{o}, \mathbf{z}] \models \varphi(\bar{\mathbf{b}})$. Choose $j > 0$ and a tuple $\tilde{\mathbf{c}}$ of elements in $[\mathbf{o}, \mathbf{z}_j]$ such that $[\mathbf{o}, \mathbf{z}] \models \psi(\bar{\mathbf{b}}, \tilde{\mathbf{c}})$. By (3.6), $[\mathbf{o}, \mathbf{z}_j] \models \psi(\bar{\mathbf{b}}, \tilde{\mathbf{c}})$. Then, because $[\mathbf{o}, \mathbf{z}_j] \prec_{\Sigma_k} \mathcal{R}_m^-$, $\mathcal{R}_m^- \models \psi(\bar{\mathbf{b}}, \tilde{\mathbf{c}})$.
2. Now suppose that $\mathcal{R}_m^- \models \varphi(\bar{\mathbf{b}})$. Because $\mathcal{R}_m^- \cong [\mathbf{o}, F_0(\mathbf{z}_j)]$ via an isomorphism which acts as the identity on $[\mathbf{o}, \mathbf{z}_j]$, there is a tuple of witnesses $\tilde{\mathbf{c}}$ in $[\mathbf{o}, F_0(\mathbf{z}_j)] \subseteq [\mathbf{o}, \mathbf{z}]$ such that $\mathcal{R}_m^- \models \psi(\bar{\mathbf{b}}, \tilde{\mathbf{c}})$. By (3.6), $[\mathbf{o}, \mathbf{z}] \models \psi(\bar{\mathbf{b}}, \tilde{\mathbf{c}})$.

Finally, let $\mathbf{e} > \mathbf{a}$ be such that $[\mathbf{o}, \mathbf{e}] = \bigcup_{k \geq 0} [\mathbf{o}, F_k(\mathbf{a})]$. Since $[\mathbf{o}, F_l(\mathbf{a})] \prec_{\Sigma_k} \mathcal{R}_m^-$ for all $l \geq k$, we conclude that $[\mathbf{o}, \mathbf{e}] \prec \mathcal{R}_m^-$ by the elementary chain principle. \diamond

Notice that in fact $[\mathbf{o}, \mathbf{e}] \cong \mathcal{R}_m^-$, because $[\mathbf{o}, \mathbf{e}]$ satisfies the characterization of \mathcal{R}_m^- given in [13]. However, the isomorphism cannot be Δ_3^0 (let alone effective), because by construction of \mathbf{e} we have a u.c.e. chain $(F_k(\mathbf{a}))$ such that $\mathbf{x} < \mathbf{e} \Leftrightarrow \exists k \mathbf{x} \leq F_k(\mathbf{a})$. Such a chain converging to $\mathbf{1}$ cannot exist, because $\{i : W_i \equiv_m K\}$ is Σ_3^0 -complete.

Chapter 4

C.e. sets under inclusion

4.1 Outline

We first give a proof of Harrington’s ideal definability lemma explained in Section 2.2, (2.8). Based on this lemma we develop a direct coding with parameters of a standard model of arithmetic and thereby give a new proof of Harrington’s result that true arithmetic can be interpreted in $\text{Th}(\mathcal{E})$.

Recall that \mathcal{E}^* is the lattice of c.e. sets modulo finite differences. Both \mathcal{E} and \mathcal{E}^* are distributive lattices. The coding methods can be used as well to give a uniform coding of finite graphs in \mathcal{E}^* via a Σ_4 -scheme, which proves the undecidability of $\Pi_6 - \text{Th}(\mathcal{E}^*)$. Furthermore they yield elementary differences between relativized versions of \mathcal{E} . A natural question due to E. Herrmann is if, for $0 < p < q$, the relativization of \mathcal{E} to $\emptyset^{(p-1)}$ (i.e. the Σ_p^0 -sets under inclusion) and to $\emptyset^{(q-1)}$ are elementarily equivalent. Evidence for an affirmative answer would come from the fact that constructions of c.e. sets which show that \mathcal{E} possesses certain first-order properties, like the construction of a maximal set in Friedberg [18], relativize and therefore show that for each $Z \subseteq \mathbb{N}$, \mathcal{E}^Z , the lattice of sets c.e. in Z , has the same property. However, we answer the question negatively. Roughly speaking, an elementary difference between the lattice of Σ_p^0 - and the lattice of Σ_q^0 -sets ($0 < p < q$) is obtained by considering the “coding power” in the structure of a scheme of formulas intended to code models of PA^- with an extra unary predicate. This coding power increases with the complexity of the oracle \mathcal{E} is relativized to.

Recall that $\mathcal{L}^*(A)$ is the lattice of c.e. supersets of A modulo finite differences and that A is *quasimaximal* if $\mathcal{L}^*(A)$ is finite or, equivalently, if A is the intersection of finitely many maximal sets. In Soare [56] it is asked if the class

of quasimaximal sets is definable in \mathcal{E} . We answer this question affirmatively. The definability of “quasimaximal” and of further classes of hh-simple sets can be obtained from the ideal definability lemma and certain isomorphism properties of boolean algebras which are coded in \mathcal{E} with parameters.

The lattice \mathcal{E} is set apart from other structures studied in computability theory by the fact that many results restricting coding and definability can be obtained. We show that no infinite linear order can be coded (without parameters) even in the most general way, namely on equivalence classes of n -tuples. Moreover we give an example of a subclass of \mathcal{E} which is nondefinable, but has an arithmetical index set and is invariant under automorphisms.

For any class $\mathcal{C} \subseteq \mathcal{E}$, \mathcal{C}^* will denote the class \mathcal{C}/\equiv^* . We state our results for \mathcal{E} instead of \mathcal{E}^* mostly for notational convenience. For definability and coding concerns, it does not matter whether the setting of \mathcal{E} or of \mathcal{E}^* is used, unless we study fragments of the theory. The reason is that from the methods in Lachlan [28] one can derive that, if $C \subseteq \mathcal{E}^n$ is closed under finite variants, then

$$(4.1) \quad \mathcal{C} \text{ definable in } \mathcal{E} \iff \mathcal{C}^* \text{ definable in } \mathcal{E}^*,$$

via a uniform translation between formulas, and similarly for definability with parameters. Now our coding and definability results do not refer to membership of particular elements. So one can easily transfer all the results from \mathcal{E} to \mathcal{E}^* , e.g. one can prove that $\{A^* : \mathcal{L}^*(A) \text{ finite}\}$ is definable in \mathcal{E}^* or that the Σ_2^0 -sets modulo finite variants are not elementarily equivalent to \mathcal{E}^* .

Intervals play an important role in the study of the lattice \mathcal{E} . Several interesting properties of a c.e. set can be given alternative definitions in terms of the structure of $\mathcal{L}(A)$, the lattice of c.e. supersets of A . For instance, a coinfinite c.e. set A is *hyperhypersimple* iff $\mathcal{L}(A)$ is a boolean algebra, and A is *r-maximal* if and only if $\mathcal{L}(A)$ has no nontrivial complemented elements. Unlike to the case of \mathcal{R}_m , the possible structure of intervals of \mathcal{E} and \mathcal{E}^* is still not very well understood. Lachlan [28] shows that the boolean algebras which can be represented as \mathcal{L}^* , A hh-simple, are precisely the Σ_3^0 -boolean algebras (see Section 5.1 for a definition). The class of r -maximal sets is much more elusive. Cholak and Nies [12] have shown that infinitely many non-isomorphic lattices $\mathcal{L}^*(A)$, A r -maximal, exist.

We now review the notation and terminology used in this chapter. All subsets of \mathbb{N} are c.e. unless otherwise mentioned.

Notation 4.1.1 • Capital letters A, B, C, X, Y range over r.e. sets, letters R, S, T over computable sets.

- $X \sqsubset A \Leftrightarrow (\exists Y)[X \cap Y = \emptyset \ \& \ X \cup Y = A]$,
 $\mathcal{B}(A) = \{X : X \sqsubset A\}$ and
 $\mathcal{R}(A) = \{R : R \sqsubset A\}$.
- An ideal I of $\mathcal{B}(A)$ is k -acceptable if $\mathcal{R}(A) \subseteq I$ and $\{e : W_e \in I\}$ is Σ_k^0 . If we say “ I is acceptable” we mean that I is k -acceptable, where k is a fixed number which depends only on the context in which I is defined (e.g. on formulas in some coding scheme or on arithmetical constructions).
- Given an r.e. set A define a Δ_3^0 -enumeration $(U_e)_{e \in \mathbb{N}}$ of $\mathcal{B}(A)$ as follows: if $e = \langle i, j \rangle$, $W_i \cap W_j = \emptyset$ and $W_i \cup W_j = A$ let $U_e = W_i$ and write \overline{U}_e for W_j . Else let $U_e = \emptyset$ and $\overline{U}_e = A$.

Recall that the major subset relation is defined as follows: for $A, B \in \mathcal{E}$,

$$B \subset_m A \Leftrightarrow B \subset_\infty A \wedge (\forall W \text{ c.e.})[A \cup W = \mathbb{N} \Rightarrow B \cup W =^* \mathbb{N}].$$

A set B is a *small subset* of A , denoted $B \subset_s A$, if $B \subseteq A$ and

$$(4.2) \quad (\forall U, V)[U \cap (A - B) \subseteq^* V \Rightarrow (U - A) \cup V \text{ r.e.}].$$

We will make use of the following well-known facts.

Lemma 4.1.2 (i) If $B \subset_s A$, then each $Y \sqsubset A$ such that $Y \subseteq^* B$ must be computable.

(ii) If $B \subset_m A$, then for each computable $R \subseteq A$, $R \subseteq^* B$.

(iii) If $B \subset_s A$ and $B \subset_m A$ (this is also denoted by $B \subset_{sm} A$) and the set $X \sqsubset A$ is non-computable, then $X - B$ is non-c.e.

Proof.

(i). Let $U = \mathbb{N}, V = A - Y$. Then $U \cap (A - B) =^* A - B \subseteq^* V$, so $\overline{A} \cup V = \overline{Y}$ is c.e.

(ii). Immediate because $A \cup (\mathbb{N} - R) = \mathbb{N}$ and $\mathbb{N} - R$ is c.e.

(iii). If $X - B$ is c.e., then $Y := X \cap B \sqsubset A$, because $A - (X \cap B) = (A - X) \cup (X - B)$. So by (i), $X \cap B$ is computable. Since X is non-computable, $X - B$ is non-computable, so we can choose an infinite computable $R \subseteq X - B$. This contradicts $B \subset_m A$. \diamond

4.2 The ideal definability lemma

Ideal Definability Lemma 4.2.1 *For each $n \geq 1$ the class of $2n + 1$ -acceptable ideals is uniformly definable. More precisely, there is a formula with parameters $\varphi_n(X : D, \overline{C}, A)$ ($|\overline{C}| = n$) with the following property.*

If A is non-computable, for D, \overline{C} ranging over tuples where the correctness condition

$$D \subseteq C_0 \subseteq C_1 \subseteq \dots \subseteq C_{n-1} \subseteq A \ \& \ D \subset_m C_1$$

($D \subset_m A$ in the case that $n = 1$) is satisfied,

$$\{X : \mathcal{E} \models \varphi_n(X; D, \overline{C}, A)\}$$

ranges precisely over the class of $2n + 1$ -acceptable ideals of $\mathcal{B}(A)$.

Proof. The formulas φ_n are defined recursively, by reducing the problem of defining a $2n + 3$ -acceptable ideal to the problem to define a $2n + 1$ -acceptable one.

The Case $n = 1$. Let

$$(4.3) \quad \varphi_1(X; D, C, A) \equiv X \sqsubset A \ \& \ X \cap C \subseteq^* D].$$

Clearly, the index set of any ideal I defined via φ is a Σ_3^0 -ideal of $\mathcal{B}(A)$. Moreover, since $D \subset_m A$ is a , $\mathcal{R}(A) \subseteq I$ by Lemma 4.1.2 (ii). We now prove that, whenever $D \subset_{sm} A$, then each 3-acceptable ideal of $\mathcal{B}(A)$ has the form $\{X : X \cap C \subseteq^* D\}$ for some C . To do so, we will in fact prove a slightly more general fact about intervals $[D, A]$, where $D \subset_m A$, which will be used again in Section 6.1. Consider the set

$$(4.4) \quad \mathcal{B} = \{X \cup D^* : X \sqsubset A\}$$

(we will write $X \cup Y^*$ instead of $(X \cup Y)^*$). By the reduction principle (1.2), \mathcal{B} equals the set of complemented elements in the lattice $[D^*, A^*]$ and therefore is a boolean algebra. Then $(U_e \cup D^*)_{e \in \mathbb{N}}$ is a Δ_3^0 listing of \mathcal{B} , (see Notation 4.1.1 for the sequence (U_e)), and we obtain a notion of index sets of subsets of \mathcal{B} with respect to that listing, and especially of Σ_3^0 subsets of \mathcal{B} . *In the following we will identify subsets of \mathcal{B} with their index sets.*

Lemma 4.2.2 *If $D \subset_m A$ and I is a Σ_3^0 -ideal of \mathcal{B} , then there is $C, D \subseteq^* C \subseteq^* A$ such that*

$$(4.5) \quad I = \{U_j \cup D^* : U_j \cap C \subseteq^* D\}.$$

Proof. First we give an effective representation of the filter of complements of elements of I , using the following uniformization fact.

Fact 4.2.3 *If $(W_{g(i)})_{i \in \mathbb{N}}$ is a sequence of splits of A , $g \leq_T \emptyset''$, then there is a uniformly c.e. sequence of splits (Z_i) of A such that $\forall i \ W_{g(i)} \Delta Z_i \subseteq^* D$.*

To prove this, choose a u.c.e. sequence (V_k) of initial segments of \mathbb{N} such that $W_p = W_{g(i)} \Leftrightarrow \exists n V_{\langle i, p, n \rangle} = \mathbb{N}$ (this is possible since “ $W_p = W_{g(i)}$ ” is Σ_3^0). The desired u.c.e. sequence is

$$Z_i = \{a : \exists s \exists q = \langle i, p, n \rangle \\ \max \bigcup_{\langle i, p', n' \rangle < q} V_{\langle i, p', n' \rangle, s} < a \leq \max V_{q, s} \ \& \ a \in W_{p, s}\}.$$

Given i , let $p = g(i)$ and let $q = \langle p, n \rangle$ be the least such that $V_{\langle i, p, n \rangle} = \mathbb{N}$. Then $Z_i =^* R \cup W_p$, where R is the computable set $\{a : \exists s \ a \in Z_{i, s} \ \& \ a > \max V_{q, s}\}$. Therefore $W_p \Delta Z_i \subseteq^* R \subseteq^* D$. This proves the fact.

Clearly, the indices of c.e. sets which are complements of elements in I ,

$$S = \{i : \exists k \in I \ W_i \cap U_k \subseteq^* D \ \& \ W_i \cup U_k =^* A\}$$

is Σ_3^0 and therefore S is the range of a function $g \leq_T \emptyset''$. Applying the preceding fact we obtain a sequence (\tilde{Z}_i) . Let $Z_n = \bigcap_{i \leq n} \tilde{Z}_i$. Then the u.c.e. sequence $(Z_n \cup D)_{n \in \mathbb{N}}$ generates the filter of complements of elements in I .

To build C , we meet for each n the following requirement:

$$P_n : |W_e \cap Z_n \cap \overline{D}| = \infty \Rightarrow |W_e \cap C \cap \overline{D}| \geq k \ (k = \langle e, n \rangle).$$

The construction of C is the following. Let $C_0 = \emptyset$. At a stage $s + 1$, for each $\langle e, k \rangle = n < s$, act as follows. If P_n is currently unsatisfied, namely $|W_{e, s} \cap C_s \cap \overline{D}_s| < k$, and there is an $x \in Z_{n, s} - D_s$ such that $x \in W_{e, s}$, then enumerate the least such x into C .

We verify that C satisfies (4.5). Notice that at most $k + 1$ elements which are permanently in \overline{D} are enumerated into C for the sake of $P_{\langle e, k \rangle}$. Therefore $C \subseteq^* D \cup Z_m$ for each m . Now, if $j \in I$, then choose an m such that $Z_m - (A - U_j) \subseteq^* D$, i.e. $Z_m \cap U_j \subseteq^* D$. Since $C \subseteq^* D \cup Z_m$, $C \cap U_j \subseteq^* D$.

If $j \notin I$, then $D \cup (A - U_j)^*$ is not in the filter dual to I , so $D \cup (A - U_j)$ does not $*$ -include Z_n for any n . Thus if $W_e = U_j$, the hypothesis of all the requirements $P_{\langle e, k \rangle}$ is satisfied. Hence $C \cap W_e \cap \overline{D}$ is infinite. This proves Lemma 4.2.2. \diamond

Now assume that $D \subset_{sm} A$ and let I be a 3-acceptable ideal of $\mathcal{B}(A)$. To show that I has the form $\{X : X \cap C \subseteq^* D\}$ for some C , consider the ideal $\hat{I} = \{(X \cup D)^* : X \in I\}$ of \mathcal{B} . Then $D \subset_s A$ implies $X \in I \Leftrightarrow D \cup X^* \in \hat{I}$: the direction " \Rightarrow " is immediate, and " \Leftarrow " follows because $D \cup X =^* D \cup Y$ for $Y \in I$ implies that $X \Delta Y \subseteq^* D$. Hence $X \Delta Y$ is computable and $X \in I$. Since \hat{I} is Σ_3^0 , we obtain C such that $\hat{I} = \{X \cup D^* : X \cap D \subseteq^* C\}$. So $I = \{X : X \cap D \subseteq^* C\}$.

The Inductive Step. To complete the proof of the ideal definability lemma, we will show the following: if $m \geq 2$ and I is an $m+3$ -acceptable ideal of $\mathcal{B}(A)$, then there is a non-computable $C \subseteq A$ and an $m+1$ -acceptable ideal J of $\mathcal{B}(C)$ such that, for each $X \sqsubset A$,

$$(4.6) \quad X \in I \Leftrightarrow \exists R \subseteq A \forall S \subseteq A - R [X \cap S \cap R \in J].$$

Then, if $\overline{C} = (C_0, \dots, C_{n-1})$, let

$$(4.7) \quad \begin{aligned} \varphi_{n+1}(X; D, \overline{C}, C_n, A) \equiv & X \sqsubset A \ \& \\ & \exists R \subseteq A \forall S \subseteq A - R \\ & \varphi_n(X \cap S \cap C_n; D, \overline{C}, C_n). \end{aligned}$$

(Recall that the variables R, S range over computable sets. Notice that C_n plays the role of C in (4.6).) For instance,

$$\begin{aligned} \varphi_2(X; D, C_0, C_1, A) \equiv & X \sqsubset A \ \& \\ & \exists R \subseteq A \forall S \subseteq A [X \cap S \cap C_1 \cap D \subseteq^* C_0]. \end{aligned}$$

We first check that this formula only defines $2n+3$ -acceptable ideals. Firstly, if $X \sqsubset A$ is computable, then (4.7) holds via $R = X$. Secondly, the class of X satisfying φ_n is downward closed, and if X, Y satisfy φ_{n+1} via R_X and R_Y respectively, then $X \cup Y$ satisfies φ_{n+1} via $R_X \cup R_Y$, by inductive hypothesis on φ_n . Finally, to see that the ideals defined by φ_{n+1} are Σ_{2n+3}^0 , we write $\varphi_{n+1}(X; D, \overline{C}, C_n, A)$ more explicitly (for the moment, let R, S range over arbitrary c.e. sets):

$$\begin{aligned}
& (\exists R \subseteq A)(\exists \tilde{R})[R \cap \tilde{R} = \emptyset \ \& \ R \cup \tilde{R} = \mathbb{N} \ \& \\
& (\forall S \subseteq A \cap \tilde{R})[S \text{ noncomputable } (\Pi_3^0) \vee \\
& \varphi_n(X \cap S \cap C_n; D, \overline{C}, C_n)(\Sigma_{2n+1}^0)].
\end{aligned}$$

Because $n \geq 1$, this shows that the corresponding index set is Σ_{2n+3}^0 . To prove (4.6), we need some facts. First, we describe an appropriate set C for (4.6). Each noncomputable splitting X of A effectively obtains a *trace* $T \cap C$ in $\mathcal{B}(C)$, where $T \subseteq X$ is computable.

Trace Lemma 4.2.4 *Let A be non-computable. Then there is $C \subseteq A$ such that $(\forall X \sqsubset A \text{ non-computable}) (\exists T \subseteq X \text{ computable})$*

$$T \cap C \text{ non-computable.}$$

A strictly increasing (finite or infinite) computable sequence $b_0 < b_1 < \dots$ such that $T = \{b_0, b_1, \dots\}$ can be obtained effectively in (an index for) X . We write $T = T_X$.

Proof. Let $B \subset_{sm} A$. By an infinitary version of the proof of the Friedberg splitting theorem in Soare [56], obtain a u.c.e. partition (B_k) of B such that

$$(\forall W)(\forall k)[W - B \text{ non-computable} \Rightarrow W - B_k \text{ non-computable}].$$

Let $C = \bigcup\{B_n : n \in K\}$. We claim that C is the desired set. First we show that for each k and each non-computable $X \sqsubset A$, $X \cap B_k$ is infinite. By Lemma 4.1.2 (iii), $X - B$ is non-c.e. So, by (4.2), $X - B_k$ is non-c.e., thus $X \cap B_k$ must be infinite.

Now define $T_X = \{b_0, b_1, \dots\}$, where (b_k) is an effective strictly increasing sequence and $b_k \in X \cap B_k$. To do so, by induction over k , enumerate $X \cap B_k$ until a new element is found. If X is non-computable, then T_X will be an infinite computable subset of X . Moreover, $T_X \cap C \equiv_m K$, so $T_X \cap C$ is non-computable. \diamond

We now give a lemma on how to approximate Σ_3^0 sets. This lemma will be relativized to $\emptyset^{(m)}$ in order to obtain (4.6).

Lemma 4.2.5 *If P is a Σ_3^0 set, then there is a u.c.e. sequence (Z_i) such that $Z_i \subseteq \{0, \dots, i\}$ and*

$$a) (\forall b \in P) (a.e.i) [b \in Z_i]$$

b) $(\exists^\infty i) [Z_i \subseteq P]$

Remark. If $b \notin P$, then $b \notin Z_i$ infinitely many i , so

$$b \in P \Leftrightarrow \text{for almost every } i \ b \in Z_i.$$

Note that the right hand side is in Σ_3^0 -form.

Proof. We first assume that P is a Σ_2^0 and show that there exist a sequence (Y_i) of strong indices for finite sets with the properties required in the lemma (this was first proved by Jockusch). For the general case, we will relativize to \emptyset' .

If P is Σ_2^0 , there is a c.e. set C such that $P = \{(X)_0 : x \in \overline{C}\}$. Suppose $C = \bigcup C_i$, where (C_i) is an effective sequence of strong indices and $C_i \subseteq \{0, \dots, i\}$. Let $d(i) = \min(C_{i+1} - C_i)$ if $C_{i+1} - C_i \neq \emptyset$ and $d(i) = i+1$ else. Note that at most two arguments for the map d can yield the same value. Let Z_i be a strong index for

$$\{c < d(i) : c \notin C_i\}.$$

Then $a \notin C \Rightarrow a \in Z_i$ for almost every i and, if j is a non-deficiency state, i.e.

$$d(j) = \min\{d(i) : i \geq j\},$$

then $Z_j \subseteq \overline{C}$. Now let

$$Y_i = \{(x)_0 : x \in Z_i\}.$$

Then $Y_i \subseteq \{0, \dots, i\}$ (because this holds for Z_i). For (a), if $b \in P$, say $b = (c)_0$ for $c \in \overline{C}$, then $b \in Y_1$ whenever $d_1 > c$, so almost every $i \ b \in Y_i$. For (b), note that $Y_i \subseteq P$ whenever $Z_i \subseteq \overline{C}$.

Now suppose P is Σ_3^0 . By relativization to \emptyset' , there is a Δ_2^0 -sequence of strong indices for finite sets $Y_i \subseteq \{0, \dots, i\}$ such that (a) and (b) hold. By the Limit Lemma [56], there is a computable array of strong indices $(Y_{i,k})$ such that for each i and for almost every k , $Y_{i,k} = Y_i$. Let

$$Y_{\langle i,k \rangle}^* = \{0, \dots, i\} \cap \bigcup_{t \geq k} Y_{i,t},$$

and let f be a 1-1 computable function such that

$$rg(f) = \{\langle i,k \rangle : k = 0 \vee Y_{i,k} \neq Y_{i,k-1}\}.$$

Note that, for each i , there are only finitely many j such that $(f(j))_0 = i$. We claim that $Z_j = Y_{f(j)}^* \cap \{0, \dots, j\}$ is the desired u.c.e. sequence. For (a),

if $b \in P$, then for almost every i , $b \in Y_i$. Since $Y_i = Y_{i,k}^*$ for almost every k , by the above property of f , $b \in Z_j$ for almost every j .

For (b), if i is such that $Y_i \subseteq B$ and s is maximal such that $s = 0$ or $Y_{i,s-1} \neq Y_{i,s}$, then, for j such that $f(j) = \langle i, s \rangle$, $Z_j = Y_i \subseteq B$. So for infinitely many j $Z_j \subseteq B$. \diamond

We are now ready to prove (4.6). Let $P = \{e : U_e \in I\}$ (recall (U_e) is a listing of the splits of A). By applying Lemma 4.2.5 relativized to $\emptyset^{(m)}$, we obtain a sequence of sets (Z_i) which are uniformly Σ_{m+1}^0 such that $Z_i \subseteq \{0, \dots, i\}$ and

$$(a') \quad U_e \in I \Leftrightarrow (a.e. i)(e \in Z_i]$$

$$(b') \quad \exists^\infty i \forall e \in Z_i [U_e \in I].$$

Let $C \subseteq A$ be the set obtained by the Trace Lemma 4.2.4. Moreover let $\mathcal{B}(A)_{\leq i}$ be the boolean algebra generated by $\{U_e : e \leq i\}$ (assume $\mathcal{B}(A)_{\leq 0} = \{\emptyset, A\}$).

Claim 4.2.6 *There is a Δ_3^0 -sequence $(S_i)_{i \in \mathbb{N}}$ of computable subsets of A such that the sets S_i are pairwise disjoint,*

$$\forall R \subseteq A \exists i [R \subseteq S_0 \cup \dots \cup S_i]$$

and

$$\forall i \forall V \in \mathcal{B}(A)_{\leq i} [V \text{ non-computable} \Rightarrow V \cap S_i \cap C \text{ non-computable}].$$

Then we will define J essentially as the ideal on $\mathcal{B}(C)$ generated by the intersections $U_e \cap S_i \cap C$, where $e \in Z_i$. Let (R_i) be a Δ_3^0 listing of $\mathcal{R}(A)$.

Proof of the Claim. Let $S_0 = \emptyset$ and, if $\widehat{S}_i = S_0 \cup \dots \cup S_i$,

$$S_{i+1} = (R_i - \widehat{S}_i) \cup \bigcup \{T_{V-\widehat{S}_i} : V \in \mathcal{B}(A)_{\leq i+1}\}$$

Then $R_i \subseteq S_0 \cup \dots \cup S_{i+1}$. Moreover, if $V \in \mathcal{B}(A)_{\leq i+1}$ is non-computable, then, by the Trace Lemma 4.2.4, $T_{V-\widehat{S}_i} \cap C$ is non-computable (where $T_{V-\widehat{S}_i} \subseteq V$), so, since S_{i+1} computably splits into computable sets $T_{V-\widehat{S}_i}$ and $\widehat{S}_{i+1} - T_{V-\widehat{S}_i}$, $V \cap S_{i+1} \cap C$ must be non-computable.

Let

$$J = \text{the ideal of } \mathcal{B}(C) \text{ generated by } \mathcal{R}(C) \text{ and } \{U_e \cap S_i \cap C : e \in Z_i\}.$$

Since the relation “ $e \in Z_i$ ” is Σ_{m+1}^0 , (S_i) is a Δ_3^0 sequence and $m \geq 2$, J is a Σ_{m+1}^0 -acceptable ideal. It remains to verify (4.6). Suppose $U = U_{\tilde{e}}$.

“ \Rightarrow ” If $U_{\tilde{e}} \in I$, choose i_0 such that $e \in Z_i$ for all $i > i_0$. We claim that $R = S_0 \cup \dots \cup S_{i_0}$ is a witness for the right hand side in 4.6. If $S \subseteq A - R$, then $S \subseteq S_{i_0+1} \cup \dots \cup S_j$ for some $j > i_0$. Now $U_{\tilde{e}} \cap S_i \cap C \in J$ for any $i > i_0$, so, $U_{\tilde{e}} \cap S \cap C \in J$.

“ \Leftarrow ” Suppose $U_{\tilde{e}} \notin I$. Given any $R \subseteq A$, choose k such that $R \subseteq S_0 \cup \dots \cup S_k$. By (b'), there is an $i > k$ such that $Z_i \subseteq \{e : U_e \in I\}$, and also $U_{\tilde{e}} \in \mathcal{B}(A)_{\leq i}$. We show that S_i is a counterexample to the right hand side in 4.6, i.e. $U_{\tilde{e}} \cap S_i \cap C \notin J$. Let $V = U_{\tilde{e}} - \bigcup_{e \in Z_i} U_e$. Since $U_{\tilde{e}} \notin I$, V is a non-computable element of $\mathcal{B}(A)_{\leq i}$. So $V \cap S_i \cap C$ is not computable by the claim above. But, if $U_{\tilde{e}} \cap S_i \cap C \in J$, then, by the disjointness of the sets (S_j) ,

$$U_{\tilde{e}} \cap S_i \cap C \subseteq R \cup \left(\bigcup_{e \in Z_i} U_e \cap S_i \cap C \right)$$

for some computable subset R of C . So $V \cap S_i \cap C$ is computable as a split of C which is contained in a computable subset of C .

This concludes the proof of the ideal definability lemma. \diamond

4.3 Defining classes of hh-simple sets

Recall that A is *hyperhypersimple* (*hh-simple*) if $\mathcal{L}^*(A)$ forms a boolean algebra. In that case, $\mathcal{L}(A) = \{A \cup R : R \text{ computable}\}$. We consider parameterless definability in \mathcal{E}^* of classes of hh-simple sets, based on the ideal definability lemma. For instance the class of quasimaximal sets is definable in \mathcal{E} . Recall that, by (4.1), we can disregard the difference between \mathcal{E} and \mathcal{E}^* . We need two facts.

Fact 4.3.1 *If $\mathcal{L}^*(A)$ is a boolean algebra, then there is a Δ_3^0 -isomorphism $\Phi : \mathcal{L}^*(A) \mapsto \mathcal{B}_{rec}^*(A)$, where $\mathcal{B}_{rec}^*(A) = \{(R \cap A) : R \text{ computable}\}$.*

Proof. Let $\Phi(B^*) = (R \cap A)^*$, where $B = A \cup R$. Note that it takes an oracle \emptyset'' to find R from an input e such that $B = W_e$. \diamond

Observe that $\mathcal{B}_{rec}^*(A)$ is a subalgebra of $\mathcal{B}^*(A)$ containing $\mathcal{R}^*(A)$. Thus we also obtain an isomorphism of the lattice of Σ_k^0 -ideals I of $\mathcal{L}^*(A)$ ($k \geq 3$) onto the lattice of Σ_k^0 -ideals \tilde{I} of $\mathcal{B}_{rec}^*(A)$ which contain $\mathcal{R}^*(A)$. The ideal definability lemma now implies that the Σ_k^0 -ideals of $\mathcal{L}^*(A)$ ($k \geq 3$ odd)

are uniformly definable, because $\tilde{I} = [\tilde{I}]_{\text{id}} \cap \mathcal{B}_{\text{rec}}^*(A)^*$, where $[\tilde{I}]_{\text{id}}$ is the (k -acceptable) ideal of $\mathcal{B}^*(A)$ generated by \tilde{I} .

Fix a hh -simple A as a parameter. We consider definability of ideals of $\mathcal{L}^*(A)$ with parameter A^* in \mathcal{E}^* . The *derivative* of a boolean algebra \mathbf{B} is \mathbf{B}/U , where U is the ideal generated by the atoms of \mathbf{B} . If I is an ideal of $\mathcal{L}^*(A)$, let $\mathcal{A}(I)$ be the ideal of $\mathcal{L}^*(A)$ generated by the atoms of $\mathcal{L}^*(A)/I$ (i.e; $\mathcal{L}^*(A)/\mathcal{A}(I)$ is the derivative of $\mathcal{L}^*(A)/I$).

Fact 4.3.2 *If I is an ideal of $\mathcal{L}^*(A)$ which is definable in \mathcal{E}^* with parameter A^* , then so is $\mathcal{A}(I)$. The formula defining $\mathcal{A}(I)$ only depends on the formula defining I , not on A .*

Proof. If I is in a Σ_k^0 ideal ($k \geq 3$), then $\mathcal{A}(I)$ is Σ_{k+2}^0 . So we can define $\mathcal{A}(I)$ as the least Σ_{k+2}^0 ideal of $\mathcal{L}^*(A)$ which contains all the elements of I and all $B^* \geq A^*$ such that B^*/I is an atom in $\mathcal{L}^*(A)/I$. \diamond

Note that we can also express that $\mathcal{A}(I)$ contains infinitely many atoms of $\mathcal{L}^*(A)/I$: this is the case iff $\mathcal{A}(I)$ describes a nonprincipal ideal in $\mathcal{L}^*(A)/I$, i.e. if there is no $B^* \geq A^*$ such that, for each $C^* \supseteq A^*$, $C^* \in \mathcal{A}(I) \Leftrightarrow (C - B)^* \in I$.

In the following Theorem, (i) for $n = 1$ and $\mathbf{B} = \{0\}$ gives a first-order definition of quasimaximality. In (ii), we refer to Tarski's classification of the completions T of the theory of boolean algebras, in the form presented in Chang and Keisler [11], Section 5.5. They assign invariants $m(\mathbf{B}), n(\mathbf{B}) \in \omega + 1$ to Boolean algebras and prove that two boolean algebras are elementarily equivalent iff they have the same invariants. Thus if T is a completion of the theory of boolean algebras, we can also write $m(T), n(T)$ for $m(\mathbf{B}), n(\mathbf{B})$, where \mathbf{B} is some model of T .

Theorem 4.3.3 *The following classes of hh -simple sets are definable in \mathcal{E}^* without parameters.*

- (i) $\{A^* : \text{the } n\text{-th derivative of } \mathcal{L}^*(A) \text{ is } \mathbf{B}\}$, where \mathbf{B} is a fixed finite boolean algebra or $\mathbf{B} = \{0\}$
- (ii) $\{A^* : \mathcal{L}^*(A) \models T\}$, where T is any completion of the theory of boolean algebra's except the one with the invariants $m(T) = \infty, n(T) = 0$.

Note that (ii) is non-trivial since some completions are not finitely axiomatizable.

Proof. (i). Let I_0^A be the least ideal of $\mathcal{L}^*(A)$, and for each n let $I_{n+1}^A = \mathcal{A}(I_n^A)$. Then, by Fact 4.3.2, there is a formula ψ_n which uniformly for each A^* defines I_n^A in $\mathcal{L}^*(A)$. So we can express that the quotient boolean algebra of $\mathcal{L}^*(A)$ through I_n^A is isomorphic to \mathbf{B} . (ii). is left as an exercise to the reader. \diamond

Corollary 4.3.4 *The following classes of hypersimple sets are definable in \mathcal{E}^* .*

- (i) $\{A^* : A \text{ is quasimaximal}\}$
- (ii) $\{A^* : \mathcal{L}^*(A) \text{ is isomorphic to the boolean algebra of finite or cofinite subsets of } \mathbb{N}\}$

Proof. (i) and (ii) follow from (i) of the preceding theorem with $\mathbf{B} = \{0\}$ and $\mathbf{B} = \{0, 1\}$, respectively. \diamond

4.4 Coding a recursive graph in \mathcal{E}

We will develop a scheme

$$(4.8) \quad \varphi_{dom}(x; \bar{p}), \varphi_{\equiv}(x, y; \bar{p}), \varphi_E(x, y; \bar{p})$$

for coding a recursive directed graph (\mathbb{N}, E) into \mathcal{E} . Applying this to the graph (2.6), we obtain a scheme as in Example 2.1.3. In particular, $\varphi_{num}(x; \bar{p})$ expresses that x is a minimal element in the copy of the partial order $(V_{\mathbb{N}}, E_{\mathbb{N}})$ coded in \mathcal{E} , and the equality formulas φ_{\equiv} are the same.

Let A be any c.e. set such that $\mathcal{L}^*(A)$ is a boolean algebra with infinitely many atoms. Each atom of $\mathcal{L}^*(A)$ has the form $(A \cup H)^*$ for some computable set H . Now let

$$\mathcal{H} = \{H : H \text{ computable} \ \& \ (A \cup H)^* \text{ atom in } \mathcal{L}^*(A)\}.$$

Since the index set of \mathcal{H} is computable in $\emptyset^{(4)}$, there is a $\emptyset^{(4)}$ -sequence (H_i) of computable sets such that $(A \cup H_i)_{i \in \mathbb{N}}^*$ lists the atoms of $\mathcal{L}^*(A)$ without repetitions. *The variable H will range over sets in \mathcal{H} .*

Remark 4.4.1 Lachlan [28] proved that each Σ_3^0 -boolean algebra \mathcal{B} is isomorphic to some $\mathcal{L}^*(A)$ (also see Soare [56]). If \mathcal{B} is the boolean algebra of finite or cofinite subsets of \mathbb{N} , then the set A obtained by his construction has the property that there is in fact a \emptyset'' -list (H_i) as above.

We will introduce all-together six acceptable ideals of $\mathcal{B}(A)$. The parameters needed to define them via the ideal definability lemma will constitute the list of parameters coding the graph (\mathbb{N}, E) . For a set $\mathcal{C} \subseteq \mathcal{B}(A)$, let

$$[\mathcal{C}]_{id} = \text{the ideal of } \mathcal{B}(A) \text{ generated by } \mathcal{R}(A) \cup \mathcal{C}.$$

Hence $[\mathcal{C}]_{id}$ consists of the sets in $\mathcal{B}(A)$ which are contained in a set $R \cup X_1 \cup \dots \cup X_n$, where $R \in \mathcal{R}(A)$ and $X_i \in \mathcal{C}$. Clearly, if \mathcal{C} is Σ_k^0 ($k \geq 3$) then $[\mathcal{C}]_{id}$ is k -acceptable.

We obtain a u.c.e. partition

$$(4.9) \quad (A_k)_{k \in \mathbb{N}}$$

of A by modifying the proof of the Friedberg Splitting Theorem in Soare [56] so that a splitting of A into infinitely many sets is produced. We intend to use A_k / \equiv to represent the vertex k , where \equiv is the equivalence relation defined via φ_{\equiv} . By the argument in [56], for each c.e. W and each k ,

$$W - A \text{ non-c.e.} \Rightarrow W - A_k \text{ non-c.e.}$$

In particular, $H - A_k$ is non-c.e. for each $H \in \mathcal{H}$, and hence $H \cap A_k$ is not computable.

For a better understanding, we first consider a simplified version of the coding scheme, ignoring the necessity of a nontrivial equivalence relation \equiv , at the cost of obtaining coding of the recursive graph only in the structure in \mathcal{E} enriched by an additional unary predicate for $\{A_k : k \in \mathbb{N}\}$. Think of $H_{\langle k, h \rangle}$ as representing the pair A_k, A_h . Given a recursive graph (\mathbb{N}, E) , we define a copy of E on $\{A_k : k \in \mathbb{N}\}$ by using two acceptable ideals \tilde{J}_0 and \tilde{J}_1 . Let

$$(4.10) \quad \begin{aligned} \tilde{J}_0 &= [\{H_{\langle k, h \rangle} \cap A_j : \neg Ekh \vee (Ekh \ \& \ k \neq j)\}]_{id} \\ \tilde{J}_1 &= [\{H_{\langle k, h \rangle} \cap A_j : \neg Ekh \vee (Ekh \ \& \ h \neq j)\}]_{id}. \end{aligned}$$

Thus all the intersections $H_{\langle k, h \rangle} \cap A_j$ go into \tilde{J}_0 [\tilde{J}_1], unless Ekh holds and $k = j$ [$h = j$]. Then one can recover E from \tilde{J}_0, \tilde{J}_1 because

$$(4.11) \quad Eij \Leftrightarrow (\exists H \in \mathcal{H})[H \cap A_i \notin \tilde{J}_0 \ \& \ H \cap A_j \notin \tilde{J}_1].$$

This can be verified using the facts that for $H, N \in \mathcal{H}$, $H \cap A_k \notin \mathcal{R}(A)$ for each k , and either $H \cap A_k, N \cap A_k$ are disjoint on the complement of a set in $\mathcal{R}(A)$ or they are equal.

With an additional unary predicate for $\{A_k : k \in \mathbb{N}\}$, a copy of E on this set can be defined with parameters by (4.11), since \mathcal{H} , J_0 and J_1 are definable with parameters.

We now describe how to obtain a definable equivalence relation \equiv such that $A_k \not\equiv A_h$ for $k \neq h$ and $\{X : \exists k A_k \equiv X\}$ is definable. After this we will come back to the coding of edges. The proof was inspired by Rabin's uniform coding of finite graphs in *boolean pairs* (i.e. boolean algebras with a distinguished subalgebra, see Burris and Sankappanavar [10]). However, we don't make an explicit use of boolean pairs.

We picture the sets A_k as *columns* and the sets $H_i \cap A$ as *rows*. The intersection of a column and a row is a nontrivial splitting of A . Our goal is to find a parameter definable collection of splittings of A including the columns, and to define an equivalence relation such that each split in the collection is equivalent to just one column. We call this collection of splittings the *approximations to columns*.

Observe that, for each noncomputable Q , one can uniformly in an index for Q choose a maximal ideal J of $\mathcal{B}(Q)$ (i.e. $|\mathcal{B}(Q)/J| = 2$) which is 4-acceptable, as follows. Let $(U_e)_{e \in \mathbb{N}}$ be a Δ_3^0 -listing of $\mathcal{B}(Q)$ as described in Notation 4.1.1. One builds an ascending sequence $(X_k)_{k \in \mathbb{N}}$ of elements of $\mathcal{B}(Q)$ which generates J , ensuring that $(\forall e)[U_e \in J \vee \overline{U}_e \in J]$ (to make J maximal) and $(\forall k)[Q - X_k \text{ noncomputable}]$ (to ensure $\mathcal{R}(Q) \subseteq J \neq \mathcal{B}(Q)$). Let $X_0 = \emptyset$. Inductively, for $k > 0$, one has to make a decision, computably in $\emptyset^{(3)}$, if

$$X_n = X_{n-1} \cup U_n \text{ or } X_n = X_{n-1} \cup \overline{U}_n.$$

If one of these sets has a computable complement R in Q , one has to take the other (i.e. R is added to X_{n-1}). If both are non-computable, one can decide either way. This procedure guarantees $\mathcal{R}(Q) \subseteq J$: if \overline{U}_n is computable, then the first set has the computable complement $\overline{X_{n-1}} \cap \overline{U}_n$, so one opts for $\overline{U}_n \in J$. This shows $\mathcal{R}(Q) \subseteq J$, so J is 4-acceptable.

Now choose such a maximal 4-acceptable ideal $J_{i,k}$ for each $Q = H_i \cap A_k$ ($i, k \in \mathbb{N}$) in a uniform way, and let

$$(4.12) \quad I = [\bigcup \{J_{i,k} : i, k \in \mathbb{N}\}]_{id}.$$

Also let

$$(4.13) \quad I_A = [\{A_k : k \in \mathbb{N}\}]_{id}.$$

The approximations to columns will be in I_A . The two ideals are 5-acceptable, and $I \subseteq I_A$. To define a set of approximations to columns we let

$$(4.14) \quad \varphi_{dom}(X) \equiv X \in I_A \ \& \ \forall H X \cap H/I \text{ is atom in } \mathcal{B}(A)/I].$$

This property can be expressed as a first-order property of parameters coding the acceptable ideals via the formulas of the ideal definability lemma. Moreover it is satisfied by each A_k . Now, to express that X, Y approximate the same column, one is tempted to use the formula

$$(4.15) \quad \forall H (X \Delta Y) \cap H \in I.$$

However, some $X \sqsubset A$ could satisfy (4.14) without approximating a column, because it may happen that for two different A_k 's there are infinitely many H_i such that $X \cap H_i = A_k \cap H_i$. Thus, X seems to choose two different columns at the same time. To avoid this, we restrict the set of H considered in (4.15). Let $S \subseteq \mathbb{N}$ be any Σ_5^0 -set which is maximal in the lattice of \mathcal{E}^5 of Σ_5^0 sets (i.e., S^* is a co-atom in $(\mathcal{E}^5)^*$). The existence of S follows by relativizing Friedberg's maximal set construction (see Soare [56]) to $\emptyset^{(4)}$. Let

$$(4.16) \quad I_S = [\{H_i \cap A : i \in S\}]_{id}.$$

Then I_S is a 5-acceptable ideal representing the set of atoms of $\mathcal{L}^*(A)$ which are "in S ". Finally let

$$(4.17) \quad I_H = [\{H_i \cap A : i \in \mathbb{N}\}]_{id}.$$

In Table 4.4 we summarize the definitions of acceptable ideals which are needed for the coding of (\mathbb{N}, E) .

Symbol	Defined in	Function	k -acc. for $k =$
I	4.12	$H_i \cap A_k$ atom in $\mathcal{B}(A)/I$	4
I_A	4.13	auxiliary	3
I_S	4.16	represents S , a maximal set in \mathcal{E}^5	5 [3]
I_H	4.17	auxiliary	5 [3]
J_0, J_1	4.19	Code edge relation on $\{A_k/\equiv : k \in \mathbb{N}\}$	6 [4]

Table 4.4. Numbers [k] are for A as in Remark 4.4.1

Modify (4.15) as follows:

$$(4.18) \quad \varphi_{\equiv}(X, Y; \overline{P}) \equiv \exists U \in I_H \\ \forall H[H \cap A \notin I_S \ \& \ H \cap U \in \mathcal{R}(A) \Rightarrow \\ (X \Delta Y) \cap H \in I].$$

(Recall that the variable H ranges over \mathcal{H} .) This formula expresses that except for on finitely many relevant rows, X and Y behave similarly. It clearly defines an equivalence relation. We claim the following (omitting the list of parameters).

Claim 4.4.2 (i) for each $k \neq h$, $\varphi_{dom}(A_k)$ and $\neg\varphi_{\equiv}(A_k, A_h)$.

$$(ii) \ \forall X[\varphi_{dom}(X) \Rightarrow \exists!k \ \varphi_{\equiv}(X, A_k)].$$

Proof. (i). $\varphi_{dom}(A_k)$ is obvious. If $k \neq h$, given $U \in I_H$ choose an $r \notin S$ such that $H_r \cap A \notin^* V$. Then $H_r \cap A_k$ and $H_r \cap A_h$ represent different atoms in $\mathcal{B}(A)/I$.

(ii). Clearly there can be at most one k such that $\varphi_{\equiv}(X, A_k)$. For the existence of k , since $X \in I_A$, $X \subseteq A_0 \cup \dots \cup A_n \cup R$ for some $n \in \mathbb{N}$, $R \in \mathcal{R}(A)$. So, in $\mathcal{B}(A)/I$ the following relation between atoms holds for each H :

$$(X \cap H)/I \leq \sup_{k \leq n} (A_k \cap H)/I.$$

Thus there is a $k \leq n$ such that $X \cap H_i/I = A_k \cap H_i/I$ for infinitely many i , $H_i \cap A \notin I_S$. The set $\mathcal{D} = \{i : X \cap H_i/I = A_k \cap H_i/I\}$ is Σ_5^0 and $\mathcal{D} \not\subseteq^* S$. Since S is maximal, $\mathcal{D} \cup S =^* \mathbb{N}$. If $U = \bigcup\{H_i \cap A : i \notin \mathcal{D} \cup S\}$, then $U \in I_H$ and, for each H such that $H \cap U \in \mathcal{R}(A)$ and $H \cap A \notin I_S$, $X \cap H/I = A_k \cap H/I$. \diamond

Summarizing, the splittings X satisfying φ_{dom} are those which, for some k on almost all rows $H \notin I_S$ behave like A_k . The finitely many exceptions must be taken into consideration when determining φ_E . We have to represent an edge from A_k/\equiv to A_h/\equiv on infinitely many rows $\notin I_S$. So pick a sequence $N_{\langle k, h, i \rangle}$ without repetitions of elements H_j such that $H_j \cap A \notin I_S$. Such a sequence can be chosen computably in $\emptyset^{(5)}$ because S is Σ_5^0 . Modifying (4.10), let

$$(4.19) \quad J_0 = [I \cup \{N_{\langle k, h, i \rangle} \cap A_j : \neg Ekh \vee (Ekh \ \& \ k \neq j)\}]_{id} \\ J_1 = [I \cup \{N_{\langle k, h, i \rangle} \cap A_j : \neg Ekh \vee (Ekh \ \& \ h \neq j)\}]_{id}.$$

Then J_0, J_1 are 6-acceptable. Let

$$\varphi_E(X, Y; \overline{P}) = \forall U \in I_H \exists H [H \cap A \notin I_S \ \& \ H \cap U \cap A \in \mathcal{R}(A) \ \& \\ H \cap X \notin J_0 \ \& \ H \cap Y \notin J_1].$$

Claim 4.4.3 *If $\varphi_{\equiv}(X, A_k; \overline{P}) \ \& \ \varphi_{\equiv}(Y, A_h; \overline{P})$, then $\varphi_E(X, Y; \overline{P}) \Leftrightarrow Ekh$.*

Proof. Pick $V \in I_H$ such that $\forall H [H \cap A \in I_S \ \& \ H \cap V \in \mathcal{R}(A) \Rightarrow (X \Delta A_k) \cap H \in I \ \& \ (Y \Delta A_h) \cap H \in I]$.

“ \Leftarrow ”. Given $U \in I_H$, pick any $H, H \cap A \notin I_S$ such that $H \cap (U \cup V) \in \mathcal{R}(A)$. Because $(X \Delta A_k) \cap H \in I$ and $(Y \Delta A_h) \cap H \in I$, $H \cap X \notin J_0$ and $H \cap Y \notin J_1$.

“ \Rightarrow ”. If not Ekh , we show $\neg \varphi_E(X, Y; \overline{P})$: for each $H, H \cap A \notin I_S$ such that $H \cap V \in \mathcal{R}(A)$, $H \cap A_k \in J_0$ implies $H \cap X \in J_0$, and $H \cap A_h \in J_1$ implies $H \cap Y \in J_1$. This concludes the construction of the scheme (4.8) and the proof that (\mathbb{N}, E) can be coded via this scheme. \diamond

4.5 Interpreting true arithmetic in $\text{Th}(\mathcal{E})$

Theorem 4.5.1 ([21]) *$\text{Th}(\mathbb{N})$ can be interpreted in $\text{Th}(\mathcal{E})$.*

This theorem was first proved by L. Harrington. A simpler proof appeared in Harrington and Nies [21]. The present proof is a simplification once again because of an improved coding scheme S_M .

Proof. We will apply Fact 2.2.2. First we must develop a scheme S_M as in Example 2.1.3 to satisfy (2.3). Combining the coding of a copy of $(\mathbb{N}, +, \times)$ in a recursive graph $(V_{\mathbb{N}}, E_{\mathbb{N}})$ in (2.6) with the coding of any recursive graph in \mathcal{E} from the preceding Section, we obtain formulas (2.2) with parameters \overline{P} . The list \overline{P} consists of a parameter A (the base set) and parameters to define ideals as in Table 4.4. As before let the variable H range over computable sets such that $A \cup H^*$ is an atom in $\mathcal{L}^*(A)$. Beyond the correctness condition $\alpha_0(\overline{P})$ from Example 2.1.3 we add some more s which enable us to quantify over Σ_k^0 subsets of $M_{\overline{P}}$ (in the sense of Definition 2.2.3), for a sufficiently large k , in order to satisfy (2.4).

First, using Corollary 4.3.4 (ii) we require in a first-order way that $\mathcal{L}^*(A)$ is isomorphic to the boolean algebra of finite or cofinite subsets of \mathbb{N} and that I_H^* is the ideal generated by the set of atoms in $\mathcal{L}^*(A)$. As before let (H_i) be a $\emptyset^{(4)}$ -sequence of computable sets such that $(A \cup H_i)_{i \in \mathbb{N}}^*$ lists the atoms of $\mathcal{L}^*(A)$ without repetitions. Then the collection Σ_5^0 ideals of

$\mathcal{L}^*(A)$ contained in the ideal \tilde{I}_H generated by the atoms under inclusion is canonically isomorphic to \mathcal{E}^5 . Moreover, as described in Section 4.3 we can quantify over this collection of ideals. *In the following we identify those ideals with Σ_5^0 -sets.* Let $\tilde{I}_S = \{(A \cup H)^* : A \cap H \in I_S\}$. As a correctness condition we require that

$$(4.20) \quad \tilde{I}_S \text{ (viewed as a } \Sigma_5^0 \text{- set) is maximal in } \mathcal{E}^5.$$

This completes the description of the scheme S_M .

For $X, Y \in \mathcal{B}(A)$, let

$$\mathcal{D}_{X,Y} = \{i : (H_i \cap X)/I = (H_i \cap Y)/I\}.$$

Since the sequence (H_i) is $\emptyset^{(4)}$ and I is 5-acceptable, $\mathcal{D}_{X,Y}$ is a Σ_5^0 -set. So, under the identification we make,

$$(4.21) \quad \mathcal{D}_{X,Y} \subseteq^* \tilde{I}_S \vee \mathcal{D}_{X,Y} \cup \tilde{I}_S =^* \mathbb{N}.$$

Moreover, $\mathcal{D}_{X,Y} \cup \tilde{I}_S =^* \mathbb{N} \Leftrightarrow \varphi_{\equiv}(X, Y; \overline{P})$.

The following lemma will allow us to quantify over Σ_k^0 -subsets of $M_{\overline{P}}$: if $\mathcal{C} \subseteq M_{\overline{P}}$ and $\widehat{\mathcal{C}} \subseteq \mathcal{B}(A)$ represents \mathcal{C} (in the sense of Definition 2.2.3), then \mathcal{C} can be recovered from $[\widehat{\mathcal{C}}]_{id}$, the ideal generated by $\widehat{\mathcal{C}} \cup \mathcal{R}(A)$.

Lemma 4.5.2 *Suppose that $\widehat{\mathcal{C}} \subseteq \mathcal{B}(A)$ represents $\mathcal{C} \subseteq M_{\overline{P}}$. Then for each $X \in \mathcal{B}(A)$,*

$$\varphi_{num}(X; \overline{P}) \ \& \ X \in [\widehat{\mathcal{C}}]_{id} \Rightarrow X \in \widehat{\mathcal{C}}.$$

Proof. Since $\widehat{\mathcal{C}}$ represents $\mathcal{C} \subseteq M_{\overline{P}}$, $\varphi_{num}(X; \overline{P})$ holds for each $X \in \widehat{\mathcal{C}}$. We now use an argument similar as the one to prove Claim 4.4.2 (ii). Suppose $X \in [\widehat{\mathcal{C}}]_{id}$, then $X \subseteq Y_0 \cup \dots \cup Y_n \cup R$ for some $n, R \in \mathcal{R}(A)$. Then $\forall i \exists k \leq n (X \cap H_i)/I = (Y_k \cap H_i)/I$. If $\mathcal{D}_{X,Y_j} \subseteq^* \tilde{I}_S$ for each $j \leq n$, then $\mathbb{N} = \bigcup_{j \leq n} \mathcal{D}_{X,Y_j} \subseteq^* \tilde{I}_S$, contrary to the correctness condition (4.20). So, by (4.21), there is $j \leq n$ such that $\mathcal{D}_{X,Y_j} \cup \tilde{I}_S =^* \mathbb{N}$, hence $\varphi_{\equiv}(X, Y_j; \overline{P})$. Since $\widehat{\mathcal{C}}$ is closed under \equiv , this implies that $X \in \widehat{\mathcal{C}}$. \diamond

By the ideal definability lemma 4.2 and since $[\mathcal{C}]_{id}$ is k -acceptable for any Σ_k^0 -set $\mathcal{C} \subseteq \mathcal{B}(A)$ if $k \geq 3$, the set of Σ_k^0 -subsets of $M_{\overline{P}}$ is weakly uniformly definable (in fact, uniformly definable if k is sufficiently large). By Fact 2.2.2, this completes the proof. \diamond

4.6 Fragments of $\text{Th}(\mathcal{E}^*)$

We will investigate decidability and undecidability for fragments of $\text{Th}(\mathcal{E}^*)$ as a lattice. Lachlan [27] proved that $\Pi_2\text{-Th}(\mathcal{E}^*)$ is decidable. Here we will use the coding methods developed in Section 4.4 in order to prove that $\Pi_6\text{-Th}(\mathcal{E}^*)$ is undecidable. While seemingly far from optimal, this result improves the bound one obtains from the coding in Harrington, Nies [21] by two quantifier alternations. For an optimal result, one would wish to develop a coding of a sufficiently complex class, like the class of finite undirected graphs (see Theorem 2.3.1), using a Σ_1 -scheme with parameters. By the methods of Section 2.3, this would imply the undecidability of the Π_3 -theory of \mathcal{E}^* . However, such a proof is not possible, since it would show that the class of finite distributive lattices with the reduction property (1.2) (also called *separated* distributive lattices) has an undecidable theory, contrary to a result of Gurevich [20]. The argument is as follows: Suppose that, via some Σ_1 -scheme in the sense of Section 2.3 (or even a scheme (2.1) which consists solely of Σ_1 formulas) we can code each finite undirected graph (V, E) (say), using appropriate parameters \bar{p} . Let G be a finite distributive sublattice of \mathcal{E}^* which contains \bar{p} , all the elements of \mathcal{E}^* representing the vertices in V and also witnesses for all Σ_1 -formulas involved to code (V, E) . Then G , and in fact any distributive lattice D such that $G \subseteq D \subseteq \mathcal{E}^*$ codes (V, E) via the same scheme and parameters. Now by Lachlan [27] let D be such a lattice which is also finite and satisfies the reduction principle. In this way, we have obtained a uniform coding of a complex class in the class of finite distributive lattices with the reduction property.

We conclude that the best we can hope for by the standard coding methods is the undecidability of the Π_4 -theory of \mathcal{E}^* , which still would require a far more direct coding than the one presented here.

Theorem 4.6.1 *The Π_6 -theory of \mathcal{E}^* as a lattice is undecidable.*

Proof. We will show that the class of finite directed graphs (V, E) can be uniformly Σ_4 -coded in \mathcal{E}^* . Then, by Theorem 2.3.1 and (ii) of the Transfer Lemma 2.3.2, $\Pi_6\text{-Th}(\mathcal{E}^*)$ is undecidable. The coding is based on the formulas in Section 4.4, but all the formulas will now be interpreted in \mathcal{E}^* . We use same-type lower case letters to indicate this difference. For instance, the formula (4.3) becomes

$$x \sqsubset a \ \& \ x \wedge c \leq d.$$

We use the abbreviations “ $u = v \times w$ ” for “ $u \wedge v = 0 \ \& \ u \vee v = w$ ” (so $u \sqsubset w \Leftrightarrow \exists v \ u \times v = w$).

The advantage of working in \mathcal{E}^* is that the formula (4.3) is essentially quantifier free, since the Σ_1 -condition “ $x \sqsubset a$ ” can be stated independently of the rest. Moreover, (4.7) is Σ_{2n-2} . For example,

$$\begin{aligned} \varphi_2(x; d, c_0, c_1, a) &\Leftrightarrow x \sqsubset a \ \& \\ &\exists r \leq a \exists \tilde{r} [r \times \tilde{r} = 1 \ \& \ \forall s \leq a \ \forall \tilde{s} (s \times \tilde{s} = 1 \Rightarrow \\ &x \wedge s \wedge c_1 \wedge d \leq c_0)]. \end{aligned}$$

Thus, a $2n + 1$ -acceptable ideal of $\mathcal{B}(A)$ is defined in \mathcal{E}^* by a Σ_{2n-2} -formula with parameters. We will also work with the particular hh-simple set obtained from Lachlan’s construction (see Remark 4.4.1) and fix $D \subseteq_{sm} A$. This is possible since here we are satisfied with a *particular* list of parameters in \mathcal{E}^* and containing A^*, D^* which codes a finite directed graph. Suppose we are given a finite directed graph (V, E) , where $V = \{0, \dots, v\} \subseteq \mathbb{N}$. We follow the definitions in Section 4.4, but with a finite partition $(A_k)_{k \leq v}$ instead of the infinite one used in (4.9). Since we work with the particular set A , we can assume that S is a maximal set in \mathcal{E}^3 , and the sequence $(N_{\langle k, h, i \rangle})$ introduced before (4.19) is computable in $\emptyset^{(3)}$. Then the acceptable ideals in Table 4.4 have the (lower) complexities indicated in cornered brackets. The formula $\varphi_{dom}(X; \overline{P})$, rewritten for \mathcal{E}^* , becomes:

$$\begin{aligned} \varphi_{dom}(x; \overline{p}) &\Leftrightarrow x \in I_A^* \ \& \\ &\forall h \\ &(x \wedge h) / I^* \text{ is atom in } \mathcal{B}^*(a) / I^*. \end{aligned}$$

Now I_A is 3-acceptable, so I_A^* is defined in \mathcal{E}^* by a Σ_1 - formula. Moreover, I is 4-acceptable, so I^* is defined in \mathcal{E}^* by a Σ_2 - formula. Then “ $(x \wedge h) / I^*$ is an atom in $\mathcal{B}^*(a) / I^*$ ” can be expressed by a Π_3 -formula, and $\varphi_{dom}(x; \overline{p})$ is Σ_4 .

Next we look at $\varphi_E(X, Y; \overline{P})$ in \mathcal{E}^* and obtain

$$\begin{aligned} \varphi_E(x, y; \overline{p}) &\Leftrightarrow \forall u \in I_M^* \ \exists h [h \wedge a \notin I_S^* \ \& \ h \wedge u \wedge a \leq d \ \& \\ &h \wedge x \notin J_0^* \ \& \ h \wedge y \notin J_1^*]. \end{aligned}$$

This describes a Π_4 -formula, since J_0, J_1 are 4-acceptable. In order to obtain a Σ_4 -scheme, we use the Σ_4 -formula $\neg \varphi_E(x, y; \overline{p})$ to code the complement $\overline{E} = \{0, \dots, v\}^2 - E$ of the edge relation E on $\{x : \varphi_{dom}(x; \overline{p})\}$. Moreover, from (4.19) for \overline{E} we obtain two further 4-acceptable ideals $\overline{J}_0, \overline{J}_1$ and a Π_4 -formula $\varphi_{\overline{E}}(x, y; \overline{p})$, which is like φ_E but uses the parameters for $\overline{J}_0, \overline{J}_1$.

Now use $\neg\varphi_{\overline{E}}(x, y; \overline{p})$ to code E . This completes the description of the Σ_4 -scheme and thereby the proof. \diamond

4.7 The theories of relativized versions of \mathcal{E}

In this section we investigate and compare the theories of the lattices \mathcal{E}^Z of sets c.e. in an oracle set Z , in particular for sets $Z \subseteq \mathbb{N}$ with the following property: Z is called *implicitly definable* in arithmetic if there is a formula ψ_Z in the language $L(+, \times)$ extended by a unary predicate R such that, for each $X \subseteq \mathbb{N}$,

$$(\mathbb{N}, +, \times) \models \psi_Z(X) \Leftrightarrow X = Z.$$

Note that a set which is implicitly definable in arithmetic is Δ_1^1 , hence hyperarithmetical, and that implicit definability of Z only depends on the arithmetical degree of Z . Hence each Z which is in the same arithmetical degree as some $\emptyset^{(\alpha)}$, α a recursive ordinal, is implicitly definable in arithmetic. However, "most" hyperarithmetical sets are not implicitly definable in arithmetic, since both arithmetically generic sets and arithmetically random sets Z cannot be implicitly definable (this is described in more detail in Nies[43]).

We prove that, if Z is implicitly definable in arithmetic, then $\text{Th}(\mathbb{N}, +, \times, Z)$ can be interpreted in $\text{Th}(\mathcal{E}^Z)$. Since an interpretation in the other direction exists as well, the two theories have the same m -degree. To do so, we exploit the coding power of a specific collection of formulas in \mathcal{E}^Z to show that for some fixed $c \in \mathbb{N}$, if Z is implicitly definable in arithmetic and $Z^{(c)} \neq W^{(c)}$, then \mathcal{E}^Z is not elementarily equivalent to \mathcal{E}^W . (In Shore [53], similar ideas were first applied to relativizations of the structure of Δ_2^0 Turing-degrees.) In particular, if $Z = \emptyset^{(\alpha)}$, $W = \emptyset^{(\beta)}$, where $\beta < \alpha$ are recursive ordinals, then $\mathcal{E}^Z \not\equiv \mathcal{E}^W$. For finite α, β , this negatively answers the question mentioned in the introduction to this chapter whether (Σ_p^0, \subseteq) is elementarily equivalent to (Σ_q^0, \subseteq) for $p < q$. As a further application, if Z is sufficiently complex, namely $Z \notin \text{Low}_c$ (c as above), then \mathcal{E}^Z is not elementarily equivalent to \mathcal{E} . This includes the case that Z is arithmetically generic. We note that, for all arithmetically generic Z , the relativization \mathcal{E}^Z has the same theory. Similar remarks apply to arithmetically random sets (Nies [43]).

We make some observations which will enable us to interpret true arithmetic in $\text{Th}(\mathcal{E}^Z)$ for each Z and $\text{Th}(\mathbb{N}, +, \times, Z)$ in $\text{Th}(\mathcal{E}^Z)$ if Z is implicitly definable in arithmetic. In \mathcal{E}^Z , define $\mathcal{B}(A)$ as in 4.1.1 and let $\mathcal{R}(A) \subseteq \mathcal{B}(A)$ be the collection of subsets of A which are computable in Z . An ideal I of

$\mathcal{B}(A)$ is k -acceptable (relative to Z) if $\mathcal{R}(A) \subseteq I$ and I has a Σ_k^Z index set. The proof of the ideal definability lemma relativizes to Z , so in \mathcal{E}^Z the class of k -acceptable ideals is uniformly definable for all odd $k \geq 3$. Thus the scheme S_M also works in \mathcal{E}^Z , and, relativizing the considerations in Section 4.5, we obtain:

Corollary 4.7.1 *Th(N) can be interpreted in Th(\mathcal{E}^Z) for each Z . \diamond*

Moreover, we observe

Fact 4.7.2 (i) *If $\varphi(\overline{X}; \overline{P})$ is a Σ_k^0 formula with parameters in the language of \mathcal{E} , then for each Z , the index set with respect to the indexing of \mathcal{E}^Z , $(W_e^Z)_{e \in \mathbb{N}}$, of the relation defined by φ with a fixed parameter list is computable in $Z^{(k+2)}$.*

(ii) *For some fixed number h (which does not depend on Z), for each M , there is $g \leq_T Z^{(h)}$ such that*

$$(\forall i)[W_{g(i)}^Z / \equiv = i^M].$$

Proof. (i) is immediate since “ $W_i^Z \subseteq W_j^Z$ ” is computable in $Z^{(2)}$. For (ii), suppose that $M = M(\overline{P})$. Let $\varphi_S(X, Y; \overline{P})$ be a formula defining the successor function in (any) $M(\overline{P})$. By (i), the corresponding binary relation on indices is computable in $Z^{(h)}$ for some fixed number h , so there is a partial “choice” map f which can be computed with the oracle $Z^{(h)}$ such that, in \mathcal{E}^Z ,

$$\varphi_S(W_i^Z, W_j^Z; \overline{P}) \text{ for some } j \Rightarrow \varphi_S(W_i^Z, W_{f(i)}^Z; \overline{P}).$$

Fix i_0 such that $W_{i_0}^Z / I = 0^M$. Then, by iterating f with i_0 as an initial value, obtain g as desired. \diamond

From (ii) one immediately obtains the relativization of Fact 2.2.1: for each structure M coded in \mathcal{E}^Z via S_M , $\{e : W_e^Z / I \text{ is a standard number of } M\}$ is $\Sigma_p^0(Z)$ for some fixed p .

Theorem 4.7.3 *If Z is implicitly definable in arithmetic, then there are interpretations of theories which show $\text{Th}(\mathcal{E}^Z) \equiv_m \text{Th}(\mathbb{N}, +, \times, Z)$.*

Proof. Suppose that Z is implicitly definable in arithmetic. To interpret $\text{Th}(\mathbb{N}, +, \times, Z)$ in $\text{Th}(\mathcal{E}^Z)$ we need an extended scheme which enables us to

encode structures (M, Z_M) , where $M = M_{\overline{P}}$ is a coded copy of \mathbb{N} and Z_M is Z , viewed as a subset of M . Let ψ_Z be a formula describing Z as in (4.7). Given M as above, suppose that $\widehat{Z}_M \subseteq \mathcal{B}(A)$ represents Z and let $I_Z = [\widehat{Z}_M]_{id}$ (the ideal generated by \widehat{Z}_M and $\mathcal{R}(A)$). Then, using the map g from Fact 4.7.2 (ii),

$$I_Z = [\{W_{g(n)}^Z : n \in Z\}^{\equiv}]_{id}.$$

Since $g \leq_T Z^{(h)}$ for some h , I_Z is q -acceptable (in \mathcal{E}^Z) for some q . Suppose that M is standard. Since Lemma 4.5.2 also holds in \mathcal{E}^Z , $\varphi_{num}(P; \overline{P})$ implies that

$$P/\equiv \in Z_M \Leftrightarrow P \in I_{\widehat{Z}_M}.$$

In the extended scheme, expand the list of parameters by parameters defining a q -acceptable ideal J of $\mathcal{B}(A)$. As an additional for the scheme we require that

$$[\varphi_{num}(X; \overline{P}) \ \& \ \varphi_{num}(Y; \overline{P}) \ \& \ X \in J \ \& \ X \equiv Y] \Rightarrow Y \in J.$$

Let W be the subset of M represented by $J \cap \{X : \varphi_{num}(X; \overline{P})\}$ (the intended meaning is that $W = \widehat{Z}_M$).

The interpretation of $\text{Th}(\mathbb{N}, +, \times, Z)$ in $\text{Th}(\mathcal{E}^Z)$ is now given by $(\mathbb{N}, +, \times, Z) \models \varphi \Leftrightarrow$

for some (M, W) coded via the extended scheme, M is standard, W (as a subset of M) is represented by $J \cap \{X : \varphi_{num}(X; \overline{P})\}$, $M_{\overline{P}} \models \psi_Z(W)$ and $(M, W) \models \varphi$.

The right-hand side can be expressed by a first-order sentence effectively obtained from φ . \diamond

Let $T \subseteq \mathbb{N}$. Given an M coded in \mathcal{E}^T , let g be the function from Fact 4.7.2 (ii). For $Q \subseteq \mathbb{N}$, let $J_Q = [\{W_{g(k)}^T : k \in Q\}^{\equiv}]_{id}$. The following is a key technical fact.

Lemma 4.7.4 *For a sufficiently big number p and any M coded in \mathcal{E}^T , the following holds: if $Q \subseteq \mathbb{N}$ then*

$$Q \text{ is } \Sigma_p^0(T) \Leftrightarrow J_Q \text{ is } p\text{-acceptable.}$$

Proof. Let $p \in \mathbb{N}$ be such that all ideals needed for the coding of $(V_{\mathbb{N}}, E_{\mathbb{N}})$ in \mathcal{E}^T are $p-1$ -acceptable, the function g in Fact 4.7.2 is computable in $\emptyset^{(p-1)}$ and $X \equiv Y$ is recursive in $T^{(p-1)}$ as a relation between indices.

For the direction “ \Rightarrow ”, note that

$$\begin{aligned} W_e^T \in J_Q &\Leftrightarrow \exists r \exists k_0, \dots, k_r \exists X_0, \dots, X_r \\ &W_e^T \subseteq \bigcup_{i=0, \dots, r} X_i \ \& \ \forall i \leq r (k_i \in Q \ \& \ W_{g(k_i)}^T \equiv X_i). \end{aligned}$$

It is easy to check that this can be expressed as a $\Sigma_p^0(T)$ property of e . For the direction “ \Leftarrow ”, if J_Q has a $\Sigma_p^0(X)$ index set, then, because

$$\begin{aligned} n \in Q &\Leftrightarrow W_{g(n)}^T \in J_Q \\ &\Leftrightarrow (\exists e)[g(n) = e \ \& \ W_e^T \in J_Q] \end{aligned}$$

and $g \leq_T \emptyset^{(p-1)}$, Q is $\Sigma_p^0(T)$ (this uses Lemma 4.5.2). \diamond

Now assume in addition that p is odd, and let $c = p-1$.

Theorem 4.7.5 *If $Z^{(c)} \not\equiv_T W^{(c)}$ and Z or W is implicitly definable in arithmetic, then $\mathcal{E}^Z \not\equiv \mathcal{E}^W$.*

Corollary 4.7.6 *If α is a recursive ordinal and $\beta < \alpha$, then $\mathcal{E}^{\emptyset^{(\alpha)}} \not\equiv \mathcal{E}^{\emptyset^{(\beta)}}$.* \diamond

Proof of the theorem. Assume that $Z^{(c)} \not\equiv_T W^{(c)}$. Then, if $Z^{(p)} \in \Sigma_p^0(Z) - \Sigma_p^0(W)$. Let $\varphi(Y; D, \overline{C}, A)$ be the formula obtained from the ideal definability lemma to define uniformly in \mathcal{E}^T for a set A which is c.e., but not computable in T all p -acceptable ideals of $\mathcal{B}(A)$.

First suppose that Z is implicitly definable in arithmetic, via the description ψ_Z . Then the following is true in \mathcal{E}^Z , but not in \mathcal{E}^W .

There is a structure (M, Y) coded by the extended scheme such that M is standard, $(M, Y) \models \psi_Z(Y)$ and, for some list D, \overline{C} , the “intersection” of M and the ideal coded by D, \overline{C}, A equals $Y^{(p)}$, i.e.

$$(4.22) \quad \forall P[\varphi_{num}(P; \overline{P}) \Rightarrow ((M, Y) \models P / \equiv \in Y^{(p)} \Leftrightarrow \varphi(P; D, \overline{C}, A))].$$

The statement holds in \mathcal{E}^Z via any standard M and $Y = Z_M$ (i.e. Z viewed as a subset of M), for in this case $J_{Z^{(p)}}$ is p -acceptable by Lemma 4.7.4. In \mathcal{E}^W , either does $\psi_Z(Y)$ hold in no structure (M, Y) , M standard, defined by the extended scheme, or, if (M, Y) is such a structure, then (4.22) fails. For, in \mathcal{E}^W , $\{P \in \mathcal{B}(A) : \varphi(P, D, \overline{C}, A)\}$ is an ideal with $\Sigma_p^0(W)$ index set by the easy direction of the ideal definability lemma relativized to W . So, if (4.22) holds, by 4.7.4, $Z^{(p)} \in \Sigma_p^0(W)$, a contradiction to $Z^{(c)} \not\leq_T W^{(c)}$.

Now suppose that W is implicitly definable via ψ_W . The case that $W^{(c)} \not\leq_T Z^{(c)}$ has already been covered above. Otherwise there is an index e such that $\{e\}^{Z^{(c)}} = W$. Then the following is true in \mathcal{E}^Z , but not in \mathcal{E}^W .

There is a coded standard model M and a list D, \overline{C} coding a p -acceptable ideal of K of $\mathcal{B}(A)$ such that

$$U_0 = K \cap \{X : \varphi_{num}(X; \overline{P})\}$$

is closed under \equiv and if $U = U_0/\equiv$, then for some index $e \in M$,

$$M \models \psi_W(\{e\}^U) \ \& \ U \notin \Sigma_p^0(\{e\}^U).$$

This statement holds in \mathcal{E}^Z via the ideal $K = J_{Z^{(p)}}$, but fails in \mathcal{E}^W , once again by the easy direction of the ideal definability lemma. \diamond

4.8 Non-coding and Non-definability Theorems

In the last section of this chapter we investigate the limits of definability and coding in \mathcal{E} . We show that no infinite linear order can be coded (without parameters) even in the most general way, namely on equivalence classes of n -tuples. The proof makes use of the fact that for each partition of \mathbb{N} into three infinite computable sets R, S, T there is a canonical isomorphism $\mathcal{E} \rightarrow \mathcal{E}^3$ given by $X \rightarrow (X \cap R, X \cap S, X \cap T)$, combined with a model theoretic result due to Feferman and Vaught [16] that a first-order property of a tuple in a model of the form \mathbf{A}^n can be expressed as a certain boolean combination of first-order properties of the components. First we prove a noncoding theorem in the context of uniform first-order definability with parameters, which can be considered as a weak version of the model-theoretic notion of stability for \mathcal{E} : there is no uniform way to define, even with parameters, a linear order on arbitrarily large classes $\{R_1, \dots, R_k\}$ of pairwise disjoint

computable sets. This implies that infinite no linear order can be coded in a first-order way on atoms of $\mathcal{L}^*(A)$, if $\mathcal{L}^*(A)$ is a boolean algebra with infinitely many atoms.

Hodges and Nies [26] have shown that in fact no infinite linear order can be coded without parameters in any structure isomorphic to a structure $\mathbf{A} \times \mathbf{A}$ (as \mathcal{E} is one). However, the proof given here for \mathcal{E} contains interesting insights into further self-similarity properties of \mathcal{E} and also puts an effective upper bound on the cardinality of a linear order which can be coded by a given formula.

If \mathbf{A} can be coded in $(\mathbb{N}, +, \times)$, then each relation on \mathbf{A} which is definable without parameters must be invariant under automorphisms and has an arithmetical index relation. The question arises if a *maximum definability property* holds, namely if these two properties actually characterize the definable relations. The question has been answered affirmatively for the structure of Δ_2^0 T -degrees by Slaman and Woodin [54]. In Harrington and Nies [21] it was proved that the maximum definability property fails for \mathcal{E}^* (and hence for \mathcal{E}) by giving a binary relation as a counterexample. The counterexample provided here is in fact a subclass of the class of quasimaximal sets.

Let the variable \mathcal{R} range over finite classes of pairwise disjoint infinite computable sets. We use the variable \tilde{X} for tuples of c.e. sets (X_0, \dots, X_{n-1}) .

Theorem 4.8.1 *For each formula $\varphi(X, Y; \tilde{P})$ one can find in an effective way a number k such that for each \mathcal{R} , $|\mathcal{R}| \geq k$, and for each list of parameters \tilde{A} , the relation*

$$\{(X, Y) : X, Y \in \mathcal{R} \ \& \ \mathcal{E} \models \varphi(X, Y; \tilde{A})\}$$

is not a linear ordering of \mathcal{R} .

Corollary 4.8.2 *If $\mathcal{L}^*(A)$ is a boolean algebra with infinitely many atoms, then it is not possible to code, even with parameters, an infinite linear ordering on atoms of $\mathcal{L}^*(A)$.*

Proof of the Corollary. If F is a set of atoms and $|F| = k$, then for some \mathcal{R} such that $|\mathcal{R}| = k$, $F = \{A \cup R^* : R \in \mathcal{R}\}$. Hence, if $\psi(X, Y; \tilde{P})$ defines a linear order on the atoms, then $\varphi(X, Y; \tilde{P}, A) \equiv \psi(X \cup A, Y \cup A; \tilde{P})$ defines a linear order on sets \mathcal{R} of arbitrarily large cardinality. \diamond

Proof of the Theorem. Note that, if R, S and $T = \overline{R \cup S}$ are infinite, then $\mathcal{E} \cong \mathcal{E}^3$ via the map

$$X \mapsto (X \cap R, X \cap S, X \cap T).$$

By a result of Feferman and Vaught [16], if \mathbf{A} is a structure $k \geq 0$ and $\varphi(X^0, \dots, X^{n-1})$ is a formula in the language of \mathbf{A} , then

$$(4.23) \quad \mathbf{A}^{k+1} \models \varphi \left(\begin{array}{ccc} a_0^0 & & a_0^{n-1} \\ a_1^0 & \dots & a_1^{n-1} \\ & \dots & \\ a_k^0 & & a_k^{n-1} \end{array} \right) \Leftrightarrow \bigvee_{\alpha=1, \dots, r} \bigwedge_{i=0, \dots, k} \mathbf{A} \models \varphi_i^\alpha(a_i^0, \dots, a_i^{n-1})$$

for some formulas φ_i^α which only depend on φ and can be found effectively. Thus, whether $\varphi(\bar{a}^0, \dots, \bar{a}^{n-1})$ holds in \mathbf{A}^{k+1} only depends on finitely many effectively determined first-order properties of the components $a_i^0, \dots, a_i^{n-1} \in \mathbf{A}$ ($i \leq k$). This can be proved by induction on $|\varphi|$.

Now suppose that $\varphi(X, Y; \tilde{P})$ defines a linear order $<_L$ on a set \mathcal{R} . By the isomorphisms $\mathcal{E} \cong \mathcal{E}^3$ above, an element $A \in \mathcal{E}$ corresponds to the vector

$$\begin{pmatrix} A \cap R \\ A \cap S \\ A \cap T \end{pmatrix}.$$

Hence, if $R, S \in \mathcal{R}$, $R \neq S$, then

$$\begin{aligned} R <_L S &\Leftrightarrow \bigvee_{\alpha=1, \dots, r} (\mathcal{E}(R) \models \varphi_0^\alpha(R, \emptyset, \tilde{P} \cap R) \\ &\quad \& \mathcal{E}(S) \models \varphi_1^\alpha(\emptyset, S, \tilde{P} \cap S) \\ &\quad \& \mathcal{E}(T) \models \varphi_2^\alpha(\emptyset, \emptyset, \tilde{P} \cap T)), \end{aligned}$$

where $T = \overline{R \cup S}$ and $\tilde{P} \cap X = (P_0 \cap X, \dots, P_{k-1} \cap X)$. Note that “ $\mathcal{E}(T) \models \varphi_2^\alpha(\emptyset, \emptyset, \tilde{P} \cap T)$ ” does not depend on the order of R, S . We say that $R <_L S$ via α if the disjunct corresponding to α holds. Now, we can compute a number M such that, for $|\mathcal{R}| \geq M$, there exist α and $A, B, C, D \in \mathcal{R}$ such that $A <_L B <_L C <_L D$ and the ordering relations hold all via α . This is verified by using Ramsey’s Theorem: assign one of r possible colors to $\{X, Y\} \subseteq \mathcal{R}$, $X \neq Y$, according to the minimum $\alpha \leq r$ such that $X <_L Y$ or $Y <_L X$ holds via α . For $k = |\mathcal{R}|$ large enough, there exists a homogeneous

set F for this coloring of cardinality 4. Since either $X <_L Y$ or $Y <_L X$ for each $X, Y \in \mathcal{R}, X \neq Y$, there must be α such that, for $X, Y \in F$,

$$X <_L Y \Leftrightarrow \bar{X} <_L \bar{Y} \quad \text{via } \alpha.$$

Now let $F = \{A, B, C, D\}$, $A <_L B <_L C <_L D$. We show $C <_L B$, a contradiction. $\mathcal{E}(C) \models \varphi_0^\alpha(C, \emptyset, \tilde{P} \cap C)$ holds since $C <_L D$ via α , and $\mathcal{E}(B) \models \varphi_1^\alpha(\emptyset, B, \tilde{P} \cap B)$ because $A <_L B$ via α . Finally $\mathcal{E}(\overline{B \cup C}) \models \varphi_2^\alpha(\emptyset, \emptyset, \tilde{P} \cap (\overline{B \cup C}))$ is true since $B <_L C$. This shows $C <_L B$ via α . \diamond

Theorem 4.8.3 *It is not possible to code an infinite linear ordering in \mathcal{E} without parameters.*

Proof. We write $\tilde{X} <_L \tilde{Y}$ for $\tilde{X} \leq_L \tilde{Y}$ & $\tilde{Y} \not\leq_L \tilde{X}$. Suppose for a contradiction that there is an \mathcal{E} -definable $2n$ -ary relation \leq_L which is a linear pre-ordering on \mathcal{E}^n such that the equivalence relation $\tilde{X} \equiv_L \tilde{Y} \Leftrightarrow \tilde{X} \leq_L \tilde{Y} \leq_L \tilde{X}$ has infinitely many equivalence classes. We say that a computable set R *supports* A if $A \subseteq R$ or $\overline{R} \subseteq A$. R *supports* (A_0, \dots, A_{n-1}) if R supports each set A_i . Let

$$\mathcal{C} = \{R : |R| = |\overline{R}| = \infty\}.$$

Lemma 4.8.4 *For each tuple $\tilde{A} = (A_0, \dots, A_{n-1})$ of sets there exists $R \in \mathcal{C}$ such that R supports \tilde{A} .*

Proof. We say that S *co-supports* A if \overline{S} supports A , i.e. $S \subseteq A$ or $A \subseteq \overline{S}$. This notion is closed downwards in S . We define inductively sets $S_k \in \mathcal{C}$ co-supporting A_0, \dots, A_k . Then $R = \overline{S_{n-1}}$ is as required.

Let S_0 be a set in \mathcal{C} which is a subset of A_0 if A_0 is infinite and of $\overline{A_0}$ else. If $k < n - 1$ and $S_k \cap A_{k+1}$ is infinite let $S_{k+1} \in \mathcal{C}$ be a computable subset of $S_k \cap A_{k+1}$. Else let $S_{k+1} = S_k - A_{k+1}$. \diamond

We now derive an effective bound on $|\mathcal{E}^n / \equiv_L|$ (depending on the defining formula for \leq_L). First we show that each equivalence class of \equiv_L is large in the following sense: for each $\tilde{A} \in \mathcal{E}^n$,

$$(4.24) \quad (\forall S \in \mathcal{C})(\exists \tilde{B} \equiv_L \tilde{A})[S \text{ supports } \tilde{B}]$$

Fix $R \in \mathcal{C}$ supporting \tilde{A} , and let $S \in \mathcal{C}$ be arbitrary. First suppose that $R \cap S = \emptyset$, and let π be a computable permutation of order 2 which exchanges R and S and is the identity on $\overline{R \cup S}$. Let $B_i = \pi(A_i)$ ($i < n$). Then S

supports \tilde{B} . Now $\tilde{A} \leq_L \tilde{B}$ is equivalent to $\tilde{B} = \pi(\tilde{A}) \leq_L \pi(\tilde{B}) = \tilde{A}$, since \leq_L is definable. So $\tilde{A} \equiv_L \tilde{B}$.

If $R \cap S$ is finite, proceed as above, replacing S by $S - R$. Then \tilde{B} is supported by $S - R$ and hence by S . If $R \cap S$ is infinite, obtain first $\tilde{B}_0 \equiv_L \tilde{A}$ supported by \overline{R} and then $\tilde{B} \equiv_L \tilde{B}_0$ supported by $R \cap S$. Then $\tilde{B} \equiv_L \tilde{A}$ and \tilde{B} is supported by S .

Suppose $|\mathcal{E}^n / \equiv_L| \geq p$. We derive a bound on p . By (4.24), let $S_0, \dots, S_{p-1} \in \mathcal{C}$ be pairwise disjoint sets and let \tilde{B}^i , $i < p$, be n -tuples of sets supported by S_i such that

$$\tilde{B}^0 <_L \dots <_L \tilde{B}^{p-1}.$$

If a tuple $\tilde{X} = (X_0, \dots, X_{n-1})$ is supported by S , we assign a signature $\beta \in \{0, 1\}^n$ to (\tilde{X}, S) by $\beta(k) = 0 \Leftrightarrow X_k \subseteq S$ ($k < n$). Fix an arbitrary number q . If $p \geq 2^n q$, then there is a subsequence $(\tilde{A}^j, R_j)_{j < q}$ of $(\tilde{B}^i, S_i)_{i < p}$ such that all (\tilde{A}^i, R_i) have the same signature β . Let

$$A_k = \bigcup_{j < q} (A_k^j \cap R_j) \quad (k < n).$$

We show that the parameters A_0, \dots, A_{n-1} can be used to define in a first-order way a linear order on $\{R_0, \dots, R_{q-1}\}$. Clearly one can decode each \tilde{A}^j in a uniform first-order way from R_j and this list of parameters, because $A_k^j = A_k \cap R_j$ if $\beta(k) = 0$ and $A_k^j = (A_k \cap R_j) \cup \overline{R_j}$ if $\beta(k) = 1$. Thus for the formula $\psi(R, S; A_0, \dots, A_{n-1})$ expressing $\tilde{C} <_L \tilde{D}$, where C_k is $A_k \cap R$ if $\beta(k) = 0$ and $(A_k \cap R) \cup \overline{R}$ else, and D_k is $A_k \cap S$ if $\beta(k) = 0$ and $(A_k \cap S) \cup \overline{S}$ else,

$$\psi(R_i, R_j; A_0, \dots, A_{n-1}) \Leftrightarrow \tilde{A}^i <_L \tilde{A}^j,$$

so ψ defines a linear order on $\{R_0, \dots, R_{q-1}\}$ with the parameters A_0, \dots, A_{n-1} . By Theorem 4.8.1, this gives an effective bound on q depending on ψ (where ψ was obtained in an effective way from φ and β , but did not depend on q). Hence $|\mathcal{E}^n / \equiv_L|$ cannot exceed 2^n times this bound. Since we can take the maximum over all possible β , we effectively obtain a bound which only depends on φ . \diamond

For quasimaximal A , let

$$n(A) = \text{number of atoms in } \mathcal{L}^*(A).$$

Corollary 4.8.5 *The following relation, which is arithmetical and invariant under automorphisms, is not definable in \mathcal{E}^* :*

$$\{\langle A^*, B^* \rangle : n(A) \leq n(B)\}.$$

Proof. Definability would enable us to code without parameters (\mathbb{N}, \leq) on equivalence classes. \diamond

Let $A \approx B$ denote that A is automorphic to B in \mathcal{E} . Soare [55] proves that, for quasimaximal A, B , $A \approx B \Leftrightarrow n(A) = n(B)$.

Therefore, $n(A) \leq n(B) \Leftrightarrow (\exists B')[B \subseteq B' \wedge B' \approx A]$. In fact the automorphism obtained in [55] can be represented by a Σ_3^0 map on indices.

Corollary 4.8.6 *The following relations (which are invariant under $=^*$) are non-definable in \mathcal{E} :*

(i) $A \approx B$

(ii) $A \approx B$ via a Σ_3^0 automorphism.

Proof. Definability of either one of the relations, together with (i) of Corollary 4.3.4, would imply the definability of

$$\{(A, B) : A, B \text{ quasimaximal} \ \& \ n(A) \leq n(B)\},$$

so (\mathbb{N}, \leq) could be coded in \mathcal{E} without parameters. \diamond

We conclude this Section with an example of a *unary* relation on \mathcal{E}^* which is arithmetical and invariant under automorphisms: the class

$$\{A^* : n(A) \geq 2 \ \& \ n(A) \text{ is a power of } 2\}$$

is not definable.

Theorem 4.8.7 *Let $X \subseteq \mathbb{N}$ be an infinite set of even numbers such that for each distinct $n, m \in X$, $(n + m)/2$ is not in X (for example let $X = \{n \geq 2 : n \text{ is a power of } 2\}$). Then $\{A^* : n(A) \in X\}$ is not definable in \mathcal{E}^* .*

Notice that for each X , $\{A^* : n(A) \in X\}$ is invariant under automorphisms of \mathcal{E}^* . Moreover, if X is arithmetical, then this class has an arithmetical index set.

Proof. Let $P = \{A : n(A) \in X\}$. Since P is closed under finite differences, by a result of Lachlan [28] described in (4.1), it suffices to prove nondefinability of P in \mathcal{E} (however, one could also perform some notational changes

below to give a direct proof for \mathcal{E}^*). If A is quasimaximal and R is an infinite coinfinite computable set, then $A \cap R$ is quasimaximal in $\mathcal{E}(R) = [\emptyset, R]_{\mathcal{E}}$. Let $n_R(A)$ denote $n(A \cap R)$ (evaluated in $\mathcal{E}(R)$). If B^* is an atom above A^* in $\mathcal{L}^*(A)$, then either $B \subseteq^* A \cup R$, in which case $(B \cap R)^*$ is an atom above $(A \cap R)^*$, or $B \subseteq^* A \cup \overline{R}$, in which case $(B \cap \overline{R})^*$ is an atom above $(A \cap \overline{R})^*$. Conversely, each atom above $(A \cap R)^*$ gives rise to one above A^* , and similarly for atoms above $(A \cap \overline{R})^*$. Thus

$$n(A) = n_R(A) + n_{\overline{R}}(A).$$

We use the result of Feferman and Vaught (4.23) for $k = 1$. If R is an infinite coinfinite computable set, then $\mathcal{E} \cong \mathcal{E} \times \mathcal{E}$ via the map

$$X \mapsto (X \cap R, X \cap \overline{R}).$$

Thus, if P is definable in \mathcal{E} by a formula $\varphi(x)$, then

(4.25)

$$\mathcal{E} \models \varphi(X) \Leftrightarrow \bigvee_{\alpha=1, \dots, r} (\mathcal{E}(R) \models \varphi_0^\alpha(X \cap R) \ \& \ \mathcal{E}(\overline{R}) \models \varphi_1^\alpha(X \cap \overline{R})).$$

For each $C \in P$, choose some computable set R_C such that $n(R_C) = n(C)/2$. By the pigeonhole principle, there are sets $A, B \in P$, $n(A) \neq n(B)$ so that (4.25) holds via the same α , if R is $R(A)$ ($R(B)$, respectively). After applying an appropriate computable permutation, we can assume that $R = R_A = R_B$. Let $D = (A \cap R) \cup (B \cap \overline{R})$. Then $\mathcal{E} \models \varphi(D)$, because

$$\mathcal{E}(R) \models \varphi_0^\alpha(D \cap R) \ \& \ \mathcal{E}(\overline{R}) \models \varphi_1^\alpha(D \cap \overline{R}).$$

But $n(D) = (n(A) + n(B))/2 \notin X$, contradiction. \diamond

Chapter 5

Ideal lattices

We prove that if \mathcal{B} is an effectively dense boolean algebra, then the theory of the ideal lattice $\mathcal{I}(\mathcal{B})$ is undecidable. The next chapter contains applications of this result: we present a coding of a lattice $\mathcal{I}(\mathcal{B})$ in various structures, in many cases even without parameters. Thus the theory of those structures is undecidable. In a forthcoming paper [40], the author proves by a much harder argument that $\text{Th}(\mathcal{I}(\mathcal{B}))$ actually interprets true arithmetic.

5.1 Computably enumerable boolean algebras

First we define in detail the concepts introduced in Section 1.1.2. We specify the notion of a c.e. boolean algebra as follows. A c.e. boolean algebra is represented by a model

$$(5.1) \quad (\mathbb{N}, \preceq, \vee, \wedge)$$

such that \preceq is a c.e. relation which is a preordering, \vee, \wedge are total computable binary functions, and the quotient structure

$$(5.2) \quad \mathcal{B} = (\mathbb{N}, \preceq, \vee, \wedge) / \approx$$

is a boolean algebra (where $n \approx m \Leftrightarrow n \preceq m \ \& \ m \preceq n$.) We require that 0 is an index for the least element of \mathcal{B} , and 1 is an index for the greatest element. Then $0 \not\approx 1$ by the definition of boolean algebras. Note that, in an effective way, for each n we can find an index for a complement of n/\approx in \mathcal{B} , denoted by

$$(5.3) \quad \text{Cpl}(n).$$

At stage s of the algorithm, see if there is $b \leq s$ such that $n \wedge b \approx 0$ and $n \vee b \approx 1$, and these equivalences can be verified in $\leq s$ steps. If so, return b as an output. We write $b - c$ for $b \wedge \text{Cpl}(c)$ and $b \prec c$ if $b \preceq c$ & $c \not\preceq b$. In general, “ $b \prec c$ ” is not decidable.

We will often relativize our results to $\emptyset^{(k-1)}$. To define the notion of a Σ_k^0 -boolean algebras, one requires that \preceq be Σ_k^0 and that \wedge, \vee be computable in $\emptyset^{(k-1)}$.

For a Σ_k^0 -boolean algebra \mathcal{B} , let

$$(5.4) \quad \mathcal{I}(\mathcal{B}) = \text{the lattice of } \Sigma_k^0\text{-ideals of } \mathcal{B}.$$

In the following we will mostly use the terminology of c.e. boolean algebras. It will be clear how to relativize the notions to the Σ_k^0 -cases for $k > 1$.

A c.e. boolean algebra \mathcal{B} is called *effectively dense* if there is a computable F such that $\forall x [F(x) \preceq x]$ and

$$(5.5) \quad \forall x \not\preceq 0 \ [0 \prec F(x) \prec x].$$

More generally, a Σ_k^0 -boolean algebra \mathcal{B} is effectively dense if (5.5) holds with some $F \leq_T \emptyset^{(k-1)}$. All effectively dense boolean algebras are dense and hence isomorphic, but not necessarily effectively isomorphic. Thus our study of boolean algebras is in the spirit of recursive model theory, and not along the lines of Feiner [17], where (classical) isomorphism types of c.e. boolean algebras are investigated. Feiner proves that there is a c.e. boolean algebra which is not isomorphic to a recursive one.

Example 5.1.1 *Let T be a consistent recursively axiomatizable theory, and let \mathcal{B}_T be Lindenbaum algebra of sentences over T . If T contains Robinson’s Q , then \mathcal{B}_T is effectively dense.*

Proof. We use Rosser’s Theorem which asserts that, from an index of a c.e. theory $S \supseteq Q$ one can effectively obtain a sentence α such that

$$S \text{ consistent} \Rightarrow S \not\vdash \alpha \text{ and } S \not\vdash \neg\alpha.$$

Given $\varphi \in \mathcal{B}_T$, to determine $F(\varphi)$ let $S = T \cup \{\varphi\}$. If $\varphi \not\preceq 0$ (in \mathcal{B}_T), then S is consistent, so $\varphi \not\preceq \alpha$ and $\varphi \not\preceq \neg\alpha$. Thus let $F(\varphi) = \varphi \wedge \alpha_S$. \diamond

Notice that, by a result of Montagna and Sorbi [39], the boolean algebras for all such theories are effectively isomorphic. In that paper the notion of effective density (for general c.e. lattices) is apparently mentioned for the first time.

5.2 The theory of ideal lattices

This section will be devoted to the following result.

Main Theorem 5.2.1 *Suppose \mathcal{B} is an effectively dense Σ_k^0 -boolean algebra. Then $\text{Th}(\mathcal{I}(\mathcal{B}))$ is hereditarily undecidable.*

The main component of the proof is an uniform definability lemma for the Σ_3^0 -ideals of \mathcal{B} which contain a certain “separating” c.e. ideal I_0 , where $|\mathcal{B}/I_0| = \infty$. This proof uses some ideas from Section 4.2 in the context of c.e. boolean algebras.

In what follows, for notational reasons we will actually give codings in the two sorted structure $(\mathcal{B}, \mathcal{I}(\mathcal{B}))$. This structure can be interpreted in the lattice $\mathcal{I}(\mathcal{B})$ in a natural way: represent $b \in \mathcal{B}$ by the principal ideal $\widehat{b} = [0, b]_{\mathcal{B}}$. Since the principal ideals are just the complemented elements in $\mathcal{I}(\mathcal{B})$, the set of ideals in $\mathcal{I}(\mathcal{B})$ representing elements of \mathcal{B} is definable in $\mathcal{I}(\mathcal{B})$ without parameters. Moreover, the membership relation “ $b \in I$ ” can be translated into “ $\widehat{b} \subseteq I$ ”.

We will find a formula with parameters $\psi(x; L, I_0)$ such that, as L varies over c.e. ideals, $\{x : (\mathcal{B}, \mathcal{I}(\mathcal{B})) \models \psi(x; L, I_0)\}$ ranges over the Σ_3^0 -ideals of \mathcal{B} containing I_0 . Then, intuitively speaking because Σ_3^0 is far beyond the level of complexity of the c.e. structure \mathcal{B} itself, it will be possible to give an interpretation of \mathcal{E}^3 in $\mathcal{I}(\mathcal{B})$, using I_0 as a parameter.

We say that a c.e. ideal I_0 of \mathcal{B} is *separating* if the following holds in \mathcal{B} :

$$(5.6) \quad \forall x \exists y \preceq x \ y \in I_0 \ \& \ (x \notin I_0 \Rightarrow y \not\approx 0)$$

and, moreover, y can be determined effectively in x . The intuition is that a separating ideal nontrivially meets all the principal ideals $\neq \{0\}$, in an effective way.

Lemma 5.2.2 *\mathcal{B} possesses a separating c.e. ideal I_0 such that the boolean algebra \mathcal{B}/I_0 is infinite.*

Proof. We write b_n instead of n if we think of the number n as determining an element of the boolean algebra under consideration, and call b_n an *index* for the element b_n/\approx . First we consider the easier problem how to build a separating ideal I_0 such that \mathcal{B}/I_0 has at least two elements. Recall that F is the function from (5.5). Let $y_0 = F(b_0)$ (so $y_0 \not\approx 1$). If y_0, \dots, y_n have already been defined, then let $y_{n+1} = y_n \vee F(b_{n+1} - y_n)$. Let I_0 be the ideal generated by $\{y_i : i \in \mathbb{N}\}$. Then I_0 is c.e. and separating, because $b_{n+1} - y_n \not\approx 0$ if $b_{n+1} \notin I_0$. Also $I_0 \neq \mathcal{B}$: otherwise suppose that n is the least number such that $y_{n+1} \approx 1$. Then $F(b_{n+1} - y_n) \geq \text{Cpl}(y_n)$, which is impossible by our hypothesis on F and since $\text{Cpl}(y_n) \not\approx 0$.

We now refine the construction in order to make \mathcal{B}/I_0 infinite. To this end, we also define elements $z_0 < z_1 < \dots$ of \mathcal{B} such that $(z_n/I_0)_{n \in \mathbb{N}}$ is a strictly ascending sequence in \mathcal{B}/I_0 . As above, let $y_0 = F(b_0)$, and let $z_0 = 0$. Now, if y_0, \dots, y_n and $z_0 < \dots < z_n$ have already been defined, then consider the “partition”

$$p_0 = z_1 - z_0, \dots, p_{n-1} = z_n - z_{n-1}, p_n = \text{Cpl}(z_n).$$

Our intention is never to put so much into I_0 that one of the components of the partition goes completely into I_0 . Let $c_{n+1} = b_{n+1} - y_n$. Note that, if $c_{n+1} \not\approx 0$, then the same must hold for $c_{n+1} \wedge p_i$ for some i . Thus if we let

$$y_{n+1} = y_n \vee \bigvee_{i \leq n} F(c_{n+1} \wedge p_i),$$

we make sure that (5.6) is satisfied for $x = b_{n+1}$ via y_{n+1} . To make progress on the ascending sequence, also let

$$(5.7) \quad z_{n+1} = F(\text{Cpl}(y_{n+1})) \vee z_n.$$

Again, let I_0 be the ideal generated by $\{y_i : i \in \mathbb{N}\}$. We verify that I_0 has the required properties. Since the sequence (y_n) is effective, I_0 is c.e. Moreover, I_0 is separating because, if $b_{n+1} \notin I_0$, then $c_{n+1} \not\approx 0$, and therefore $y_{n+1} \not\approx 0$. Furthermore, y_{n+1} was determined effectively from b_{n+1} . If n is least such that $y_{n+1} \approx 1$, then

$$\bigvee_{i \leq n} (F(b_{n+1} \wedge \text{Cpl}(y_n) \wedge p_i)) \succeq \text{Cpl}(y_n),$$

contrary to the fact that $F(b_{n+1} \wedge \text{Cpl}(y_n) \wedge p_i) < b_{n+1} \wedge \text{Cpl}(y_n) \wedge p_i$ for some i . Thus $y_n \not\approx 1$ for each n .

Fix n . We now show that $d = z_{n+1} - z_n \notin I_0$. Since $y_{n+1} \not\approx 1$, $d \not\approx 0$, so $d \not\leq y_0$. Suppose k is least such that $d \leq y_{k+1}$. Then $k > n$, because $d \leq \text{Cpl}(y_{n+1})$, but $y_{k+1} \leq y_{n+1}$ for $k \leq n$. We now argue as above, but restrict ourselves to the interval $[0, d]$. By the minimality of k , $d \wedge y_k \prec d$, so

$$0 \prec d - y_k \leq \bigvee_{i \leq k} F(c_{k+1} \wedge p_i).$$

Since the $(p_i)_{i \leq k}$ form a partition and $d = p_n$, in the supremum above only the term $F(c_{k+1} \wedge p_n)$ matters. Thus (recall that $c_{k+1} = b_{k+1} - y_k$)

$$d - y_k \leq F(c_{k+1} \wedge d) = F((d - y_k) \wedge b_{k+1}).$$

Since $d - y_k \not\approx 0$, this contradicts the properties of F . \diamond

Lemma 5.2.3 *Suppose that I_0 is a c.e. separating ideal. For each Σ_3^0 -ideal J , $I_0 \subseteq J$, there is a c.e. ideal $L \subseteq I_0$ such that*

$$(5.8) \quad x \in J \Leftrightarrow \exists r \in I_0 \forall s \in I_0 [s \wedge r \approx 0 \Rightarrow x \wedge s \in L].$$

We write $\psi(x, L; I_0)$ for the right hand side in (5.8).

Proof. Choose a computable function $x \mapsto y(x)$ such that, given x , $y(x)$ is a witness for (5.6). We first define a computable sequence (s_n) which generates I_0 as an ideal and has further useful properties. To start with, since I_0 is c.e. there is some computable sequence (y_i) generating I_0 . Let $\mathcal{B}_{\leq e}$ be a finite set of indices for the boolean algebra generated by $\{0, \dots, e\}$ ($\mathcal{B}_{\leq e}$ can be obtained effectively from e). Moreover let $s_0 = y_0$ and

$$(5.9) \quad s_{n+1} = (y_{n+1} - \widehat{s}_n) \vee \bigvee \{y(z - \widehat{s}_n) : z \in \mathcal{B}_{\leq n}\},$$

where $\widehat{s}_n = s_0 \vee \dots \vee s_n$. Clearly $s_i \wedge s_j \approx 0$ for $i \neq j$.

Applying Lemma 4.2.5 to $P = J$ (viewed as an index set), we obtain a u.c.e. sequence (Z_i) such that $Z_i \subseteq \{0, \dots, i\}$ and

- $e \in J \Rightarrow$ a.e. $i [e \in Z_i]$
- $\exists^\infty i [Z_i \subseteq J]$.

Let L be the ideal generated by

$$\{e \wedge s_i : e \in Z_i\}.$$

Clearly $L \subseteq I_0$ and L is c.e. We now verify (5.8).

“ \Rightarrow ” Suppose that $x \in J$. Choose \tilde{i} such that $\forall i > \tilde{i} (x \in Z_i)$ and let $r = s_0 \vee \dots \vee s_{\tilde{i}}$. If $s \in I_0$ and $s \wedge r = 0$, then, for some $j > \tilde{i}$, $s \preceq s_{\tilde{i}+1} \vee \dots \vee s_j$.

But $x \wedge s_k \in L$ for all $k > \tilde{i}$. Therefore $x \wedge s \in L$.

“ \Leftarrow ” Now suppose that $x \notin J$. Given $r \in I_0$, choose k such that $r \preceq \widehat{s}_k$. Choose $i > k$ such that $Z_i \subseteq J$ and also $i > x$. We show that the witness s_i is a counterexample to (5.8), i.e. $x \wedge s_i \notin L$.

Let $v = x - \bigvee_{e \in Z_i} e - \widehat{s}_{i-1}$. Then $v \notin I_0$: else, since $\widehat{s}_{i-1} \in I_0 \subseteq J$ and $\bigvee_{e \in Z_i} e \in J$, we could infer that $x \in J$. Therefore $y(v) \not\approx 0$. Also $z = x - \bigvee_{e \in Z_i} e \in \mathcal{B}_{\leq i-1}$, so $v = z - \widehat{s}_{i-1}$ occurs in the disjunction (5.9) where s_n is defined. Hence $y(v) \preceq s_i \wedge v$ and therefore $s_i \wedge x - \bigvee_{e \in Z_i} e \not\approx 0$. But this implies that s_i is a counterexample: if $x \wedge s_i \in L$, then by the fact that the (s_k) are pairwise disjoint, $x \wedge s_i \preceq \bigvee_{e \in Z_i} e \wedge s_i$. This means that $s_i \wedge (x - \bigvee_{e \in Z_i} e) \approx 0$, a contradiction. \diamond

Lemma 5.2.4 $\mathcal{E}^3 = (\Sigma_3^0, \subseteq)$ can be coded in $\mathcal{I}(\mathcal{B})$.

Proof. By Lemma 5.2.2, fix a separating ideal I_0 of \mathcal{B} such that \mathcal{B}/I_0 is infinite. By the previous lemma, the lattice \mathbf{L} of Σ_3^0 -ideals of \mathcal{B} which contain I_0 can be coded in $(\mathcal{B}, \mathcal{I}(\mathcal{B}))$, using I_0 as a parameter. We represent a ideal J , $I_0 \subseteq J$ by any $L \subseteq I_0$ such that (5.8) is satisfied.

For completeness' sake we include the coding scheme in the language of the one-sorted structure $\mathcal{I}(\mathcal{B})$. Let lower case letters range over principal (i.e., complemented) ideals. The scheme is

$$\begin{aligned} \varphi_{dom}(L; I_0) &\equiv L \leq I_0 \\ \varphi_{\subseteq}(L, H; I_0) &\equiv \forall x [\psi(x, L; I_0) \Rightarrow \psi(x, H; I_0)] \\ \varphi_{\equiv}(L, H; I_0) &\equiv \varphi_{\subseteq}(L, H; I_0) \ \& \ \varphi_{\subseteq}(H, L; I_0). \end{aligned}$$

It is now sufficient to show that $(\Sigma_3^0, \subseteq) \simeq [C, D]_{\mathbf{L}}$ for some $C, D \in \mathbf{L}$, since $\text{Th}(\Sigma_3^0, \subseteq)$ is h.u. by Section 4.4 and Corollary 2.1.2. We distinguish two cases.

Case A: \mathcal{B}/I_0 has infinitely many atoms. Let $C = I_0$ and let D be the ideal generated by I_0 and the preimages in \mathcal{B} of atoms of \mathcal{B}/I_0 . Notice that “ x/I_0 is an atom of \mathcal{B}/I_0 ” is a Π_2^0 -property of indices, so there is a function $f \leq_T \emptyset''$ such that $(f(n)/I_0)_{n \in \mathbb{N}}$ is an enumeration of the atoms of \mathcal{B}/I_0 without repetition. This implies that D is a Σ_3^0 -ideal and, moreover,

$$J \mapsto \{n \in \mathbb{N} : f(n) \in J\}$$

is an isomorphism between $[C, D]_{\mathbf{L}}$ and (Σ_3^0, \subseteq) .

Case B: \mathcal{B}/I_0 has only finitely many atoms. If \mathcal{B}/I_0 has only finitely many atoms, let b be a preimage in \mathcal{B} of their supremum. Replacing I_0 by the separating ideal $I_0 \vee [0, b]$ if necessary, we can in fact assume that \mathcal{B}/I_0 is dense and hence free. Note that \mathcal{B}/I_0 is c.e. The standard step-by-step construction of a free generating sequence for a dense countable boolean algebra produces in the case of \mathcal{B}/I_0 a \emptyset' -sequence (a_i) such that (a_i/I_0) is a free generating sequence for \mathcal{B}/I_0 . Now let \mathcal{F} be the boolean algebra of finite or cofinite subsets of \mathbb{N} , and consider the natural map $g : \mathcal{B}/I_0 \mapsto \mathcal{F}$ defined by $g(a_i/I_0) = \{i\}$. Clearly g is computable in \emptyset' if viewed as a map from indices for \mathcal{B} into an effective representation for \mathcal{F} . Let C be the ideal $\{x : g(x) = 0\}$ and let D be the ideal generated by the a_i 's and I_0 . Then C, D are Σ_3^0 -ideals of \mathcal{B} , contain I_0 , and the Σ_3^0 -ideals X of \mathcal{B} such that $C \subseteq X \subseteq D$ correspond to the Σ_3^0 -ideals of \mathcal{F} which are contained in the ideal generated by the atoms. So again $(\Sigma_3^0, \subseteq) \simeq [C, D]_{\mathbf{L}}$. This concludes the proof. \diamond

Chapter 6

Coding Ideal lattices

Lattices $\mathcal{I}(\mathcal{B})$ can be coded without parameters in a natural way into three interesting types of structures. For the first type, \mathcal{B} is a c.e. boolean algebra, next a Σ_2^0 , and finally a Σ_3^0 -boolean algebra. Since $\text{Th}(\mathcal{I}(\mathcal{B}))$ interprets true arithmetic (Nies [40]), so do their theories. Here we contend ourselves with proving undecidability.

1. Lattices of c.e. theories $\{T' : T \subseteq T'\}$ under inclusion, where T is a c.e. consistent theory containing Robinson arithmetic Q
2. Initial intervals $[\mathbf{o}, \mathbf{a}]$ of $\text{Rec}_r^{\mathcal{P}}$, where \leq_r is a polynomial time reducibility and \mathbf{a} is the r -degree of a “super sparse” set
3. All intervals of \mathcal{E}^* which are not boolean algebras.

Our first result follows directly from the Main Theorem 5.2.1. Let \mathcal{L}_T be the lattice of c.e. extensions of T closed under logical inference.

Theorem 6.0.1 *$\text{Th}(\mathcal{L}_T)$ is undecidable.*

Proof. Let $\mathcal{B} = \mathcal{B}_T$. In Example 5.1.1 it was proved that \mathcal{B}_T is effectively dense. Notice that elements of \mathcal{L}_T are the c.e. filters in $\mathcal{B} = \mathcal{B}_T$. So $\mathcal{L}_T \cong \mathcal{I}(\mathcal{B})$ via negation. \diamond

6.1 Intervals of \mathcal{E}^* and \mathcal{E}

In Section 4.1 we discussed intervals of \mathcal{E} and \mathcal{E}^* and gave several examples. A further type of intervals is obtained by considering the major subset relation. Maass and Stob [36] proved that for each pair A, B such that $A \subset_m B$,

up to an (effective) isomorphism one obtains the same lattices $[A, B]_{\mathcal{E}}$ and $[A^*, B^*]_{\mathcal{E}^*}$. These structures are denoted by \mathcal{M} and \mathcal{M}^* . From the Maass–Stob result, it follows that \mathcal{M}^* (say) is a distributive lattice with strong homogeneity properties: all nontrivial closed intervals are isomorphic to the whole structure, and all nontrivial complemented elements are automorphic within \mathcal{M}^* . However, \mathcal{M}^* is not a boolean algebra.

A natural question to ask is which intervals $[A, B]_{\mathcal{E}}$ have an undecidable theory. For instance, Maass and Stob pose this question for \mathcal{M} , as a part of a programme to analyze the structure of \mathcal{M} . A complete answer is given by the following result.

Theorem 6.1.1 *Suppose $D \subseteq A$, where $D, A \in \mathcal{E}$. If $[D^*, A^*]_{\mathcal{E}^*}$ is not a boolean algebra, then $\text{Th}([D^*, A^*]_{\mathcal{E}^*})$ and in $\text{Th}([D, A]_{\mathcal{E}})$ are undecidable.*

Thus \mathcal{E}^* differs considerably from \mathcal{R}_m and also the Δ_2^0 -Turing degrees, where intervals of a very different type with a decidable theory exist. For instance, in both degree structures there are initial intervals which form linear orders of order type $\omega + 1$ (Lachlan [29] and Lerman [35]).

Proof. We will reduce the problem to the case of \mathcal{M}^* . First, we can assume that $A = \mathbb{N}$ since each interval of \mathcal{E} is isomorphic to an end interval. Moreover we use the following fact, due to Lachlan.

Fact 6.1.2 *If $\mathcal{L}(D)$ is not a boolean algebra, then there exist sets \tilde{D}, \tilde{A} such that $D \subseteq \tilde{D} \subseteq \tilde{A}$ and $\tilde{D} \subset_m \tilde{A}$.*

Proof. Since $\mathcal{L}(D) = [D, \mathbb{N}]$ is not a boolean algebra, we can choose $\tilde{A} \supset D$ such that \tilde{A} is not complemented in $\mathcal{L}(D)$, i.e., $\mathbb{N} - \tilde{A} \cup D$ is not c.e. Pick $E \subset_{sm} \tilde{A}$. Then $\tilde{D} := E \cup D \subset_{\infty} \tilde{A}$, because, by the definition of small subsets (4.2),

$$\tilde{A} =^* E \cup D \Rightarrow \mathbb{N} \cap (A - E) \subseteq^* D \Rightarrow (\mathbb{N} - \tilde{A}) \cup D \text{ c.e.}$$

It is sufficient to prove the following.

Claim 6.1.3 *If $D \subset_m A$, then for some effectively dense Σ_3^0 -boolean algebra \mathcal{B} , $\mathcal{I}(\mathcal{B})$ can be coded in $[\tilde{D}^*, \tilde{A}^*]$*

(Of course, by [36], all these intervals are isomorphic. But we don't make use of the Maass–Stob result, since we directly see that the interpretation is independent of the particular choices of D, A .) For the case of \mathcal{E}^* , to see the Claim suffices recall that $\text{Th}(\mathcal{I}(\mathcal{B}))$ is h.u. by the Main Theorem 5.2.1,

and $\mathcal{I}(\mathcal{B})$ can be coded also in $[D^*, A^*]$ if we use \tilde{D}^*, \tilde{A}^* as parameters. By Fact 2.1.1, $\text{Th}([\tilde{D}^*, \tilde{A}^*])$ is undecidable.

To obtain the result for $\text{Th}([\tilde{D}, \tilde{A}])$ (and hence for $\text{Th}([D, A])$), note that $\text{Th}([\tilde{D}^*, \tilde{A}^*])$ can be interpreted in $\text{Th}([\tilde{D}, \tilde{A}])$: since $D \subset_m A$, for $\tilde{D} \subseteq P, Q \subseteq \tilde{A}$, $P =^* Q \Leftrightarrow [P \cap Q, P \cup Q]$ is a boolean algebra.

To prove the Claim for \mathcal{E}^* , we will code without parameters the lattice $\mathcal{I}(\mathcal{B})$, for the Σ_3^0 -boolean algebra

$$\mathcal{B} = \{X \cup \tilde{D}^* : X \sqsubset \tilde{A}\}$$

which was already introduced in (4.4) (with sets $D \subset_m A$ instead of $\tilde{D} \subset_m \tilde{A}$). Recall that $(U_e \cup \tilde{D}^*)_{e \in \mathbb{N}}$ is a Δ_3^0 -listing of \mathcal{B} (see Notation 4.1.1), via which \mathcal{B} becomes a Σ_3^0 -boolean algebra. More precisely, the induced ordering on indices

$$e \preceq i \Leftrightarrow U_e \subseteq^* U_i \cup \tilde{D}$$

is Σ_3^0 and \emptyset'' -computable functions \vee, \wedge as in (5.1) can be defined in the appropriate way. Moreover \mathcal{B} is \emptyset'' -effectively dense, by the Owings Splitting Theorem (see Soare [56]): given e , the Theorem provides V_0, V_1 such that $\tilde{D} \cup U_e = V_0 \cup V_1$ and

$$U_e - \tilde{D} \text{ not co-c.e.} \Rightarrow V_i - D \text{ not co-c.e.} \quad (i = 0, 1).$$

Let $F(e)$ be such that $U_{F(e)} = V_0$. If $U_e \not\subseteq^* \tilde{D}$, then $\tilde{D} \subset_m \tilde{D} \cup U_e$, so $U_e - \tilde{D}$ not co-c.e. Thus, in \mathcal{B} , $0 \prec F(e) \prec e$. In fact, the Owings Splitting Theorem is effective, but it takes \emptyset'' to determine $U_k \cup \tilde{D}^*$ from k .

By Lemma 4.2.2, if I is a Σ_3^0 -ideal of \mathcal{B} , then there is C_I , $\tilde{D} \subseteq^* C_I \subseteq^* \tilde{A}$ such that

$$I = \{j : U_j \cap C_I \subseteq^* \tilde{D}\}.$$

Conversely, an ideal I satisfying this for some C must be Σ_3^0 . Now, for the desired coding of $\mathcal{I}(\mathcal{B})$, one represents Σ_3^0 -ideals I of \mathcal{B} ambiguously by elements $c = C_I^*$. Thus, to specify the scheme for this coding, vacuously let

$$\varphi_{\text{dom}}(c) \Leftrightarrow c = c.$$

Inclusion of Σ_3^0 -ideals can be defined within $[\tilde{D}^*, \tilde{A}^*]$ using the formula

$$(6.1) \quad \varphi_{\leq}(c_1, c_2) \equiv \forall x (x \text{ complemented} \Rightarrow (x \wedge c_1 = 0 \Rightarrow x \wedge c_2 = 0)),$$

where $\tilde{d} = \tilde{D}^*$ etc., and $\varphi_{\equiv}(c_1, c_2) \Leftrightarrow \varphi_{\leq}(c_1, c_2) \& \varphi_{\leq}(c_2, c_1)$. Here, as usual 0 stands for the least element in the p.o. under consideration, namely \tilde{D}^* . \diamond

6.2 Complexity theory

We proceed to applications to complexity theory of the method to code lattices $\mathcal{I}(\mathcal{B})$. Since polynomial time reducibilities are Σ_2^0 on effectively presented collections of computable sets, the effectively dense boolean algebras we deal with will be Σ_2^0 .

Definition 6.2.1 *A is super sparse via f if*

1. *f is a strictly increasing, time constructible function $\mathbb{N} \mapsto \mathbb{N}$*
2. *$A \subseteq \{0^{f(k)} : k \in \mathbb{N}\}$ and “ $0^{f(k)} \in A$?” can be determined in time $O(f(k+1))$*

(Ambos-Spies [2]). Moreover we require that

3. *$(\forall r \in \mathbb{N})(a.e. n) [f(n)^r < f(n+1)]$.*

A string w is relevant if $w = 0^{f(k)}$ for some k.

Because of the time-constructibility of f, we obtain

Fact 6.2.2 *The set of relevant strings is in PTIME.*

Given a reducibility \leq_r^p , we denote the degree of a set X by \mathbf{x} and also write $deg_r^p(X)$ for \mathbf{x} . $Rec_r^p(\leq \mathbf{a})$ denotes the initial interval of r-degrees $\leq \mathbf{a}$.

A polynomial time 1-*tt* reduction of X to Y is a polynomial time Turing reduction where in a computation at most one oracle question is asked. Thus

Definition 6.2.3 *$X \leq_{tt}^1 Y$ if there are polynomial time computable functions $g : \Sigma^{<\omega} \times \{0, 1\} \mapsto \{0, 1\}$ and $h : \Sigma^{<\omega} \mapsto \Sigma^{<\omega}$ such that*

$$\forall w \in \Sigma^{<\omega} [X(w) = g(w, Y(h(x)))].$$

Polynomial time 1-*tt* reducibility is a reducibility of more technical interest. Here is one application of the notion, due to Ambos-Spies.

Theorem 6.2.4 ([2]) *Suppose A is super sparse. Then the polynomial time T-degree of any set $B \leq_T^p A$ consists of a single 1-*tt*-degree.* \diamond

Supersparse sets exist in the time classes we are interested in here.

Lemma 6.2.5 ([2]) *Suppose that $h : \mathbb{N} \mapsto \mathbb{N}$ is an increasing time constructible function with $\text{PTIME} \subset \text{DTIME}(h)$, so that $h(n) \geq n + 1$ and h eventually dominates all polynomials. Then there is a super sparse computable $A \in \text{DTIME}(h) - \text{PTIME}$.*

Sketch of Proof. Let $f(n) = h^{(n)}(0)$. Since h eventually dominates all polynomials, we can construct $A \subseteq \{0^{f(k)} : k \in \mathbb{N}\}$ such that $A \in \text{DTIME}(h)$, but still diagonalize against all polynomial time machines. \diamond

Theorem 6.2.6 *If $A \subseteq \{0\}^*$ is super sparse, $A \notin \text{PTIME}$ and $\mathbf{a} = \text{deg}_r^p(A)$, then*

$$\text{Th}(\text{Rec}_r^p(\leq \mathbf{a})) \text{ is undecidable.}$$

Proof. In a sequence of lemmas, we will code $\mathcal{I}(\mathcal{B})$ into $[\mathbf{o}, \mathbf{a}]$ without parameters, for an appropriate effectively dense Σ_2^0 -boolean algebra \mathcal{B} . We make \mathcal{B} a very easy, well controlled part of $[\mathbf{o}, \mathbf{a}]$, but use all of $[\mathbf{o}, \mathbf{a}]$ to sort out Σ_2^0 -ideals of \mathcal{B} . We begin by introducing \mathcal{B} . For an r -degree \mathbf{c} , we let $\mathcal{B}(\mathbf{c})$ be the set of complemented elements in $[\mathbf{o}, \mathbf{c}]$, i.e.

$$(6.2) \quad \mathcal{B}(\mathbf{c}) = \{\mathbf{x} \leq \mathbf{c} : \exists \mathbf{y} \mathbf{x} \wedge \mathbf{y} = \mathbf{o} \ \& \ \mathbf{x} \vee \mathbf{y} = \mathbf{c}\}.$$

A *splitting* (or *split*) of a set B is a set X such that for some $R \in \text{PTIME}$, $X = B \cap R$. We denote this by $X \sqsubset B$ (via R). The advantage of taking a super sparse \mathbf{a} is that not only is $\mathcal{B}(\mathbf{a})$ a boolean algebra, but in fact it is effectively isomorphic to the boolean algebra of splittings of A , modulo the equivalence relation under which two splittings are identified if their symmetric difference is in PTIME . The isomorphism is obtained by mapping a split $A \cap R$ (represented by an index of the PTIME set R) to its degree. In this way, \mathcal{B} is well controlled as desired. (We could in fact easily ensure that A has no infinite PTIME subsets. In that case \mathcal{B} is isomorphic to the boolean algebra of splits modulo finite sets.)

We first show that decomposing a super sparse set A into splits gives complements in the degrees.

Lemma 6.2.7 *Suppose that A is super sparse and via f and $A_1 = A \cap R$, $A_2 = A \cap \overline{R}$ for $R \in \text{PTIME}$. Then A_1 and A_2 form a T -minimal pair, in the sense that if $Q \leq_T^P A_1, A_2$, then $Q \in P$.*

Proof. By Theorem 6.2.4, it is sufficient to prove that

$$Q \leq_{1-tt}^p A_1, A_2 \Rightarrow Q \in \text{PTIME}.$$

Suppose that $Q \leq_{1-tt}^p A_i$ via g_i, h_i ($i = 1, 2$). The idea to show $Q \in \text{PTIME}$ is that, if both $h_1(w), h_2(w)$ are relevant oracle queries, then one of them must be much shorter than the other, so that membership of the shorter one in the appropriate oracle set can be determined in time polynomial in the input. The procedure is as follows. Given w , compute $h_1(w)$ and $h_2(w)$. If for some i $h_i(w)$ is not relevant, then $Q(w) = g_i(w, 0)$. Else,

1. if $k = |h_1(w)| = |h_2(w)|$, then see whether $0^k \in R$. If so, then $Q(w) = g_2(w, 0)$, else $Q(w) = g_1(w, 0)$.
2. Otherwise, say $|h_1(w)| < |h_2(w)|$. Evaluate $Q(w) = h_1(w, A_1(v))$, where $v = h_1(w)$. This is possible in polynomial time, because, by the definition of super sparseness, the computation for $A(v)$ takes at most $O(|h_2(w)|)$ many steps. \diamond

Next we show that, conversely, each pair of complements is represented by a decomposition into splits.

Lemma 6.2.8 *Suppose that $\mathbf{a}_1 \vee \mathbf{a}_2 = \mathbf{a}$ and $\mathbf{a}_1 \wedge \mathbf{a}_2 = \mathbf{o}$. Then there exists a split $A_1 \sqsubset A$ such that $A_1 \in \mathbf{a}_1$ and $A_2 = A - A_1 \in \mathbf{a}_2$.*

Proof. It is sufficient to consider the case that $r \in \{m, 1-tt\}$. It is well known that \leq_m^p and \leq_{1-tt}^p induce distributive uppersemilattices on the computable sets. This is because, if $X \leq_r^p Y \oplus Z$, then there is $R \in \text{PTIME}$ such that $X \cap R \leq_r^p Y$ and $X \cap \overline{R} \leq_r^p Z$ (provided that $r \in \{m, 1-tt\}$). Now, pick sets $B_i \in \mathbf{a}_i$ and apply this to $A \leq_r^p B_1 \oplus B_2$ in order to obtain R . It is sufficient to show that in fact $A_1 = A \cap R \equiv_r^p B_1$ and $A_2 = A \cap \overline{R} \equiv_r^p B_2$. Notice that since $B_1 \leq_r^p A_1 \oplus A_2$, there is $Q \in \text{PTIME}$ such that $B_1 \cap Q \leq_r^p A_1$ and $B_1 \cap \overline{Q} \leq_r^p A_2$. But B_1, A_2 form an r -minimal pair, so $B_1 \cap \overline{Q} \in \text{PTIME}$ and therefore $B_1 \equiv_r^p B_1 \cap Q \leq_r^p A_1$. \diamond

Finally, we show that the order is preserved when passing from splits modulo PTIME -subsets of A to degrees.

Lemma 6.2.9 *Let $P, Q \in \text{PTIME}$. Then*

$$A \cap P \leq_r^p A \cap Q \Leftrightarrow A \cap (P - Q) \in \text{PTIME}.$$

Proof. The implication from right to left is immediate. For the other implication, notice that $A \cap P$ splits into $A \cap P \cap Q$ and $A \cap (P - Q)$. But $A \cap (P - Q)$ and $A \cap Q$ form a T -minimal pair by Lemma 6.2.7. Therefore if $A \cap P \leq_r^p A \cap Q$, then $A \cap (P - Q) \in \text{PTIME}$. \diamond

Let $(P_e)_{e \in \mathbb{N}}$ be an effective listing of the polynomial time sets. We have obtained a representation of \mathcal{B} in the sense of Section 5.1: let $e \in \mathbb{N}$ represent $\text{deg}_r(A \cap P_e)$. The computable functions \vee, \wedge on \mathbb{N} are obtained by taking unions and intersections of polynomial time sets. Clearly, “ $A \cap P_e \subseteq A \cap P_i$ ” is Σ_2^0 in e, i .

Lemma 6.2.10 \mathcal{B} is an effectively dense Σ_2^0 -boolean algebra.

Proof. By the uniform diagonalization technique from Landweber e.a. [32], given a splitting $A \cap P_e$, we can effectively obtain $Q = P_{F(e)} \subseteq P_e$ such that $A \cap P_e \notin \text{PTIME}$ implies that $A \cap Q, A \cap (P - Q) \notin \text{PTIME}$. For details, see the proof of Theorem 7.3 in Balcazar e.a. [9]. \diamond

This concludes our analysis of \mathcal{B} . Next we show how to obtain a coding of $\mathcal{I}(\mathcal{B})$ in $[\mathbf{o}, \mathbf{a}]$. The idea is to represent a Σ_2^0 -ideal I by a degree \mathbf{c}_I such that

$$I = \{\mathbf{x} \in \mathcal{B} : \mathbf{x} \leq \mathbf{c}_I\}.$$

Clearly any ideal defined in this way must be Σ_2^0 (even if \mathbf{c}_I is just the degree of any computable set, not necessarily in $[\mathbf{o}, \mathbf{a}]$). The final lemma will show that, conversely, each Σ_2^0 ideal can be represented in that way by a degree $\mathbf{c}_I \leq \mathbf{a}$. Then one obtains the desired parameter free coding of $\mathcal{I}(\mathcal{B})$ in $[\mathbf{o}, \mathbf{a}]$, using the same framework as we did for intervals $[\tilde{D}^*, \tilde{A}^*]$ of \mathcal{E}^* where $\tilde{D} \subset_m \tilde{A}$ in Section 6.1: the scheme is given by the formulas $\varphi_{\text{dom}}(c) \Leftrightarrow c = c$,

$$\varphi_{\leq}(c_1, c_2) \equiv \forall x \text{ complemented } (x \leq c_1 \Rightarrow x \leq c_2)$$

and $\varphi_{\equiv}(c_1, c_2) \equiv \varphi_{\leq}(c_1, c_2) \ \& \ \varphi_{\leq}(c_2, c_1)$.

Lemma 6.2.11 Suppose that A is super sparse via f . Then for each Σ_2^0 ideal $I \triangleleft \mathcal{B}(\mathbf{a})$ there is $\mathbf{c}_I \leq \mathbf{a}$ such that $\forall \mathbf{x} \in \mathcal{B}(\mathbf{a})$ ($\mathbf{x} \in I \Leftrightarrow \mathbf{x} \leq \mathbf{c}_I$).

Proof of Lemma 6.2.11 Recall that w is relevant if $w = \mathbf{0}^k$ for some $k \in \text{range}(f)$. We will build $\mathbf{c}_I \leq_m^p A$ via a g which is computable in polynomial time. By Theorem 6.2.4 it is sufficient to consider the cases of \leq_m^p and

\leq_{1-tt}^p reducibility. Since I is in Σ_2^0 , there is a function $q \leq_T \emptyset'$ such that $\text{range}(q) = \{e : \text{deg}_r^p(P_e \cap A) \in I\}$. By the Limit Lemma in Soare [56], there is a computable function $q(e, t)$ such that $q(e) = \lim_t q(e, t)$. Let (h_j) be a list of polynomial time m -reductions if we consider m -reducibility, and of polynomial time $1-tt$ reductions if we consider $1-tt$ -reducibility. We meet the coding requirements

$$K_e : A \cap P_{q(e)} \leq_m^p C_I$$

by specifying polynomial time m -reductions to C_I . To do so, we assign K_e -coding locations to certain relevant 0^s . If $s = f(m)$, a K_e -coding location for 0^s will have the form 0^n , $n = \langle e, r \rangle$, where $r \geq e$ and $f(m) \leq n < f(m+1)$. We will ensure that K_e -coding locations exists for all sufficiently long relevant 0^s . We require that in n steps one can determine that $0^s \in P_u$, where u is the current guess at $q(e) = \lim_t q(e, t)$. We define C_I by specifying a polynomial time computable g such that $C_I = g^{-1}(A)$, mapping coding locations for relevant 0^s to 0^s . Thus, eventually just the relevant $0^s \in P_{q(e)}$ are assigned a K_e -coding location, which is in C_I just if 0^s is in A . An appropriate choice of the K_e -coding locations will ensure that the requirements

$$H_{\langle i, j \rangle} : A \cap P_i \leq_r^p C_I \text{ via } [g_j,] h_j \Rightarrow A \cap P_i \leq_r^p \oplus_{m \leq k} A \cap P_{q(m)} \quad (k = \langle i, j \rangle)$$

are met. We can suppose that computing $h_j(x)$ takes at most $p_j(|x|)$ steps, where

$$p_j(n) = (n+2)^j.$$

The main idea of the proof is how to ensure that the coding of K_e does not interfere with the requirements H_i , $i < e$. We make the length of any K_e -coding location for 0^s exceed $p_{e-1}(s)$.

The algorithm for g .

Given an input x , $n = |x|$, first determine in quadratic time the maximal $s \leq n$ such that 0^s is relevant. This is possible by the time constructibility of f . Now proceed as follows.

1. See if there are e, r such that $x = 0^{\langle e, r \rangle}$
2. perform computations $q(e, 0), q(e, 1), \dots$ till n steps have passed and let u be the last value (or $u = 0$ if there was no value so far).
3. see if $0^s \in P_u$ in n steps

4. check if $p_{e-1}(s) \leq n$.

If (1) and (3) are answered affirmatively and the computation in (4) stops, then let $g(x) = 0^s$ (so x is a K_e -coding location for 0^s). Else let $g(x)$ be the string (1) $\notin A$. This completes the algorithm. Clearly the algorithm takes at most $O(n^2)$ steps.

Let $C_I = g^{-1}(A)$. We verify that C_I has the required properties.

Claim 1. Let $q(e) = \lim_t q(e, t)$. Then $A \cap P_{q(e)} \leq_m^p C_I$.

Proof. Let $p(s)$ be a polynomial which dominates $p_{e-1}(s)$ and the number of steps it takes to compute $P_{q(e)}$ on the input 0^s . Pick an $s_0 = f(m)$ such that the value returned in (2.) of the algorithm is $q(e)$ for all $s \geq s_0$ and also that, by super sparseness, $\langle e, p(f(k)) \rangle < f(k+1)$ for all $k \geq m$. Then for all $s \geq s_0$, 0^s relevant,

$$0^s \in A \cap P_{q(e)} \Leftrightarrow 0^{\langle e, p(s) \rangle} \in C_I.$$

Claim 2. The requirements $H_{(i,j)}$ are met.

Proof. We first consider the case of m -reducibility. Suppose that $A \cap P_i \leq_r^p C_I$ via h_j . We obtain an m -reduction of $A \cap P_i$ to $\bigoplus_{m < k} A \cap P_{q(m)}$ ($k = \langle i, j \rangle$) as follows. Given a relevant string 0^s , first compute $x = h_j(0^s)$. Since $0^s \in A \cap P_i \Leftrightarrow x \in C_I$, it is sufficient to determine if $x \in C_I$. Run the algorithm for g on input x . If $g(x) = (1)$ then $x \notin C_I$. Otherwise x is a coding location.

Case 1: $|x| < s$. Then give $A(g(x))$ as an answer. Since A is super sparse and $|g(x)| < s$, this answer can be found in time $O(s)$.

Case 2: $n = |x| \geq s$.

We can suppose that $s \geq s_0$ where s_0 is so large that for all relevant $t \geq s_0$ $|h_j(0^t)|$ is less than the least relevant number bigger than t (by the last condition in Definition 6.2.1), and also the computation in Step 2 of the algorithm for g with input 0^t gives the final value $q(e)$ for each $e \leq k$. By the main idea, if $x \in C_I$, then x must be a coding location for a requirement K_e , $e \leq k$. Since $s \geq s_0$, $x \in C_I \Leftrightarrow g(x) \in A \cap P_{q(e)}$.

To prove Claim 2 for 1-1t reducibility, suppose that $A \cap P_i \leq_{1-1t}^p C_I$ via g_j, h_j . In Case 2, as before obtain an answer $b \in \{0, 1\}$ to “ $x \in C_I$?”, depending on a query to the oracle set. Now give as an output $g_j(x, b)$. \diamond

Corollary 6.2.12 *Suppose that h is time constructible and hyperpolynomial. Then the degrees of (1) all sets and (2) of all tally sets in $\text{DTIME}(h)$ have an undecidable theory.*

Proof. Choose a super sparse $A \in \text{DTIME}(h) - \text{PTIME}$ and let \mathcal{B} be as before. Because h is hyperpolynomial, all the sets $A \cap P_e$, as well as the sets C_I are in $\text{DTIME}(h)$. By the preceding result, we obtain a coding of $\mathcal{I}(\mathcal{B})$ in $(\text{DTIME}(h), \leq_r^p)$ with parameter \mathbf{a} . Because of Fact 2.1.2 and the Main Theorem 5.2.1 this implies that $\text{Th}(\text{DTIME}(h), \leq_r^p)$ is undecidable. For (2), observe that all sets involved are tally sets. \diamond

Note. If $\text{PTIME} = \text{NP}$, then the polynomial time honest degrees below any super sparse set form a boolean algebra (Ambos-Spies and Yang [7]). So the dishonesty of the reduction of C_I to A in the proof of Lemma 6.2.11 appears to be inevitable.

One can relativize a polynomial time reducibility \leq_r^p to a computable oracle U by replacing the underlying Turing machine model by an oracle Turing machine. We denote this relativized reducibility by \leq_r^U . The relativization process is most natural for \leq_T^p , since

$$X \leq_T^U Y \Leftrightarrow X \oplus U \leq_T^p Y \oplus U.$$

Thus, if $\mathbf{a} = \text{deg}_T^p(A)$, then the \leq_T^U -degrees of the computable sets are isomorphic to the end interval $\{\mathbf{x} \in \text{Rec}_T^p : \mathbf{x} \geq \mathbf{a}\}$.

An interesting question arising from Corollary 6.2.12 is:

$$(6.3) \quad \text{PTIME} \neq \text{NP} \Rightarrow \text{Th}(\text{NP}, \leq_T^p) \text{ is undecidable?}$$

Let $\text{EXPTIME} = \bigcup_{k \in \mathbb{N}} \text{DTIME}(2^{(n^k)})$. We show that the conclusion holds when relativized to any computable oracle U such that $\text{NP}^U = \text{EXPTIME}^U$. Such U exist by a result of Heller [24]. Clearly EXPTIME^U is closed downwards under \leq_T^U .

Theorem 6.2.13

$$\text{NP}^U = \text{EXPTIME}^U \Rightarrow \text{Th}(\text{NP}^U, \leq_T^U) \text{ is undecidable.}$$

Proof. To relativize the notion of a super sparse sets to U , we change the second condition in Definition 6.2.1: we now require that “ $0^{f(k)} \in A$?” can be determined in time $O(f(k+1))$ with the help of the oracle U . All the arguments used in order to prove Theorem 6.2.6 are relativizable, including Ambos-Spies’ Theorem 6.2.4. For instance, Lemma 6.2.5 relativized to U states the existence of a U -super sparse $A \notin \text{PTIME}^U$ such that A can be computed in time $h(n)$ with oracle U . We apply this with $h(n) = 2^n$.

Notice that the boolean algebra \mathcal{B} remains Σ_2^0 because U is computable. So we obtain a coding of $\mathcal{I}(\mathcal{B})$ in the structure R_A^U of \leq_T^U -degrees below A . (Of course, R_A^U is isomorphic to the interval $[\mathbf{u}, \mathbf{a}]$ of polynomial time T-degrees, where $\mathbf{a} = \deg_T^p(A \oplus U)$). Since $\text{NP}^U = \text{EXPTIME}^U$, R_A^U is an initial interval of the \leq_T^U -degrees of NP^U -sets. So we obtain a coding of $\mathcal{I}(\mathcal{B})$ in (NP^U, \leq_T^U) . \diamond

Next we consider relativizations of the lattice of NP sets under inclusion (which to some extent can be considered as a complexity theoretic analog of \mathcal{E}). It is not known if $\text{NP} = \text{CoNP}$, i.e. whether this lattice is a boolean algebra. The strongest possible analog to the question (6.3) would thus be:

$$(6.4) \quad \text{NP} \neq \text{CoNP} \Rightarrow \text{Th}(\text{NP}, \subseteq) \text{ is undecidable?}$$

One can construct oracles X, U such that $\text{NP}^X = \text{CoNP}^X$ and $\text{NP}^U \neq \text{CoNP}^U$. Here we extend the second oracle result:

Theorem 6.2.14 ([14]) *There is a computable oracle U such that $\text{Th}(\text{NP}^U, \subseteq)$ is undecidable.*

Proof. We develop a coding with parameters of a lattice $\mathcal{I}(\mathcal{B})$, where \mathcal{B} is an effectively dense Σ_2^0 -boolean algebra. The proof necessarily produces an oracle U such that $\text{NP}^U \neq \text{CoNP}^U$. In fact we make \mathcal{B} a boolean algebra which is closely related to

$$\mathcal{C}^U := \text{NP}^U \cap \text{CoNP}^U,$$

and use the rest of NP^U to represent $\mathcal{I}(\mathcal{B})$. A similar idea was used in the proof of Theorem 6.2.6. *Let the variables R, S range over \mathcal{C}^U .* We use the concept of oracle nondeterministic Turing machine (oracle NTM) which is described in Balcazar e.a. [9].

Outline of the proof. The construction of U extends Baker e.a. [8]. As a parameter, we determine a set $Q \in \text{NP}^U - \mathcal{C}^U$, where for some polynomial time $S \subseteq \{0\}^*$,

$$(6.5) \quad Q = \{w \in S : \exists v \in U \ |v| = |w|\}.$$

Then we let $\mathcal{B} = \mathcal{B}(Q)/\mathcal{R}(Q)$, where

$$\begin{aligned}
(6.6) \quad \mathcal{B}(Q) &= \{Q \cap R : R \in \mathcal{C}^U\}, \\
\mathcal{R}(Q) &= \{R \in \mathcal{C}^U : R \subseteq Q\}, \\
\text{Co}\mathcal{R}(Q) &= \{Q - R : R \in \mathcal{R}(Q)\}.
\end{aligned}$$

Clearly $\mathcal{R}(Q)$ is an ideal of $\mathcal{B}(Q)$. With an appropriate numbering of NP^U , \mathcal{B} is an effectively dense Σ_2^0 -boolean algebra.

The general frame for the coding of $\mathcal{I}(\mathcal{B})$ follows Lemma 4.2.2, the $n = 1$ case of the ideal definability lemma for \mathcal{E} . However, here we prefer the language of filters. A filter \mathcal{F} of $\mathcal{B}(Q)$ is *2-acceptable* if $\text{Co}\mathcal{R}(Q) \subseteq \mathcal{F}$ and \mathcal{F} has a Σ_2^0 -index set. The construction of U will ensure that \mathcal{F} is 2-acceptable iff for some $D \subseteq Q$ in NP^U ,

$$(6.7) \quad \mathcal{F} = \{X \in \mathcal{B}(Q) : \exists R \in \mathcal{R}(Q)[D - X \subseteq R]\}.$$

Hence the class of 2-acceptable filters is uniformly definable in NP^U . Moreover it is in 1-1 correspondence with the class of Σ_2^0 -filters of $\mathcal{B} = \mathcal{B}(Q)/\mathcal{R}(Q)$, and hence to $\mathcal{I}(\mathcal{B})$. In this way we code $\mathcal{I}(\mathcal{B})$ into NP^U with a parameter Q .

The details. First we need an appropriate listing of \mathcal{C}^U . We rely on the fact that U , and therefore Q , is given by a construction which at stage s determines $U^{=s} = U \cap \Sigma^s$.

Lemma 6.2.15 *There is a uniformly computable pair of sequences $(C_e), (\overline{C}_e)$ such that*

- (i) *for each e we are effectively given oracle NTMs computing C_e, \overline{C}_e with time bound $(n + 2)^e$*
- (ii) *$C_e \cap \overline{C}_e =^* \emptyset$ and $C_e \cup \overline{C}_e =^* \Sigma^{<\omega}$.*

Proof. Fix some listing of all oracle NTM (N_k) such that N_k has time bound $(n + 2)^k$. We write N_i^U for the set accepted by N_i when the oracle is U . To determine $C_e, e = \langle i, j \rangle$, we assume that N_i^U is the complement of N_j^U until, if ever, this can be refuted in real time based on oracle queries whose answer has been already determined. Given input w , to obtain $C_e(w), \overline{C}_e(w)$, run $s = |w|$ steps of the following:

in lexicographical order, for strings x such that $(|x| + 2)^e < s$,
see whether $x \in N_i^U \Leftrightarrow x \in N_j^U$. If so, stop.

If we stop in $\leq s$ steps, then our assumption was wrong, so arbitrarily let $C_e(w) = 0, \overline{C}_e(1) = 1$. Else let $C_e(w) = N_i^U(w), \overline{C}_e(w) = N_j^U(w)$.

Clearly (i) and (ii) are satisfied. Moreover, if N_i^U actually is the complement of N_j^U , then $C_e = N_i^U$ and $\overline{C}_e = N_j^U$. \diamond

Notice that $\mathcal{B}(Q) = \{Q \cap C_e : e \in \mathbb{N}\}$, so we obtain a presentation in the sense of (5.2) for $\mathcal{B}(Q)$, and hence for \mathcal{B} . Moreover, \mathcal{B} with this presentation is a Σ_2^0 -boolean algebra, because U is computable:

$$e \preceq i \Leftrightarrow Q \cap (C_e - C_i) \in \mathcal{R}(Q) \Leftrightarrow \exists j Q \cap (C_e - C_i) = C_j \subseteq Q,$$

and the matrix of the last expression is Π_1^0 .

It remains to be proved that \mathcal{B} is effectively dense. This is implied by the following relativizable lemma.

Lemma 6.2.16 *If B is decidable and $B \notin \text{CoNP}$, then one can in an effective way from a decision procedure for B determine a set $R \in \text{PTIME}$ such that $B \cap R, B - R \notin \text{CoNP}$.*

Proof. An easy application of the delay diagonalization technique, similar to the proof of Lemma 6.2.10. \diamond

Effective density of \mathcal{B} is obtained as follows: given e , consider $B = Q \cap C_e$. Applying the preceding lemma relativized to U yields $R \in \text{PTIME}^U$ such that $B \notin \text{CoNP} \Rightarrow B \cap R, B - R \notin \text{CoNP}^U$. Using \emptyset' as an oracle one can compute $i = F(e)$ such that $B \cap R = Q \cap C_i$. So \mathcal{B} is effectively dense via F .

We next describe how to ensure $Q \notin \mathcal{C}^U$ and introduce a first version of the set S needed for (6.5). Using the technique of Baker e.a. [8], for each e , we produce a witness w such that $Q(w) = N_e^U(w)$. Thus we meet the requirements

$$R_e : Q \neq \Sigma^{<\omega} - N_e^U.$$

If w is our witness and we see an accepting computation $N_e^U(w) = 1$, we have to put a string u of the same length as w into U which is not an oracle query asked in that computation (or in accepting computations for requirements which have already been satisfied). Let $S = \{0^{s_0}, 0^{s_1}, \dots\}$, where $s_0 = 0$ and, for $k > 0$

$$(6.8) \quad s_k = \min\{s > s_{k-1} : s > (s_{k-1} + 2)^{k-1} \ \& \ 2^s > G_k(s)\}.$$

Here $G_k(s) = (s + 2)^k$, but this definition of $G_k(s)$ will be modified when we add further requirements. Clearly $S \in \text{PTIME}$ (apply the logarithm with base 2 to “ $2^s > G_k(s)$ ”).

Construction of U , Part 1.

For each string w , $U(w) = 0$ unless otherwise specified.

To determine $U^=s$ for $s = s_k$, check whether there is an $e < k$ such that R_e is not yet met, namely

$$\forall w \in S[|w| < s \Rightarrow N_e^U(w) \neq Q(w)].$$

If not, $U^=s = \emptyset$. If so for e minimal, we meet requirement R_e : see whether $N_e^U(0^s) = 1$ via some accepting computation Γ based on the current oracle. Let $v \in \Sigma^s$ be the lexicographically first string which is not an oracle query in Γ , and define $U(v) = 1$, thereby causing $Q(0^s) = 1$.

Next we describe how we obtain, for each 2-acceptable \mathcal{F} a set $D \subseteq Q$ in NP^U satisfying (6.7). We identify subsets of \mathcal{B} and their preimages under the canonical map associated with the presentation (5.2). Note that there is an effective listing $(\mathcal{F}_e)_{e>0}$ of Σ_2^0 -indices for 2-acceptable filters: let \mathcal{F}_e be the filter generated by $\text{Co}\mathcal{R}(Q)$ and the $e - 1$ -th Σ_2^0 -set. (We need $e > 0$ for notational reasons.)

Since each \mathcal{F}_e is infinite (when viewed as a subset of \mathbb{N}), there is a binary function $\alpha \leq_T \emptyset'$ such that, for all $e > 0$,

$$\mathcal{F}_e = \{\alpha_e(n) : n \in \mathbb{N}\}.$$

By the Limit Lemma in Soare [56], there is a computable β such that, for each $n, e > 0$, $\alpha(e, n) = \lim_k \beta(e, n, k)$. We can assume that

$$(6.9) \quad \beta(e, n, k) < k.$$

To obtain a good representation of \mathcal{F}_e , let

$$(6.10) \quad F_{n,k}^e = Q \cap \bigcap_{m \leq n} C_{\beta(e,m,k)}.$$

Then, for each n , $F_n^e = \lim_k F_{n,k}^e$ exists in the sense that an index for an oracle NTM obtained from (6.10) stabilizes. Moreover, the sequence $F_0^e \supset F_1^e \supset \dots$ generates \mathcal{F}_e .

For $e > 0$, let

$$(6.11) \quad D_e = \{0^{s_k} : e < k \ \& \ \exists w \in U \ |w| = s_k + e\}.$$

For the inclusion “ \subseteq ” in (6.7), we ensure that

$$(6.12) \quad \forall m \ D_e \subseteq^* F_m^e.$$

Then $X \in \mathcal{F}_e \Rightarrow \exists m F_m^e \subseteq X \Rightarrow D_e - X$ finite.

For the converse inclusion, we meet the requirements

$$P_{\langle e, m \rangle} : |F_m^e \cap \overline{C}_m| = \infty \Rightarrow D_e \cap \overline{C}_m \neq \emptyset.$$

Then, if $X = Q \cap C_i \notin \mathcal{F}_e$, we can deduce that $D_e - X \not\subseteq R$ for each $R \in \mathcal{R}(Q)$. Observe that $X \cup R \notin \mathcal{F}_e$ because $\text{Co}\mathcal{R}(Q) \subseteq \mathcal{F}_e$. Choose an m such that $X \cup R = Q \cap C_m$, and also that \overline{C}_m is the complement of C_m . Then the hypothesis of $P_{\langle e, m \rangle}$ is satisfied, thus $D_e \cap \overline{C}_m \neq \emptyset$, which means that $D_e - X \not\subseteq R$.

We extend the construction by putting at most one element of length $s_k + e$, $0 < e < k$ into U in order to meet the P-type requirements: according to (6.11) this will determine the sets D_e . After presenting the construction we will determine an appropriate choice of the function $G_k(s)$ needed in (6.8).

Construction of U , Part 2.

For $s = s_k$, after determining $U^{=s}$, if we placed some string of length s into U , we also do the following: search for a minimal $\langle e, m \rangle < k, e > 0$ such that $P_{\langle e, m \rangle}$ is not yet *satisfied*, namely

$$(6.13) \quad D_e \cap \overline{C}_m \cap \Sigma^{<s} = \emptyset,$$

and also (based on the current oracle)

$$(6.14) \quad 0^s \in F_{m,k}^e \cap \overline{C}_m.$$

If $\langle e, m \rangle < k$ exists, find a $w \in \Sigma^{s+e}$ which does not occur as an oracle query in an accepting computation in (6.14), and also not in the accepting computation Γ from Part 1, stage s . Define $U(w) = 1$. We say that $P_{\langle e, m \rangle}$ *receives attention*.

Now to make sure we can find w , we have to count relevant accepting computations and define $G_k(s)$ appropriately. For a Q -type requirement there is at most one, and to determine $0^s \in F_{m,k}^e$ we need at most $k+1$ many, see (6.10). Notice that these computations have a time bound $(s+2)^k$, by the property (6.9). There is one more accepting computation for $0^s \in \overline{C}_m$. So the definition

$$G_k(s) = (k+3)(s+2)^k$$

is as desired.

Clearly U is computable and $Q \in \text{NP}^U$. The R-type requirements are met for the same reasons as before. No requirement is ever injured by a “later” U -change by the fact that $s_k > (s_{k-1} + 2)^{k-1}$ and the construction. So by the condition (6.13), each requirement receives attention at most once. We conclude that (6.12) holds: given $e > 0$ and m , choose a k such that for $n < m$, $\beta(e, n, k)$ has reached its limit and $P_{\langle e, n \rangle}$ does not receive attention from s_k on. If a requirement causes $v \in D_e$ at a stage $s \geq s_k$, then $s = s_h + e$ for some $h \geq k$ and the requirement is $P_{\langle e, n \rangle}$ for some $n > m$. Hence $v \in F_{n,h}^e \subseteq F_m^e$.

To prove that $P_{\langle e, m \rangle}$ is met, suppose that $|F_m^e \cap \overline{C}_m| = \infty$. Choose a k such that $\beta(e, m, k)$ has reached its limit and no requirement P_u , $u < \langle e, m \rangle$ receives attention at a stage $\geq s_k$. Since $F_m^e \subseteq Q \subseteq \{0^{s_i} : i \in \mathbb{N}\}$, there is an $s = s_h \geq s_k$ such that $0^s \in F_m^e \cap \overline{C}_m$. Since $P_{\langle e, m \rangle}$ has the highest priority at s , we cause $0^s \in D_e$. So $P_{\langle e, m \rangle}$ is met. \diamond

Chapter 7

C.e. weak truth-table degrees

We give a coding without parameters of a copy of $(\mathbb{N}, +, \times)$ in \mathcal{R}_{wtt} . This implies that $\text{Th}(\mathbb{N})$ can be interpreted in $\text{Th}(\mathcal{R}_{wtt})$. As a tool we develop a theory of two sorts of parameter definable subsets, using the distributivity of \mathcal{R}_{wtt} in an essential way. One of them is the uniformly definable class of EN-sets (“EN” stands for end segment). These are relatively definable without parameters in an *end segment*, i.e. an upward closed subset E of \mathcal{R}_{wtt} , while E is definable from two parameters \mathbf{c}, \mathbf{d} . The number $n \in \mathbb{N}$ is represented by (parameters defining) any EN-set of size n (but there may also be infinite EN-sets). Using the combinatorics of EN-sets, we give first-order definitions in terms of parameters of whether two EN-sets have the same size, and of the operations $+$ and \times . For instance, for $+$, we express that an EN-set is the disjoint union of two others.

The second type of uniformly definable set, called ID-set (“ID” stands for ideal) is needed to single out the finite EN-sets. We will compare EN-sets to ID-sets, using uniformly definable maps between the first and the second. We need to introduce various schemes. To understand the formulas related to these schemes, it is vital to keep in mind the convention in 2.1.7: if a scheme S_X is given, then X, X_0, \dots are objects coded via S_X .

Notation 7.0.1 As in Soare [56, p. 49], we assume that the use of the computation $\{e\}_s^A(x)$, $u(A; e, x, s) \leq s$. For $e = \langle e_0, e_1 \rangle$ let

$$[e](x) \simeq \max_{y \leq x} \varphi_{e_1}(y).$$

Let $[e]^A(x)$ be $\{e_0\}^A(x)$ if $[e](x)$ and $\{e_0\}^A(x)$ are defined, and the compu-

tation has use $\leq [e](x)$. Otherwise $[e]^A(x)$ is undefined. In a similar way define the approximations at stage s , namely $[e]_s(x)$ and $[e]_s^A(x)$.

Note that $A \leq_{\text{wtt}} B \Leftrightarrow A = [e]^B$ for some e . This implies that

$$(7.1) \quad \{(e, i) : W_e \leq_{\text{wtt}} W_i\} \text{ is } \Sigma_3^0.$$

7.1 Uniformly definable classes in \mathcal{R}_{wtt}

We prove some facts which lead to the concepts of EN- and ID-sets. Most of the facts are algebraic. We outline the duality between the two concepts, as far as the non-symmetric framework of an upper semilattice which may not be a lattice allows this. In the following let $(D; \leq, \vee, 0, 1)$ be a distributive upper semilattice with least and greatest elements $0, 1$.

Lemma 7.1.1 *Suppose that $b, y_0, \dots, y_n \in D$.*

1. *If $b \wedge y_i = 0$ for each i , then $b \wedge \sup_i y_i = 0$*
2. *If $b \vee y_i = 1$ for each i , then there is $t \in D$ such that $b \vee t = 1$ and $t \leq y_i$ for each i .*

Proof. (i) If $0 < x \leq b, \sup_i y_i$, then by distributivity, there is an i and $r \in D$ such that $0 < r \leq x, y_i$. But then $r \leq b, y_i$, contrary to $b \wedge y_i = 0$.

(ii) If $n = 0$ let $t = y_0$. Else, since $y_1 \leq b \vee y_0$, we can choose a $t_1 \leq y_0$ and $b_1 \leq b$ such that $y_1 = b_1 \vee t_1$. Then, $1 = b \vee b_1 \vee t_1 = b \vee t_1$, so if $n \neq 1$, $y_2 \leq b \vee t_1$ implies that we can pick $t_2 \leq t_1$ and $b_2 \leq b$ such that $y_2 = b_2 \vee t_2$. Continuing in this way we obtain $t = t_n \leq y_0, \dots, y_n$ such that $b \vee t = 1$. \diamond

For $d_0, \dots, d_n \in D$, let

$$(7.2) \quad E(d_0, \dots, d_n) = \{x \in D : \forall y [\forall i \leq n (y \leq d_i) \Rightarrow y \leq x]\}.$$

Thus $E(d_0, \dots, d_n)$ is the set of upper bounds of the ideal $[0, d_0] \cap \dots \cap [0, d_n]$. Note that $d_i \in E(d_0, \dots, d_n)$ for each i . Finite EN-sets $\{p_0, \dots, p_n\}$ will be sets which are relatively definable in $E(p_0, \dots, p_n)$. First we need a characterization of the elements in such an end segment.

Lemma 7.1.2 For $d_0, \dots, d_n \in (D; \leq, \vee, 0, 1)$,

$$x \in E(d_0, \dots, d_n) \Leftrightarrow x = \inf_{i \leq n} (x \vee d_i).$$

Proof. For the direction from right to left, clearly $x \vee d_i \in E(d_0, \dots, d_n)$ for each i . Hence, if the infimum exists, it is also an upper bound for the ideal $[0, d_0] \cap \dots \cap [0, d_n]$.

For the other direction, the argument is similar to the one used in the proof of Lemma 7.1.1 (ii). If $y \leq x \vee d_i$ for each i , then by distributivity we can choose $x_0 \leq x$ and $q_0 \leq p_0$ such that $y = x_0 \vee q_0$. If $n \geq 1$, choose $x_1 \leq x, q_1 \leq p_1$ such that $q_0 = x_1 \vee q_1$. Continuing in this way we obtain $q_n \leq p_n$ such that $q_{n-1} = x_n \vee q_n$. Moreover $q_n \leq p_0, \dots, p_n$, so $q_n \leq x$. Hence $q_{n-1} \leq x, \dots, q_0 \leq x$ and finally $y \leq x$. \diamond

For $x, y \in D$, we write

$$nd[x, y]$$

if $x < y$ and the interval $[x, y]$ does not embed the 4-element boolean algebra preserving least and greatest element. Clearly $nd[x, y]$ can be expressed in the language of p.o. In the next lemma, (i) leads to the definition of EN-sets, and (ii) to the definition of ID -sets.

Lemma 7.1.3 (i) Let p_0, \dots, p_n be a finite sequence of elements of D such that for each i , $nd[p_i, 1]$ (in particular, $p_i < 1$) and for $i \neq j$, $p_i \vee p_j = 1$. Then $\{p_i : i \leq n\}$ is the set of minimal elements x in $E = E(p_0, \dots, p_n)$ such that $nd[x, 1]$.

(ii) Let (a_i) be a finite or infinite sequence of elements of D such that for each i $nd[0, a_i]$ and for $i \neq j$, $a_i \wedge a_j = 0$. Then $\{a_i\}$ is the set of maximal elements x in I such that $nd[0, x]$, where I is the ideal of D generated by $\{a_i\}$.

Proof. (i) It is sufficient to prove that

$$x \in E \ \& \ nd[x, 1] \Rightarrow \exists j \ p_j \leq x.$$

Since $x < 1$, by Lemma 7.1.2 there is j such that $x \vee p_j < 1$. Moreover, by Lemma 7.1.1 there is t such that, for all $i \neq j$, $t \leq x \vee p_i$ and $x \vee p_j \vee t = 1$. We can suppose that $x \leq t$. By Lemma 7.1.2, $x = \inf_{k \leq n} x \vee p_k$, so $(x \vee p_j) \wedge t = x$. By $nd[x, 1]$, this implies $t = 1$, so $x \vee p_i = 1$ for $i \neq j$ and $x = \inf_{k \leq n} x \vee p_k = x \vee p_j$.

(ii) It is sufficient to prove that

$$x \in I \ \& \ nd[0, x] \Rightarrow \exists j \ x \leq a_j.$$

Since $x \in I$, $x \leq \sup_{i \leq n} a_i$ for some n . By distributivity, $x = \sup_{i \leq n} \tilde{a}_i$ for some $\tilde{a}_i \leq a_i$ ($i \leq n$). Since $0 < x$, some \tilde{a}_j does not equal 0. By Lemma 7.1.1, $\tilde{a}_j \wedge \sup_{i \leq n, i \neq j} \tilde{a}_i = 0$, so $nd[0, x]$ implies that $\tilde{a}_i = 0$ for $i \leq n, i \neq j$, hence $x = \tilde{a}_j \leq a_j$. \diamond

In the context of \mathcal{R}_{wtt} , we are able to give first-order definitions with parameters of the set E in (i) of the preceding Lemma, and also of I in (ii) if (\mathbf{a}_i) is a finite or an infinite u.c.e. sequence. We use the following theorem of Ambos-Spies, Nies and Shore.

Theorem 7.1.4 ([5]) *Let I be a Σ_3^0 -ideal of \mathcal{R}_{wtt} . Then there exists $\mathbf{a}, \mathbf{b} \in \mathcal{R}_{wtt}$ such that $I = [\mathbf{o}, \mathbf{a}] \cap [\mathbf{o}, \mathbf{b}]$.* \diamond

Degrees \mathbf{a}, \mathbf{b} as above are called an *exact pair* for I . Note that, conversely, each ideal which has an exact pair is Σ_3^0 , so that the theorem constitutes a uniform definability result for the class of Σ_3^0 -ideals.

Lemma 7.1.5 (i) *Suppose $\{\mathbf{p}_0, \dots, \mathbf{p}_n\}$ is a subset of \mathcal{R}_{wtt} such that $nd[\mathbf{p}_i, \mathbf{1}]$ for each i and $\mathbf{p}_i \vee \mathbf{p}_j = \mathbf{1}$ for $i \neq j$. Then $\{\mathbf{p}_0, \dots, \mathbf{p}_n\}$ is definable from two parameters \mathbf{c}, \mathbf{d} via a formula $\varphi_P(x; c, d)$.*

(ii) *Suppose (\mathbf{a}_i) is a finite or infinite u.c.e. sequence in \mathcal{R}_{wtt} such that $nd[\mathbf{o}, \mathbf{a}_i]$ for each i and $\mathbf{a}_i \vee \mathbf{a}_j = \mathbf{o}$ for $i \neq j$. Then $\{\mathbf{a}_i\}$ is definable from two parameters \mathbf{c}, \mathbf{d} via a formula $\varphi_A(x; c, d)$.*

Proof. (i) Observe that $I = [\mathbf{o}, \mathbf{p}_0] \cap \dots \cap [\mathbf{o}, \mathbf{p}_n]$ is a Σ_3^0 -ideal by (7.1), so $I = [\mathbf{o}, \mathbf{c}] \cap [\mathbf{o}, \mathbf{d}]$ for some \mathbf{c}, \mathbf{d} . Thus $E(\mathbf{p}_0, \dots, \mathbf{p}_n) = E(\mathbf{c}, \mathbf{d})$ is definable from \mathbf{c}, \mathbf{d} via the formula $\psi(x; c, d) = \forall y[y \leq c, d \Rightarrow y \leq x]$. Let $\varphi_P(x; c, d)$ be the formula expressing that x is a minimal element in $\{z : \psi(z; c, d)\}$ such that $nd[x, \mathbf{1}]$.

(ii) Let I be the ideal generated by $\{\mathbf{a}_i\}$. It follows from (7.1) that I is Σ_3^0 . So, once again, $I = [\mathbf{o}, \mathbf{c}] \cap [\mathbf{o}, \mathbf{d}]$ for some \mathbf{c}, \mathbf{d} . Let $\varphi_A(x; c, d)$ be the formula expressing that x is a maximal element $\leq c, d$ such that $nd[0, x]$. \diamond

We are now ready to specify the notions of EN-sets and ID-sets by appropriate schemes of the same type as in Example 2.1.5.

Definition 7.1.6 (i) Let S_P the scheme given by the formula $\varphi_P(z; c, d)$ and the $\alpha(c, d)$ expressing that whenever x, y satisfy the formula and $x \neq y$, then $x \vee y = 1$. Subsets of \mathcal{R}_{wtt} coded via S_P are called EN-sets.

(ii) Let S_Z the scheme given by the formula $\varphi_Z(z; c, d)$ and the correctness condition $\beta(c, d)$ expressing that whenever x, y satisfy the formula and $x \neq y$, then $x \wedge y = 0$. Subsets of \mathcal{R}_{wtt} coded via S_Z are called ID-sets.

Notice that subsets of finite EN-sets are EN-sets themselves.

7.2 The undecidability of $\text{Th}(\mathcal{R}_{wtt})$

Undecidability of $\text{Th}(\mathcal{R}_{wtt})$ was first proved in Ambos-Spies e.a. [5]. We use the fact that there is an easy way to produce finite EN-sets in order give a quite elementary new proof. The methods will also be used to obtain a coding of a copy of $(\mathbb{N}, +, \times)$. Along the lines of Theorem 3.4.2 we develop a scheme, also denoted by S_C , to code arbitrary relations between finite EN-sets.

The abundance of EN-sets stems from the fact¹ that each low $\mathbf{p} \in \mathcal{R}_{wtt}$ satisfies $nd[\mathbf{p}, \mathbf{1}]$. Thus, whenever $\mathbf{p}_0, \dots, \mathbf{p}_n$ are low and $\mathbf{p}_i \vee \mathbf{p}_j = \mathbf{1}$ for $i \neq j$, then $\{\mathbf{p}_0, \dots, \mathbf{p}_n\}$ is an EN-set. For each n , such wtt -degrees $\mathbf{p}_0, \dots, \mathbf{p}_n$ can be obtained by the method of the Sacks splitting theorem (see Soare [56]). In view of later applications, we will prove a more general version of this in Proposition 7.2.2 below.

Theorem 7.2.1 *If $\mathbf{p} \in \mathcal{R}_{wtt}$ is low, then $nd[\mathbf{p}, \mathbf{1}]$.*

Proof. We slightly modify the proof of an extension of the Lachlan Non-Diamond Theorem in Ambos-Spies [1]. He proves that, if $\mathbf{a}_0, \mathbf{a}_1, \mathbf{b}_0, \mathbf{b}_1$ are c.e. Turing degrees such that $\mathbf{a}_0 \vee \mathbf{a}_1 = \text{deg}_T(\emptyset')$ and $\mathbf{b}_0 \vee \mathbf{b}_1$ is low, then, for some $i \leq 1$, \mathbf{a}_i is not \mathbf{b}_i -cappable. Here \mathbf{a} is \mathbf{b} -cappable if there is a $\mathbf{c} \not\leq \mathbf{b}$ such that $\mathbf{b} = \mathbf{a} \wedge \mathbf{c}$. An inspection of the proof reveals that it can be adapted to wtt -reducibility. (The T -reductions built during the construction have recursively bounded use anyway, and the proof of Lemma 6 [Lemma 9] goes through. In particular, if the reduction procedures occurring in requirement R_e are now wtt -reductions $[e_1]^{B_0}$ and $[e_2]^{B_1}$, then the step counting functions g in the proofs of those lemmas can be computed from B_0 [B_1] with recursively bounded use. So the weaker hypothesis $C_0 \not\leq_{wtt} B_0$ [$C_1 \not\leq_{wtt} B_1$] suffices.)

¹The author would like to thank Klaus Ambos-Spies for suggesting this.

Here we use only the special case of the Theorem that $\mathbf{b}_0 = \mathbf{b}_1 = \mathbf{p}$. If $nd[\mathbf{p}, \mathbf{1}]$ fails, then there are $\mathbf{a}_0, \mathbf{a}_1 < \mathbf{1}$ such that $\mathbf{a}_0 \vee \mathbf{a}_1 = \mathbf{1}$ and $\mathbf{a}_0 \wedge \mathbf{a}_1 = \mathbf{p}$. So for both $i = 0$ and $i = 1$, \mathbf{a}_i is \mathbf{p} -cappable via $\mathbf{c}_i = \mathbf{a}_{1-i}$. \diamond

We now prove the existence of appropriate EN-sets.

Proposition 7.2.2 *Suppose that $\mathbf{u}_0, \dots, \mathbf{u}_m < \mathbf{1}$. Then for each $n \geq 0$ there exist low $\mathbf{v}_0, \dots, \mathbf{v}_n \in \mathcal{R}_{wtt}$ such that $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$ is an EN-set and $\mathbf{u}_i \vee \mathbf{v}_j < \mathbf{1}$ for each $i \leq m, j \leq n$.*

Proof. Choose c.e. sets $U_i \in \mathbf{u}_i$. We construct c.e. sets V_j such that the statement of the theorem holds with $\mathbf{v}_j = \deg_{wtt}(V_j)$.

To achieve $\mathbf{v}_j \vee \mathbf{v}_{j'} = \mathbf{1}$ ($j' \neq j$) we ensure that $K = V_j \cup V_{j'}$. For $nd[\mathbf{v}_j, \mathbf{1}]$, we make each V_j low and apply Theorem 7.2.1. We meet the standard lowness requirements

$$L_{e,j} : \exists^\infty s \{e\}^{V_j}(e)[s] \text{ is defined} \Rightarrow \{e\}^{V_j}(e) \text{ converges.}$$

Finally, for $\mathbf{u}_i \vee \mathbf{v}_j < \mathbf{1}$ ($0 \leq i \leq m, 0 \leq j \leq n$) we meet the requirements

$$N_{e,i,j} : K \neq [e]^{U_i \oplus V_j},$$

by refraining from changing V_j till a permanent disagreement occurs. Let (R_k) be some priority listing of the L -type and N -type requirements. If R_k is $N_{e,i,j}$ let

$$\text{length}(k, s) = \min\{x : \forall y < x \ K(y) = [e]^{U_i \oplus V_j}(y)[s],$$

and let $r(k, s) = \max\{[e]_s(y) : y < \text{length}(k, s)\}$.

If R_k is a lowness requirement $L_{e,j}$, the restraint associated with R_k is

$$r(k, s) = u(V_{j,s}; e, e, s).$$

Construction. At stage $s + 1$, if $K_s = K_{s+1}$ do nothing. Else, say y is the unique element in $K_{s+1} - K_s$. Determine the minimal k such that $y < r(k, s)$. If k fails to exist enumerate y into all sets V_j . Else let j be the number such that $R_k = L_{e,j}$ or $R_k = N_{e,i,j}$ for some e, i . Then V_j is the set such that enumerating y into V_j would violate $r(k, s)$. So enumerate y into $V_{j'}$, for each $j' \neq j$. This completes the description of the construction.

Clearly $K = V_j \cup V_{j'}$ for $j \neq j'$. By induction on k we prove:

Lemma 7.2.3 *Let $k \geq 0$.*

- (i) The requirement R_k is met.
(ii) $r(k) = \lim_s r(k, s)$ exists and is finite.

Proof. Assume the Lemma holds for all $h < k$. Choose a stage s_0 such that for all $h < k$, $r(h, s_0)$ has reached the limit, and K does not change below $\max_{h < k} r(h)$ at any stage $s \geq s_0$. Then at no stage $s \geq s_0$ can any number $y < r(k, s)$ enter V_j , where j is determined from k as in the construction: j is the number such that $R_k = L_{e,j}$ or $R_k = N_{e,i,j}$ for some e, i . If $R_k = L_{e,j}$, then R_k is met, because if ever $\{e\}^{V_j}[s]$ converges for $s \geq s_0$, then this computation is preserved. Hence also $r(k, s)$ reaches its limit. Now suppose that $R_k = N_{e,i,j}$.

For (i), assume for a contradiction that $K = [e]^{U_i \oplus V_j}$. Then

$$\limsup \text{length}(k, s) = \infty.$$

We obtain a *wtt*-reduction of K to U_j as follows: given an input y , compute $s \geq s_0$ such that $\text{length}(k, s) > y$ and $U_i|[e](y) = U_{i,s}[e](y)$. Then $r(k, t) \geq [e](y)$ for all $t \geq s$, so (by the monotonicity of the function $[e]$) $[e]^{U_i \oplus V_j}|y + 1$ is protected from changing at stages $\geq s$. So $K(y) = [e]^{U_i \oplus V_j}(y)[s]$. Since $\mathbf{u}_i < \mathbf{1}$, we conclude that $N_{e,i,j}$ is met.

For (ii), let x be least such that $K(x) \neq [e]^{U_i \oplus V_j}(x)$. Let $s_1 \geq s_0$ be least such that, $[e](x)$ is defined, then $K(x)$ and $U_i \oplus V_j|[e](x)$ have reached their final values at s_1 . Then $\text{length}(k, s) \leq x$ from s_1 on, hence $r(k, s)$ reaches its limit. \diamond

Our next goal is to code relations between arbitrary finite EN-sets.

Proposition 7.2.4 *There is an object scheme S_C for coding objects of the form (P_0, P_1, R) in \mathcal{R}_{wtt} , where P_0, P_1 are EN-sets, which has the following property: if P_0, P_1 are finite, then for any $R \subseteq P_0 \times P_1$, (P_0, P_1, R) can be coded.*

Proof. S_C contains parameters $\mathbf{c}_0, \mathbf{d}_0, \mathbf{c}_1, \mathbf{d}_1$ coding P_0, P_1 and further parameters for the relation R . Suppose that $P_0 = \{\mathbf{p}_0, \dots, \mathbf{p}_n\}$ and $P_1 = \{\mathbf{q}_0, \dots, \mathbf{q}_m\}$. First we assume that, in addition,

$$(7.3) \quad \mathbf{p}_i \vee \mathbf{q}_j < \mathbf{1} \text{ for all } i, j.$$

We will reduce the general case to this.

As in the proof of Lemma 7.1.5(ii) there are \mathbf{g}, \mathbf{h} such that

$$E(\mathbf{g}, \mathbf{h}) = E(\{\mathbf{p}_i \vee \mathbf{q}_j : R\mathbf{p}_i\mathbf{q}_j\}).$$

We claim that

$$R\mathbf{p}_i\mathbf{q}_j \Leftrightarrow \exists \mathbf{z} \in E(\mathbf{g}, \mathbf{h}) - \{\mathbf{1}\}[\mathbf{p}_i \vee \mathbf{q}_j \leq \mathbf{z}].$$

For the direction from left to right, simply let $\mathbf{z} = \mathbf{p}_i \vee \mathbf{q}_j$. For the other direction, suppose that the right hand side holds via $\mathbf{z} < \mathbf{1}$. By Lemma 7.1.2, $\mathbf{z} = \inf\{\mathbf{z} \vee \mathbf{p}_r \vee \mathbf{q}_s : R\mathbf{p}_r\mathbf{q}_s\}$. But, if not $R\mathbf{p}_i\mathbf{q}_j$, then $\mathbf{z} \vee \mathbf{p}_r \vee \mathbf{q}_s = \mathbf{1}$ for each pair $\mathbf{p}_r, \mathbf{q}_s$ in R , since $(i, j) \neq (r, s)$ and therefore $\mathbf{p}_i \vee \mathbf{q}_r = \mathbf{1}$ or $\mathbf{p}_j \vee \mathbf{q}_s = \mathbf{1}$. This contradicts $\mathbf{z} < \mathbf{1}$.

Now let

$$\begin{aligned} \tilde{\varphi}_{rel}(x, y; c_0, d_0, c_1, d_1, g, h) &\Leftrightarrow \varphi_P(x; c_0, d_0) \ \& \ \varphi_P(y; c_1, d_1) \ \& \\ &\quad \exists z < 1[x, y \leq z \ \& \ z \in E(g, h)]. \end{aligned}$$

Then in this special case each $R \subseteq P_0 \times P_1$ can be coded via $\tilde{\varphi}_{rel}$.

To remove the restriction (7.3) we imposed, we interpolate with a third EN-set. By Proposition 7.2.2, there is an EN-set $\mathbf{v}_0, \dots, \mathbf{v}_n$ such that, for all $k \leq n$, $\mathbf{p}_i \vee \mathbf{v}_k < \mathbf{1}$ and $\mathbf{q}_j \vee \mathbf{v}_k < \mathbf{1}$ ($i \leq n, j \leq m$). Let $F : P_0 \mapsto \{\mathbf{v}_0, \dots, \mathbf{v}_n\}$ be a bijection. Consider the relation \tilde{R} given by $\tilde{R}\mathbf{v}_k\mathbf{q}_j \Leftrightarrow RF^{-1}(\mathbf{v}_k)\mathbf{q}_j$. Both $F \subseteq P_0 \times \{\mathbf{v}_0, \dots, \mathbf{v}_n\}$ and $\tilde{R} \subseteq \{\mathbf{v}_0, \dots, \mathbf{v}_n\} \times P_1$ can be coded by parameters via $\tilde{\varphi}_{rel}$. Then $R = F\tilde{R}$ can be coded via the following formula (think of z as $F(x)$):

$$\begin{aligned} \varphi_{rel}(x, y; \bar{p}) &\Leftrightarrow \exists z \ [\tilde{\varphi}_{rel}(x, z; c_0, d_0, c_2, d_2, g_0, h_0) \ \& \\ &\quad \tilde{\varphi}_{rel}(z, y; c_2, d_2, c_1, d_1, g_1, h_1)], \end{aligned}$$

where c_2, d_2 are parameters coding the auxiliary EN-set and \bar{p} consists of all 10 parameters. ◇

The following result only has the exact pair theorem 7.1.4, the technique of the Sacks splitting theorem and Theorem 7.2.1 as recursion theoretic ingredients.

Theorem 7.2.5 ([5]) *Th(\mathcal{R}_{wtt}) is undecidable.*

Proof. By Theorem 2.3.1 the class \mathcal{C} of finite directed graphs has a h.u. theory. Using Proposition 7.2.4, \mathcal{C} can be uniformly coded in $\mathcal{A} = \{\mathcal{R}_{wtt}\}$. Hence by Fact 2.1.1, $\text{Th}(\mathcal{R}_{wtt})$ is undecidable. \diamond

Refining the proof with the tools from Section 2.3 yields undecidability of $\Pi_5 - \text{Th}(\mathcal{R}_{wtt})$ as a partial order. In Lempp and Nies [33] a coding of finite bipartite graphs based on ID-sets is developed, which even yields undecidability of $\Pi_4 - \text{Th}(\mathcal{R}_{wtt})$. The Π_2 -theory of \mathcal{R}_{wtt} as a partial order is decidable (Ambos e.a. [3]).

7.3 Coding a copy of $(\mathbb{N}, +, \times)$

We use the same framework and similar notation as in the proof of Theorem 3.4.2.

Theorem 7.3.1 *A copy of $(\mathbb{N}, +, \times)$ can be coded in \mathcal{R}_{wtt} without parameters.*

We will use finite EN-sets to represent numbers. The scheme S_C from Proposition 7.2.4 enables us to express by a first-order condition on parameters that EN-sets have the same cardinality, and also the arithmetical operations. In the end we face the harder problem to single out finite EN-sets. (Note that, even if our examples were all finite, there is no reason to believe that all sets defined via the scheme for EN-sets in Definition 7.1.6 are finite.)

We introduce the scheme without parameters to code $(\mathbb{N}, +, \times)$. It consists of formulas $\varphi_{num}(\bar{x})$, $\varphi_{=}(\bar{x}, \bar{y})$, $\varphi_{+}(\bar{x}, \bar{y}, \bar{z})$ and $\varphi_{\times}(\bar{x}, \bar{y}, \bar{z})$, where \bar{w} stands for a pair of variables w_0, w_1 which represent an exact pair needed to code an EN-set. The formula $\varphi_{num}(\bar{x})$ will be dealt with last, but of course it implies the correctness condition for S_P , since \bar{x} is thought of as coding an EN-set.

Equality and the arithmetical operations

Let $\varphi_{\equiv}(\bar{x}, \bar{y})$ be a formula expressing

$$\exists C [C \text{ is bijection } P_{\bar{x}} \mapsto P_{\bar{y}}],$$

using the scheme S_C from Proposition 7.2.4. By that proposition, if $P_{\bar{a}}$ and $P_{\bar{e}}$ are finite, then

$$|P_{\bar{a}}| = |P_{\bar{e}}| \Leftrightarrow \mathcal{R}_{wtt} \models \varphi_{\equiv}(\bar{a}, \bar{e}).$$

Next let $\varphi_+(\bar{x}, \bar{y}, \bar{z})$ be a formula expressing that can be partitioned into two sets of the same size as $P_{\bar{x}}$ and $P_{\bar{y}}$:

$$\exists \bar{u} \exists \bar{v} [\varphi_{\equiv}(\bar{x}, \bar{u}) \ \& \ \varphi_{\equiv}(\bar{y}, \bar{v}) \ \& \ P_{\bar{z}} = P_{\bar{u}} \cup P_{\bar{v}} \ \& \ P_{\bar{u}} \cap P_{\bar{v}} = \emptyset].$$

It can easily be checked that, for finite $P_{\bar{a}}, P_{\bar{e}}, P_{\bar{c}}$

$$|P_{\bar{a}}| + |P_{\bar{e}}| = |P_{\bar{c}}| \Leftrightarrow \mathcal{R}_{wtt} \models \varphi_+(\bar{a}, \bar{e}, \bar{c}).$$

For the direction from left to right one uses that subsets of $P_{\bar{c}}$ are again EN-sets.

For $\varphi_{\times}(\bar{x}, \bar{y}, \bar{z})$ we express in terms of definable projection maps that $P_{\bar{z}}$ has the same size as the cartesian product $P_{\bar{x}} \times P_{\bar{y}}$. Thus $\varphi_{\times}(\bar{x}, \bar{y}, \bar{z})$ expresses

$$\begin{aligned} \exists C_1 \exists C_2 \quad & C_1 : P_{\bar{z}} \mapsto P_{\bar{x}} \text{ onto} \ \& \ C_2 : P_{\bar{z}} \mapsto P_{\bar{y}} \text{ onto} \ \& \\ & \forall a \in P_{\bar{x}} \ \forall b \in P_{\bar{y}} \ \exists ! q \in P_{\bar{z}} [C_1(q) = a \ \& \ C_2(q) = b]. \end{aligned}$$

Then, for finite $P_{\bar{a}}, P_{\bar{e}}, P_{\bar{c}}$

$$|P_{\bar{a}}| |P_{\bar{e}}| = |P_{\bar{c}}| \Leftrightarrow \mathcal{R}_{wtt} \models \varphi_{\times}(\bar{a}, \bar{e}, \bar{c}).$$

Recognizing finiteness

To recognize in a first-order way that an EN-set coded by two parameters is finite, the idea is to compare EN-sets to fragments of a uniformly definable subclass of the ID-sets. ID-sets are not as easy to construct as EN-sets, but a more involved construction actually yields a u.c.e. *infinite* ID-set

$$Z^* = \{\mathbf{a}_i : i \in \mathbb{N}\}.$$

To specify the uniformly definable subclass of the class of ID-sets we will impose conditions on parameters \mathbf{c}, \mathbf{d} coding $Z = Z_{\mathbf{c}, \mathbf{d}}$ which are satisfied by Z^* and imply that

1. when \mathbf{x} ranges through degrees $\leq \mathbf{c}, \mathbf{d}$, then $|Z \cap [\mathbf{o}, \mathbf{x}]|$ assumes all finite cardinalities
2. if $|P| = |Z \cap [\mathbf{o}, \mathbf{x}]|$, $\mathbf{x} \leq \mathbf{c}, \mathbf{d}$, then a bijection between the two sets can be uniformly defined.

ID-sets Z satisfying the conditions will be called *good*. For the special good ID-set $Z^* = Z_{\mathbf{c}, \mathbf{d}}^*$, $Z^* \cap [\mathbf{o}, \mathbf{x}]$ is finite for $\mathbf{x} \leq \mathbf{c}, \mathbf{d}$. The formula φ_{num} implies about P that for each good $Z_{\mathbf{c}, \mathbf{d}}$, a bijection between P and some $|Z \cap [\mathbf{o}, \mathbf{x}]$, $\mathbf{x} \leq \mathbf{c}, \mathbf{d}$, exist.

The set Z^* is obtained by referring to a rather hard theorem in Ambos-Spies and Soare [6]. To ensure property (2.) above, one has to make all the degrees \mathbf{a}_i low. An easier result in Lempp and Nies [33] could also be used, but has the disadvantage that the actual construction needs to be modified in order to make the degrees \mathbf{a}_i low.

Main Lemma 7.3.2 ([6]) *There exists a u.c.e. sequence $(A_i)_{i \in \mathbb{N}}$ such that each A_i is low, A_i, A_j form a T -minimal pair for $i \neq j$ and, where $\mathbf{a}_i = \text{deg}_{wtt}(A_i)$, $nd[\mathbf{o}, \mathbf{a}_i]$ for each i . Thus $Z^* = \{\mathbf{a}_i\}$ is an ID-set.*

Proof. Recall that noncomputable c.e. set C is *non-bounding* if there is no minimal pair A, B such that $A, B \leq_T C$. This definition makes sense also for *wtt*-reducibility. Clearly, C is *wtt*-non-bounding iff $nd[\mathbf{o}, \mathbf{d}]$ for each $\mathbf{d} \leq \mathbf{c} = \text{deg}_{wtt}(C)$.

In Ambos-Spies e.a. [5], Lemma 6, it is proved that each non-bounding C is also *wtt*-non-bounding. From Ambos-Spies and Soare [6] one obtains a u.c.e. sequence (A_i) such that each A_i is T -non-bounding and A_i, A_j form a T -minimal pair for $i \neq j$. Since there is a uniform construction to produce from a given c.e. set A a low set \tilde{A} such that \tilde{A} is non-computable if A is [56], we can assume that each set A_i is low. \diamond

Definition 7.3.3 *An ID-set Z defined from parameters \mathbf{c}, \mathbf{d} is good if*

$$(i) \quad \forall \mathbf{x} \leq \mathbf{c}, \mathbf{d} (Z \not\subseteq [\mathbf{o}, \mathbf{x}])$$

$$(ii) \quad \forall \mathbf{x} \leq \mathbf{c}, \mathbf{d} \exists \tilde{P}$$

$$(7.4) \quad \{ \langle \mathbf{u}, \mathbf{v} \rangle : \mathbf{u} \leq \mathbf{v} \ \& \ \mathbf{u} \in Z \cap [\mathbf{o}, \mathbf{x}] \ \& \ \mathbf{v} \in \tilde{P} \}$$

is a bijection between $Z \cap [\mathbf{o}, \mathbf{x}]$ and \tilde{P} .

Clearly being good can be expressed by a first-order condition on \mathbf{c}, \mathbf{d} . Moreover, (i) implies that Z is infinite: else $Z \subseteq [\mathbf{o}, \text{sup } Z] \leq \mathbf{c}, \mathbf{d}$.

We will prove that any u.c.e ID-set Z of low *wtt*-degrees is good, when defined from an exact pair for the Σ_3^0 -ideal generated by Z . In particular the set $Z^* = \{\mathbf{a}_i\}$ from the Main Lemma 7.3.2 is good. Assuming this fact, we now give a first order condition on parameters expressing finiteness of an EN-set P .

Lemma 7.3.4 P is finite $\Leftrightarrow \forall \mathbf{a}, \mathbf{b} [Z_{\mathbf{a}, \mathbf{b}} \text{ good} \Rightarrow \exists \mathbf{x} \leq \mathbf{a}, \mathbf{b}]$

$\exists \tilde{P}[(7.4) \text{ is a bijection} \ \& \ \exists C \ C \text{ is bijection } P \leftrightarrow \tilde{P}]$.

Proof. For the direction from left to right, assume that P is finite. Because good ID-sets are infinite, we can choose $F \subseteq Z$ such that $|F| = |P|$. Let $\mathbf{x} = \sup F$ and choose \tilde{P} satisfying (7.4). By Proposition 7.2.4, a bijection $P \leftrightarrow \tilde{P}$ can be coded via S_C .

For the other direction, let \mathbf{a}, \mathbf{b} be an exact pair coding the set Z^* obtained from the Main Lemma 7.3.2. If $\mathbf{x} \leq \mathbf{a}, \mathbf{b}$, then $\mathbf{x} \leq \mathbf{a}_0, \dots, \mathbf{a}_n$ for some n . By Lemma 7.1.1, $\mathbf{a}_k \wedge \mathbf{x} = \mathbf{o}$ for all $k > n$, so $Z \cap [\mathbf{o}, \mathbf{x}]$ is finite. Thus P is finite. \diamond

Finally we prove that any infinite u.c.e. ID-set Z of low wtt-degrees is good. Let Z such a set, coded by an exact pair \mathbf{a}, \mathbf{b} . By a similar argument as above, $Z \not\subseteq [\mathbf{o}, \mathbf{x}]$ for any $\mathbf{x} \leq \mathbf{a}, \mathbf{b}$. Since all degrees in Z are low, it is now sufficient to prove the following.

Lemma 7.3.5 Suppose that $\mathbf{a}_0, \dots, \mathbf{a}_n$ are low pairwise incomparable degrees in \mathcal{R}_{wtt} . Then there is an EN-set $\mathbf{v}_0, \dots, \mathbf{v}_n$ such that

$$\mathbf{a}_i \leq \mathbf{v}_j \Leftrightarrow i = j.$$

Proof. Choose c.e. sets $A_i \in \mathbf{a}_i$. We construct c.e. sets V_j such that the statement of the theorem holds with $\mathbf{v}_j = \deg_{wtt}(A_j \oplus V_j)$. Clearly $\mathbf{a}_i \leq \mathbf{v}_i$. To ensure $\mathbf{a}_i \not\leq \mathbf{v}_j$ for $i \neq j$, we meet the requirements

$$N_{e,i,j} : A_i \neq [e]^{A_j \oplus V_j} \ (i \neq j),$$

by the same strategy as in the proof of Proposition 7.2.2: refrain from changing V_j till a permanent disagreement occurs. We will define some priority listing $(R_k)_{k \in \mathbb{N}}$ of all the requirements. If R_k is $N_{e,i,j}$ let

$$\text{length}(k, s) = \min\{x : \forall y < x \ A_i(y) = [e]^{A_j \oplus V_j}(y)[s],$$

and let $r(k, s) = \max\{[e]_s(y) : y < \text{length}(k, s)\}$.

To achieve $\mathbf{v}_j \vee \mathbf{v}_{j'} = \mathbf{1}$ ($j' \neq j$) as in Proposition 7.2.2 we ensure that $K = V_j \cup V_{j'}$. For $nd[\mathbf{v}_j, \mathbf{1}]$, we make each $A_j \oplus V_j$ low and apply Theorem 7.2.1. Lowness is achieved by the side effects of the “pseudo- lowness requirements”

$$L_{e,j} : \exists^\infty s \ \{e\}^{A_j \oplus V_j}(e)[s] \text{ is defined} \Rightarrow \{e\}^{A_j \oplus V_j}(e) \text{ converges.}$$

While $L_{e,j}$ may fail to be met, it will produce enough restraint to ensure $(A_j \oplus V_j)' \equiv_T \emptyset'$. We use a standard technique introduced by Robinson. By the recursion theorem, we can assume that the sets V_0, \dots, V_n with specific enumerations are *given* (see comment at the end). Since each set A_i ($i \leq n$) is low, the following property of e, j and a stage number \tilde{s} can be checked with an oracle \emptyset' :

$$(7.5) \quad \exists s \geq \tilde{s} [\{e\}^{A_j \oplus V_j}(e)[s] \text{ is defined via an } A_j \text{-correct computation}].$$

By the Limit Lemma ([56]) we can fix a computable function $g(\tilde{s}, e, j, t)$ such that $\lim_t g(\tilde{s}, e, j, t)$ exists, has value 0 or 1, and the limit is 1 iff (7.5) holds. Let (R_k) be some priority listing of all the requirements.

Construction. At Stage 0 initialize all the lowness requirements.

Stage $s + 1$. First determine the restraint $r(k, s)$ for all $k < s$ such that R_k is a lowness requirement $L_{e,j}$. Let $\tilde{s} < s$ be greatest such that R_k was initialized at \tilde{s} . If $\{e\}^{A_j \oplus V_j}(e)[\tilde{s}]$ is undefined, let $r(k, s) = 0$. Else let u be the use of this computation and find the least $t \geq \tilde{s}$ such that either

- (1) $A_{j,t+1}|u \neq A_{j,t}|u$, or
- (2) $g(\tilde{s}, e, i, t) = 1$.

Since $\lim_t g(\tilde{s}, e, j, t) \Leftrightarrow (7.5)$ holds and the computation at s seems to provide a witness for (7.5), one of the two cases has to apply. In Case (1) let $r(k, s) = 0$, and in Case (2) $r(k, s) = u$.

Now, if $K_s = K_{s+1}$ terminate stage $s + 1$ here. Else, say y is the unique element in $K_{s+1} - K_s$. Determine the minimal k such that $y < r(k, s)$. If k fails to exist enumerate y into all sets V_j . Else let j be the number such that $R_k = L_{e,j}$ or $R_k = N_{e,i,j}$ for some e, i . Enumerate y into $V_{j'}$, for each $j' \neq j$. Initialize all the lowness requirements $R'_k, k' > k$. This completes the description of the construction.

Lemma 7.3.6 *Let $k \geq 0$.*

- (i) *If R_k is $N_{e,i,j}$, then the requirement R_k is met.*
- (ii) *$r(k) = \lim_s r(k, s)$ exists and is finite.*

Proof. Assume the Lemma holds for all $h < k$. Choose a stage s_0 such that for all $h < k$, $r(h, s_0)$ has reached the limit, and K does not change below $\max_{h < k} r(h)$ at any stage $s \geq s_0$.

If R_k is $N_{e,i,j}$, we can prove (i) and (ii) as in Proposition 7.2.2. In particular, if $A_i = [e]^{A_j \oplus V_j}$, then one can obtain a *wtt* reduction procedure of A_i to A_j , contrary to the assumption that $\mathbf{a}_i, \mathbf{a}_j$ are incomparable.

Now suppose that R_k is $L_{e,j}$. We have to show that $\lim_s r(k, s)$ is finite. Let \tilde{s} be the greatest stage where R_k is initialized (necessarily $\tilde{s} \leq s_0$), and pick $s \geq s_0$ where $g(\tilde{s}, e, j, s)$ has reached its limit. If the limit is 0, then $r(k, t) = 0$ for all $t \geq s$. Else, by (7.5) and the definition of g there is a least stage $t \geq \tilde{s}$ such that $\{e\}^{A_j \oplus V_j}(e)[t]$ is defined via an A_j -correct computation with use u . Then at stage t we define $r(k, t) = u$. Since R_k is not initialized at stages $> \tilde{s}$, the computation $\{e\}^{A_j \oplus V_j}(e)[t]$ is preserved. So $r(k, s) = u$ for all $s \geq t$. \diamond

Lemma 7.3.7 $A_j \oplus V_j$ is low for each $j \leq n$.

Proof. Given e , we have to determine with a \emptyset' -oracle whether $\{e\}^{A_j \oplus V_j}(e)$ converges. Let k be such that R_k is $L_{e,j}$. Note that, in the preceding argument, we can determine \tilde{s} using a \emptyset' -oracle. Then, by (7.5),

$$\lim_t g(\tilde{s}, e, j, t) = 0 \Rightarrow \{e\}^{A_j \oplus V_j}(e) \text{ diverges,}$$

and by the argument above,

$$\lim_t g(\tilde{s}, e, j, t) = 1 \Rightarrow \{e\}^{A_j \oplus V_j}(e) \text{ converges.}$$

The use of the recursion theorem deserves a comment: We are given some c.e. sets V_0, \dots, V_n via a partial recursive enumeration function ψ which maps s to a strong index for $V_0 \oplus \dots \oplus V_n[s]$. From this the construction produces a similar enumeration $\tilde{\psi}$ for sets $\tilde{V}_0, \dots, \tilde{V}_n$. By the recursion theorem, there must be ψ such that $\tilde{\psi} = \psi$, and in particular $V_j = \tilde{V}_j$ for $j \leq n$. The function g actually contains an extra argument, namely an index for ψ , and in the discussion above we assume that the extra argument is an index such that $\tilde{\psi} = \psi$. \diamond

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Bibliography

- [1] K. Ambos-Spies. An extension of the nondiamond theorem in classical and α -recursion theory. *J. Symbolic Logic*, 49:586–607, 1984[91].
- [2] K. Ambos-Spies. Inhomogeneities in the polynomial time degrees. *Information Processing Letters*, 22:113–117, 1986 [4,74].
- [3] K. Ambos-Spies, P. A. Fejer, S. Lempp, and M. Lerman. Decidability of the two-quantifier theory of the recursively enumerable weak truth-table degrees and other distributive upper semi-lattices. *Journal of Symbolic Logic*, 61(3):880–905, 1996 [95].
- [4] K. Ambos-Spies and A. Nies. The theory of polynomial many-one degrees of recursive sets is undecidable. In *STACS 92*, volume 577 of *Lecture Notes in Computer Science*, pages 209–210. Springer-Verlag, 1992 [4].
- [5] K. Ambos-Spies, A. Nies, and R. A. Shore. The theory of the recursively enumerable weak truth-table degrees is undecidable. *J. Symbolic Logic*, 57:864–74, 1992 [90,91,97].
- [6] K. Ambos-Spies and R.I. Soare. The recursively enumerable degrees have infinitely many one types. *Ann. Pure Appl. Logic*, 44:1–23, 1989 [97].
- [7] K. Ambos-Spies and D. Yang. Honest polynomial-time degrees of elementary recursive sets. In *CSL: 3rd Workshop on Computer Science Logic*. LNCS, Springer-Verlag, 1990 [80].
- [8] T. Baker, J. Gill, and R. Solovay. Relativizations of the $\mathcal{P} =? \mathcal{NP}$ question. *SIAM Journal on Computing*, 4(4):431–442, December 1975 [81,83].
- [9] J. L. Balcázar, J. Díaz, and J. Gabarró. *Structural Complexity I*, volume 11 of *EATCS Monographs on Theoretical Computer Science*. Springer-Verlag, Heidelberg, 1988.
- [10] S. Burris and H. P. Sankappanavar. *A course in universal algebra*. Springer-Verlag, Berlin, 1981 [8,44].
- [11] C. C. Chang and H. J. Keisler. *Model Theory*. North-Holland Publishing Co., Amsterdam, 1973 [28,41].

- [12] P. Cholak and A. Nies. Atomless r -maximal sets. *Israel Journal of Math.* To appear [32].
- [13] S. D. Denisov. The structure of the uppersemilattice of recursively enumerable m -degrees and related questions, I. *Algebra and Logic*, 17:418–443, 1972 [2,26,28].
- [14] R. Downey and A. Nies. Undecidability results for low complexity degree structures. In *Proceedings of the 12th Annual IEEE Conference on Computational Complexity*, pages 128–132, 1997. Extended version to appear in JCSS [4].
- [15] Y. Ershov and L. Lavrov. The upper semilattice $L(\gamma)$. *Alg. Log. (transl)*, 12:93–106, 1973 [20].
- [16] Feferman and Vaught. The first-order properties of algebraic systems. *Fund. Math.*, 47:57–103, 1959 [55,57].
- [17] L. Feiner. Hierarchies of boolean algebras. *J. Symbolic Logic*, 35:365–74, 1974 [64].
- [18] R. M. Friedberg. Three theorems on recursive enumeration. I. decomposition. II. maximal set. III. enumeration without duplication. *J. Symbolic Logic*, 23:309–316, 1958 [31,37].
- [19] K. Gödel. Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I. *Monatsh. Math. Phys.*, 38:349–360, 1931.
- [20] Y. Gurevich. Decision problem for separated distributive lattices. *J. Symbolic Logic*, 48:193–6, 1983 [49].
- [21] L. A. Harrington and A. Nies. Coding in the lattice of enumerable sets. *Adv. in Math.*, 133:133–162, 1998 [ii,47,49].
- [22] L. A. Harrington and S. Shelah. The undecidability of the recursively enumerable degrees. *Bull. Amer. Math. Soc.*, 6:79–80, 1982 [i].
- [23] L. A. Harrington and R. Soare. Post’s program and incomplete recursively enumerable sets. *Proc. Nat. Acad. Sci. U.S.A.*, 88:10242–6, 1991 [3].
- [24] H. Heller. Relativized polynomial hierarchies extending two levels. *Mathematical Systems Theory*, 17(2):71–84, 1984 [80].
- [25] E. Herrmann. The undecidability of the elementary theory of the lattice of recursively enumerable sets (abstract). *Proceedings of the 2nd Frege Conference at Schwerin, GDR*, 20:66–72, January 1984.
- [26] W. Hodges and A. Nies. Interpreting infinite linear orders. In J.A. Makowsky and E.V. Ravve, editors, *Logic Colloquium ’95*, volume 11 of *Lecture Notes in Logic*, pages 247–251, 1997 [56].
- [27] A. H. Lachlan. The elementary theory of the lattice of recursively enumerable sets. *Duke Math. J.*, 35:123–146, 1968 [49,49].

- [28] A. H. Lachlan. On the lattice of recursively enumerable sets. *Trans. Amer. Math. Soc.*, 130:1–37, 1968 [32,32,42,60].
- [29] A. H. Lachlan. Initial segments of many-one degrees. *Canad. J. Math.*, 22:75–85, 1970 [8,28,72].
- [30] Alistair H. Lachlan. Embedding nondistributive lattices in the recursively enumerable degree. In W. Hodges, editor, *Conference in Mathematical Logic, London, 1970*, volume 255 of *Lecture Notes in Mathematics*, pages 149–177, Heidelberg, 1972 [2]. Springer–Verlag.
- [31] R. Ladner. On the structure of polynomial time reducibility. *J. Assoc. Comput. Mach.*, 22(1):155–171, January 1975 [4].
- [32] L. Landweber, R. Lipton, and E. Robertson. On the structure of NP and other complexity classes. *J. Theoretical Comp. Sc.*, 15:181–200, 1981 [4,77].
- [33] S. Lempp and A. Nies. Undecidability of the 4-quantifier theory for the recursively enumerable Turing and wtt degrees. *J. Symbolic Logic*, 60:1118–35, 1985 [95,97].
- [34] S. Lempp, A. Nies, and T.A. Slaman. The Π_3 -theory of the enumerable Turing degrees is undecidable. *Trans. Amer. Math. Soc.* To appear [i].
- [35] M. Lerman. *Degrees of Unsolvability*. Perspectives in Mathematical Logic. Springer–Verlag, Heidelberg, 1983. 307 pages.
- [36] W. Maass and M. Stob. The intervals of the lattice of recursively enumerable sets determined by major subsets. *Ann. Pure Appl. Logic*, 24:189–212, 1983 [71].
- [37] D. A. Martin. Classes of recursively enumerable sets and degrees of unsolvability. *Z. Math. Logik Grundlag. Math.*, 12:295–310, 1966 [3].
- [38] D. A. Martin and M. B. Pour-El. Axiomatizable theories with few axiomatizable extensions. *J. Symbolic Logic*, 35:205–209, 1970 [3].
- [39] F. Montagna and A. Sorbi. Universal recursion theoretic properties of r.e. preordered structures. *J. Symbolic Logic*, 50:397–406, 1985 [65].
- [40] A. Nies. Effectively dense boolean algebras and their applications. In preparation [63,71].
- [41] A. Nies. Recursively enumerable equivalence relations modulo finite differences. *Math. Logic Quarterly*, 5:490–518, 1994 [14].
- [42] A. Nies. The last question on recursively enumerable many-one degrees. *Algebra i Logika*, 33(5):550–563, 1995. English Translation July 1995 [13].
- [43] A. Nies. Relativizations of structures from recursion theory. In *Proceedings of the Leeds logic year*, Lecture Notes in Logic, pages 219–232. Springer–Verlag, 1996 [51].

- [44] A. Nies. Undecidable fragments of elementary theories. *Algebra Universalis*, 35:8–33, 1996 [15].
- [45] A. Nies. Intervals of the lattice of computably enumerable sets and effective boolean algebras. *Bull. Lond. Math. Soc.*, 29:683–92, 1997.
- [46] A. Nies and R. A. Shore. Interpreting true arithmetic in the theory of the r.e. truth table degrees. *Ann. Pure Appl. Logic*, 75:269–311, 1995 [14].
- [47] A. Nies, R. A. Shore, and T. A. Slaman. Definability in the recursively enumerable degrees. *Bull. of Symbolic Logic*, 4:392–404, 1996. Full version to appear in Proc. Lond. Math. Soc. [i,14].
- [48] P. Odifreddi. *Classical Recursion Theory (Volume II)*. North-Holland Publishing Co., Amsterdam. To appear.
- [49] P. Odifreddi. Strong reducibilities. *Bull. Amer. Math. Soc.*, 4(1):37–86, 1981.
- [50] E. L. Post. Recursively enumerable sets of positive integers and their decision problems. *Bull. Amer. Math. Soc.*, 50:284–316, 1944 [2].
- [51] Gerald E. Sacks. The recursively enumerable degrees are dense. *Ann. of Math. (2)*, 80:300–312, 1964.
- [52] Juichi Shinoda and T. A. Slaman. On the theory of the *PTIME* degrees of the recursive sets. *J. Comput. System Sci.*, 40:321–336, 1991 [4].
- [53] R. A. Shore. The theory of the degrees below $0'$. *J. London Math. Soc.*, 24:1–14, 1981 [14,51].
- [54] T. A. Slaman and W. H. Woodin. Definability in the Δ_2^0 turing degrees. 1989 [19,56].
- [55] R. I. Soare. Automorphisms of the lattice of recursively enumerable sets I: maximal sets. *Ann. of Math. (2)*, 100:80–120, 1974 [3,60].
- [56] R. I. Soare. *Recursively Enumerable Sets and Degrees*. Perspectives in Mathematical Logic, Omega Series. Springer-Verlag, Heidelberg, 1987.
- [57] A. Tarski. On essential undecidability. *J. Symbolic Logic*, 14:75–76, 1949.
- [58] C. E. M. Yates. A minimal pair of recursively enumerable degrees. *J. Symbolic Logic*, 31:159–168, 1966.