NON-CUPPING AND RANDOMNESS

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ABSTRACT. Let $Y \in \Delta_2^0$ be Martin-Löf-random. Then there is a promptly simple set A such that, for each Martin-Löf-random set Z, $Y \leq_T A \oplus Z \Rightarrow Y \leq_T Z$. When $Y = \Omega$, one obtains a c.e. non-computable set A which is not weakly Martin-Löfcuppable. That is, for any Martin-Löf-random set Z, if $\emptyset' \leq_T A \oplus Z$ then $\emptyset' \leq_T Z$.

1. INTRODUCTION

The interaction between K-triviality and Martin-Löf randomness via Turing reducibility is not very well understood at present. There are numerous results indicating a strong interaction, but as many open questions remain. Recall that a set $A \subseteq \mathbb{N}$ is K-*trivial* if, up to a constant, the initial segments of A have minimum complexity, namely $(\forall n)\ K(A \restriction n) \leq K(n) + \mathcal{O}(1)$. (Here $K(x)$ is the prefix free complexity of x, and one identifies a string σ in $2^{<\omega}$ with the natural number n such that the binary representation of $n + 1$ is 1σ.) This notion is opposite to Martin-Löf-randomness (ML-randomness, for short), since Z is ML-random iff there is a constant b such that $(\forall n)\ K(Z \restriction n) \geq n - b$, namely the initial segments of Z have close to maximal complexity. K-trivial sets have been studied for instance in [3, 11]. An example of an interaction between K-triviality and Martin-Löf randomness via Turing reducibility is the following result from [4].

Theorem 1.1. *Let A be c.e. If $A \leq_T Z$ for some ML-random set Z such that $\emptyset' \not\leq_T Z$, then A is K-trivial.*

In [5], Kučera provides an injury-free solution of Post's Problem. To do so, given any Δ_2^0 ML-random set Z (and in fact any Δ_2^0 diagonally non-computable set), he builds a non-computable c.e. set $A \leq_T Z$. Then, since there is a low ML-random set, a non-computable set A satisfying the hypotheses of the theorem exists. It is an open question whether each K-trivial set is of this kind [8, Question 4.6].

Here I address the case where the interaction between the K-trivial and the ML-random set is not having join above $\emptyset'$. If A is c.e. and not K-trivial, then $A \oplus Z = \emptyset'$ for some ML-random $Z <_T \emptyset'$, see Theorem 2.2 below. We say that such a set A is ML-cuppable. Thus, in analogy to Theorem 1.1, if a c.e. set A is not ML-cuppable then A is K-trivial. (The hypotheses are quite different, though: in Theorem 1.1 the *existence* of an incomplete Martin-Löf-random set above A is required, while here we require that *no* incomplete Martin-Löf- random set cups with A to $\emptyset'$.)

How about existence of a non-computable A in the non-cupping case? That is, is there a c.e. non-computable set which is not ML-cuppable? Our main result provides an affirmative answer. In fact one may choose A promptly simple. We obtain the answer as a corollary to a more general non-cupping result: if Y is Δ_2^0 and ML-random, then there is a promptly simple set A which does not help any ML-random set Z to compute A, namely $Y \leq_T A \oplus Z \ \Rightarrow \ Y \leq_T Z$. Now let $Y = \Omega$ be Chaitin's halting probability, a Turing complete ML-random set. Note that each promptly simple set cups to $\emptyset'$ by a low c.e. set [12, Thm XIII.4.2]. So A is not ML-cuppable but cups to $\emptyset'$ in the usual sense.

It is still an open question whether *all* K-trivial sets fail to be ML-cuppable [8, Question 4.8]. We will return to this question in the last Section, after Corollary 3.5.

2. Background

The notion of ML-cuppability comes in two versions, depending on whether the cupping partner is merely not above $\emptyset'$, or strictly below $\emptyset'$.

Definition 2.1. A set A is *weakly ML-cuppable* if $A \oplus Z \geq_T \emptyset'$ for some ML-random set $Z \not\geq_T \emptyset'$. A is *ML-cuppable* if one can choose $Z <_T \emptyset'$.

Kučera drew attention to Martin-Löf cuppability during his talk at a 2004 Cordoba meeting. In particular, he raised the question of which Δ_2^0 sets are (weakly) ML-cuppable, and whether one of the notions is equivalent to K-triviality (see [8, Question 4.8]). Quite a bit is known already. Let Ω^A be Chaitin's halting probability relativized to A (see [1]). If the Δ_2^0 set A is not K-trivial, then A is weakly ML-cuppable, via $Z = \Omega^A$. For $Z \not\geq_T \emptyset'$ by Theorem 1.1, and $\emptyset' \leq_T A' \equiv_T A \oplus \Omega^A$. If A is low then in fact $\Omega^A <_T \emptyset'$. Thus, each Δ_2^0 set A with a low non-K-trivial set below it is ML-cuppable. This includes:

  a) any ML-random set A, since $A \cap 2\mathbb{N}$ is low [1, Thm. 3.4]
  b) any non-low$_2$ and any c.e.a. non K-trivial set, as each one of these is the supremum of a pair of 1-generic sets, and each 1-generic set is generalized low (see [7, Ex. IV.3.15] for the first)
  c) any c.e. non-K-trivial set A, because A is a disjoint union of c.e. low sets A_0, A_1, and at least one of them is not K-trivial [3].

By c) we have an analog of Theorem 1.1.

Theorem 2.2. *If a c.e. set A is not ML-cuppable, then A is K-trivial.*

The question whether each non K-trivial Δ_2^0 set is ML-cuppable is still not completely settled. An interesting case of a non K-trivial Δ_2^0 set which is not ML-cuppable by any of the reasons above is a set of minimal degree which is properly low$_2$ [7, Ex. IX.2.7].

Notation. We let μ denote the usual product measure in Cantor space 2^ω. The variable σ denotes a string in $2^{<\omega}$. For a set $Z \in 2^\omega$, $Z \upharpoonright n$ denotes $Z(0) \ldots Z(n-1)$. For $G \subseteq 2^{<\omega}$, we let $[G]^{\preceq}$ denote the open set generated by G, that is, $\{Z : \exists \sigma \in G \ \sigma \prec Z\}$. We write μG for $\mu([G]^{\preceq})$.

For a set A and a string σ, we let $A \oplus \sigma$ be the string ρ of length $2|\sigma|$ such that $\rho(2i) = A(i)$ and $\rho(2i+1) = \sigma(i)$. Thus if Γ is a Turing functional, then $\Gamma^{A \oplus \sigma}(x)$ is defined only when the use on the A-side is at most $|\sigma|$.

Given an expression E that is approximated during stages s, $E[s]$ denotes its value at stage s.

A *Solovay test* is a c.e. set $\mathcal{S}$ of strings such that the *weight* $\sum_{\sigma \in \mathcal{S}} 2^{-|\sigma|}$ is finite. It is known that Z is ML-random iff for each Solovay test $\mathcal{S}$, $\sigma \not\prec Z$ for almost all $\sigma \in \mathcal{S}$ [2].

A c.e. set A is *promptly simple* if A is co-infinite and there is an effective approximation $(A_s)_{s \in \mathbb{N}}$ of A such that, for each e, the requirement

$$S_e: \quad |W_e| = \infty \Rightarrow \exists s \exists x \, [x \in W_{e,s} - W_{e,s-1} \ \& \ x \in A_s]$$

is met.

3. Main result

Our main theorem implies the existence of a non-computable c.e. set A which is not ML-cuppable. By Theorem 2.2, such a set A is K-trivial.

Theorem 3.1. *Let* $Y \in \Delta_2^0$ *be Martin-Löf-random. Then there is a promptly simple set* A *such that, for each Martin-Löf-random set* Z,

$$(3.1) \qquad Y \leq_T A \oplus Z \ \Rightarrow \ Y \leq_T Z.$$

Proof.
I use [9, Lemma 4.2], in relativized form.

Lemma 3.2 ([9])**.** *Let* Y *be ML-random relative to* A*. Then for each partial computable functional* Γ*, there is a constant* c *such that for each* n,

$$(3.2) \qquad \mu\{\sigma : Y \restriction n = \Gamma^{A \oplus \sigma} \restriction n\} \leq 2^{-n+c}.$$

As the authors of [9] have pointed out, this is best proved via a supermartingale whose undergraph is c.e. in A. Let

$$M(\eta) = 2^{|\eta|} \mu\{\sigma : \Gamma^{A \oplus \sigma} \succeq \eta\}.$$

Then, for each η, $M(\eta 0) + M(\eta 1) \leq 2M(\eta)$, that is, M is a supermartingale. Moreover, $\{\langle q, \eta \rangle : q \in \mathbb{Q} \ \& \ q \leq M(\eta)\}$ is c.e. relative to A. Since Y is ML-random relative to A, there is c such that $M(\eta) \leq 2^c$ for each η, which is (3.2).

Proof of Theorem 3.1, outline. Fix some effective approximation $(Y_s)_{s \in \mathbb{N}}$ of Y. To ensure that Y is ML-random relative to A, we make A K-trivial; then A is low for ML-random (namely each ML-random set is already ML-random relative to A) by [11, Thm 6.2]. To ensure that A is promptly simple, we meet the requirements S_e introduced above. Finally, we meet the non-cuppability requirements

$$N_{\Gamma,c} : Y = \Gamma(A \oplus Z) \ \& \ c \text{ is as in (3.2)} \ \& \ Z \text{ ML-random} \ \Rightarrow \ Y \leq_T Z.$$

Throughout, N will denote some non-cuppability requirement. We define a Δ_2^0 set $\Delta = \Delta_N$, which is an attempt to build a functional for $Y \leq_T Z$ by emulating Γ. Thus, when $Y \restriction n = \Gamma^{A \oplus \sigma} \restriction n$, we put the pair $\langle \sigma, Y \restriction n \rangle$ into Δ. When $A \restriction |\sigma|$ changes later, then this pair is removed from Δ. At each stage, Δ has the monotonicity properties of a functional, namely $\sigma_0 \preceq \sigma_1 \ \& \ \langle \sigma_i, \eta_i \rangle \in \Delta \Rightarrow \eta_0 \preceq \eta_1$. We also enumerate a Solovay test $\mathcal{S} = \mathcal{S}_N$ for each non-cuppability requirement N, which represents the "mistakes" we make due to A changes: when a pair $\langle \sigma, y \rangle$ is removed from Δ, then we put σ into $\mathcal{S}$. If the hypothesis of N is correct and we can ensure that $\mathcal{S}$ indeed is a Solovay test, then, since Z is ML-random, $\sigma \notin \mathcal{S}$ for almost all $\sigma \prec Z$. Thus, eventually a pair $\langle \sigma, Y \restriction n \rangle$ such that $\sigma \prec Z$ is not

removed from Δ, and we can pretend Δ is c.e. just like a Turing functional, which shows $Y \leq_T Z$.

Fix some effective priority ordering of the requirements of the form $N = N_{\Gamma,c}$ and the prompt simplicity requirements S_e. To ensure $\mathcal{S}_N$ is a Solovay test we have to minimize the number of A-changes. A requirement S_e can at most add weight 2^{-e} to $\mathcal{S}_N$, for any requirement N of higher priority. To give S_e a chance, for each string y of length n, at each stage s we only allow a maximum measure of 2^{-n+c+2} for the set of σ's such that $\langle \sigma, y \rangle$ is in $\Delta_{N,s}$. Lemma 3.2 ensures that still we don't miss the right σ, namely the shortest $\sigma \prec Z$ such that $\Gamma^{A \oplus \sigma} \restriction n = Y \restriction n$.

Details. The *construction* consists in letting the requirements at stage s carry out their strategies below, in the order of their priority, up to $s - 1$.

Strategy for $N = N_{\Gamma,c}$. Let $\Delta_{N,0} = \mathcal{S}_{N,0} = \emptyset$. At stage $s > 0$,

- (*adding to Δ*) For $n = 0, \ldots, s-1$, do the following: while

$$(3.3) \qquad \mu\{\sigma : \langle \sigma, Y_{s-1} \restriction n \rangle \in \Delta_{N,s-1}\} \leq 2^{-n+c+2},$$

    if there is $\sigma \notin \mathrm{dom}(\Delta_{N,s-1})$ such that
    - $\langle \sigma', Y_{s-1} \restriction n-1 \rangle \in \Delta_{N,s-1}$ for some $\sigma' \preceq \sigma$ and
    - $Y_{s-1} \restriction n = \Gamma_{s-1}^{A_{s-1} \oplus \sigma} \restriction n$

    then let σ be such a string of minimal length and put $\langle \sigma, Y_{s-1} \restriction n \rangle$ into Δ_N.
- (*removing from Δ*) If $\langle \sigma, y \rangle$ entered Δ at stage $t < s$ and s is minimal such that $A_s \restriction |\sigma| \neq A_t \restriction |\sigma|$, then remove $\langle \sigma, y \rangle$ from Δ, and put σ into $\mathcal{S}_N$.

Strategy for S_e. Let $K_t(y)$ be the approximation of $K(y)$ at stage t, and let $cost(x,t) = \sum_{x < y \leq t} 2^{-K_t(y)}$ be the usual cost function from [3] (also see [11]). For each $N = N_{\Gamma,c}$, let

$$cost_N(x,t) = \mu\{\sigma : \exists y \, \langle \sigma, y \rangle \in \Delta_{N,t-1} \ \& \ x < |\sigma|\}$$

be the cost, in the sense of increasing the weight of N's Solovay test $\mathcal{S}_N$, one would incur by putting x into A at stage t. At stage s, suppose S_e is not satisfied and there is $x \in W_{e,s} - W_{e,s-1}$, $x \geq 2e$, such that $cost(x,s) \leq 2^{-e-1}$ and $cost_N(x,s) \leq 2^{-e}$ for each requirement of type N of higher priority. If $x \notin A_s$ yet then put x into A_s. Declare S_e satisfied.

Verifications. A is K-trivial by the usual argument involving the cost function (see [3] or [11, Prop 4.1]). Also $\mathcal{S}_N$ is a Solovay test for each requirement N. To see this, note that each requirement S_e acts at most once. Thus the requirements of stronger priority than N contribute a finite weight, and each requirement S_e of lower priority than N contributes at most weight 2^{-e} to $\mathcal{S}_N$, so their total is at most 2.

Lemma 3.3. *A is promptly simple.*

Proof. Clearly A is co-infinite. Thus we only need to verify that each requirement S_e is met. It suffices to show that for each requirement of type N, for sufficiently large x,

$$(3.4) \qquad \forall s \geq x \ cost_N(x,s) \leq 2^{-e}.$$

For then, if $x \geq 2e$ enters W_e at stage s then x can be enumerated into A in case x is so large that $cost(x, s) \leq 2^{-e-1}$ and, for all N type requirements of stronger priority than S_e, $cost_N(x, s) \leq 2^{-e}$.

To show that (3.4) holds for sufficiently large x, we split the cost into two parts, depending on whether a computation $\Gamma^{A \oplus \sigma}(k)[s]$ for $k < e+c+3$ would be destroyed by the enumeration of x into A or not.

Bounding the cost of the first type. Fix i, and let $r \in \mathbb{N}$ be given. We show that, for sufficiently large x the cost incurred by enumerating x into A and thereby destroying a computation $\Gamma^{A \oplus \sigma}(k)$, $k < i$ (that is, removing a pair $\langle \sigma, y \rangle$ from Δ_N where $|y| = i$) can be kept below 2^{-r+1}. We later will apply this to all $i \leq e+c+3$, for $r = e+i+2$, so that the contribution is at most 2^{-i-e-1} for sufficiently large x. The argument is similar to the one of Kučera and Terwijn [6] in their construction of a non-computable low for ML-random set. Consider the open A-c.e. set $B_i^A = \{\sigma : \Gamma^{A \oplus \sigma} \restriction i \downarrow\}$, with the usual approximation $B_i^A[t] = \{\sigma : \Gamma_t^{A_t \oplus \sigma} \restriction i \downarrow\}$, and let

$$cost_i(x, s) = \mu\{\sigma \in B_i^A[s-1] : x < |\sigma|\}.$$

(By the convention in Section 2, $x < |\sigma|$ implies $x < \mathsf{use}\, \Gamma_s^{A_s \oplus \sigma} \restriction i$. Note that there is no dependence here between the costs for different i: for smaller i we have more pairs $\langle \sigma, y \rangle$ in Δ where $|y| = i$ but the use on A is smaller; for larger i there are fewer pairs but the use is larger.) We claim that, for each r there is x such that

$$\forall s \geq x \; cost_i(x, s) \leq 2^{-r+1}.$$

To show this, recall that each S_k acts at most once. So if no S_k, $k \leq r$, acts from stage t_0 on, then for each $t \geq t_0$, for each $s \geq t$, $\mu B_i^A[s] \geq \mu B_i^A[t] - 2^{-r}$. That is, $\mu B_i^A[t]$ can only decrease by 2^{-r} after stage t_0. Thus, for each $t \geq t_0$, $\mu B_i^A[t] \leq \mu B_i^A + 2^{-r}$. Now choose $x \geq t_0$ so that $\mu B_i^A[x] \geq \mu B_i^A - 2^{-r}$, with A_x correct on the use of the relevant computations $\Gamma^{A \oplus \sigma}(k)$, $k < i$. Then for all $s \geq x$, $cost_i(x, s) \leq 2^{-r+1}$.

Bounding the cost of the second type. Let s_0 be a stage so that $\widetilde{y} = Y_s \restriction e+c+3$ is stable from s_0 on. By the usual conventions, enumeration of $x \geq s_0$ into A cannot destroy any computation $\Gamma^{A \oplus \sigma}(k)$ existing at stage s_0. So if $x \geq s_0$ we only have to consider the contribution of pairs $\langle \sigma, y \rangle$ to the cost of the second type (i.e. where $|y| \geq e+c+3$) which entered Δ after stage s_0. In this case $\widetilde{y} \preceq y$. For $s \geq s_0$, let

$$F_s = \{\rho : \langle \rho, \widetilde{y} \rangle \in \Delta_s\}.$$

Then F_s is an antichain and, by (3.3), $\mu F_s \leq 2^{-(e+c+3)+c+2} = 2^{-e-1}$.

Now consider the situation where $x \geq s_0$ and a pair $\langle \sigma, y \rangle$ entered Δ at stage $u \geq s_0$ and is in Δ_t for all t, $u \leq t \leq s$. If $|\sigma| > x$ then this pair is counted in the calculation of $cost_N(x, s)$. If $|y| > e+c+3$, then by choice of s_0, $\widetilde{y} \preceq y$. In the construction, we ensured that $\langle \rho, \widetilde{y} \rangle \in \Delta_u$ for some $\rho \preceq \sigma$. Then $\langle \rho, \widetilde{y} \rangle \in \Delta_s$ since $A_u \restriction |\rho| = A_s \restriction |\rho|$, and hence $\sigma \in [F_s]^{\preceq}$. So one can bound the cost of the second type by μF_s.

Summarizing the two separate arguments we obtain (3.4): for x sufficiently large, $cost_N(x, s) \leq \sum_{i \leq e+c+3} 2^{-i-e-1} + 2^{-e-1} \leq 2^{-e}$. $\square$

Lemma 3.4. *If $Y = \Gamma(A \oplus Z)$ and Z is ML-random then $Y \leq_T Z$.*

Proof. Since A is K-trivial, Y is ML-random relative to A [11, Thm 7.2]. Let c be as in Lemma 3.2, and let $N = N_{\Gamma, c}$. First we make two observations.

(a) *For each p, a pair $\langle \sigma, Y \restriction p \rangle$ enters Δ_N, where $\sigma \prec Z$.*

To see this, informally speaking, we have to show that the measure condition (3.3) is not too restrictive. Choose t such that $Y_s \upharpoonright p = Y_t \upharpoonright p$ for each $s \geq t$, and no requirement S_k acts after t for $k \leq p$. For any $n \leq p$, by Lemma 3.2,

$$\mu\{\sigma : \Gamma^{A \oplus \sigma} \upharpoonright n = Y \upharpoonright n\} \leq 2^{-n+c}.$$

Then at any stage $s \geq t$, $\mu\{\sigma : \langle \sigma, Y \upharpoonright n \rangle \in \Delta_{N,s}\} \leq 2^{-n+c} + 2^{-n} \leq 2^{-n+c+1}$, because a requirement S_k, $k > p$, has to ensure that $cost_N(x,s) \leq 2^{-k}$ and hence $\mu\{\sigma : \langle \sigma, Y \upharpoonright n \rangle \in \Delta_{N,s}\}$ can decrease at most 2^{-n} after stage s. So by the first stage $s \geq t$ where $\Gamma_s^{A_s \oplus Z} \upharpoonright p = Y \upharpoonright p$, for all $n = 0, \ldots, p$ we may put a pair $\langle \sigma, Y \upharpoonright n \rangle$ where $\sigma \prec Z$ into Δ_N if it is not there already.

(b) *Since $\mathcal{S}_N$ is a Solovay test and Z is ML-random, there is s_0 such that if $\langle \sigma, y \rangle$ enters Δ_N at a stage $s \geq s_0$ where $\sigma \prec Z$, then $\sigma \notin \mathcal{S}_N$, and hence $\langle \sigma, y \rangle$ remains in Δ_N forever. Thus $A_s \upharpoonright |\sigma|$ is stable and hence $y \prec Y$.*

We now give a procedure with oracle Z which for almost all inputs m returns $Y(m)$. Wait for a stage s such that $\langle \sigma, y \rangle$ enters Δ_N at s, where $\sigma \prec Z$ and $|y| > m$. Then output $y(m)$. By (a) the procedure terminates, and by (b) the output is correct for almost all m, namely, whenever $s \geq s_0$. Thus $Y \leq_T Z$. Theorem is proved. $\square$

Corollary 3.5. *There is a (necessarily K-trivial) promptly simple set A which is not weakly ML-cuppable.*

Proof. Let Y be Chaitin's halting probability Ω. Since $Y \equiv_T \emptyset'$, the result follows from Theorem 3.1. $\square$

Discussion. Let us compare our construction of a promptly simple set A in the proof of Theorem 3.1 with the usual cost function construction of a promptly simple K-trivial set ([3] or [11, Prop 4.1]). Typically, a cost functions $c(x,s)$ restricts a prompt simplicity requirement S_e by stipulating that at stage s, x can enter A for the sake of S_e only if $c(x,s) \leq 2^{-e}$. First consider the usual cost function for achieving K-triviality, $cost(x,s) = \sum_{x < y \leq s} 2^{-K_s(y)}$. Then for $x > s$, $cost(x,s) = 0$. If at a later stage $t > s$, we have $c(x,t) > 2^{-e}$, then we may as well assume that the entire interval $[x,t)$ becomes unusable for S_e (as the numbers with short descriptions at stage t might be close to t). So S_e will have to look for future candidates among the numbers $\geq t$. Now, for the usual cost function this process of intervals becoming unusable can repeat at most 2^e times, as it corresponds to an increase in the measure of the domain of the universal machine by 2^{-e}. A cost functions $cost_N$ has a similar restraining behavior on S_e, but now the process of intervals becoming unusable can repeat arbitrarily often. To see this, recall how we bounded the cost of the second type, in the proof of Lemma 3.3. When $Y \upharpoonright e+c+3$ changes another time at t, then we have to restrict S_e to numbers $\geq t$ in order to make the argument work. This is similar to Kučera's construction of a promptly simple set A below a ML-random Δ_2^0 set Z. Here, when $Z \upharpoonright e$ changes at stage t, then all numbers $< t$ become unusable for S_e (see [10, Chapter 3]).

By [11, Thm 7.3], the K-trivial sets are precisely the ones that can be obtained via a cost function construction in terms of the usual function $cost(x,s)$. Our discussion gives some evidence that the sets obtain by Kučera's construction (Theorem 1.1 and the remark after) form a proper subclass of the K-trivial sets. However, for the non-cupping construction, if $Y = \Omega$, then the process of restricting S_e to larger

intervals can only repeat 2^{e+c+3} times. So there is no clear evidence if the non-ML cuppable sets form a proper subclass of the K-trivial sets.

To what extent does A in Theorem 3.1 depend on Y? At this stage we don't even know whether there is a non-computable (c.e.) set A such that (3.1) holds for all ML-random Y, Z. The construction in the proof of the Main Theorem can be modified in order to obtain A which satisfies (3.1) for all ML-random Y, Z such that $Y \leq_{\text{wtt}} \emptyset'$ (that is, Y is ω-c.e.).

Corollary 3.6. *There is a promptly simple set A such that, for all ML-random sets $Y \leq_{\text{wtt}} \emptyset'$ and Z,*

$$Y \leq_T A \oplus Z \ \Rightarrow \ Y \leq_T Z.$$

Proof. We sketch the necessary modifications. Let $(\Psi_i)_{i \in \mathbb{N}}$ be an effective listing of wtt-reduction procedures and $Y_i = \Psi_i(\emptyset')$. We have an approximation $Y_{i,s}(x) \simeq \Psi_i(\emptyset')[s]$, which may have the value 'undefined', but can change its value only finitely often. We now satisfy requirements $N = N_{\Gamma,c,i}$ which refer to Y_i instead of Y. The strategy is as before, except that N can only put a pair $\langle \sigma, Y_{i,s} \restriction n \rangle$ into Δ at stage s when $Y_{i,s}(k)$ is defined for all $k < n$. In the proof of Lemma 3.3 (which did not use that Y is ML-random), there still is a stage s_0 from which on $Y_{i,s} \restriction e + c + 3$ is stable. If some value $Y_{i,s_0}(k)$ is undefined for $k < e + c + 3$, then there is no contribution to the cost of the second type. Else we may argue as before. $\square$

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