DEMUTH RANDOMNESS AND COMPUTATIONAL COMPLEXITY

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ABSTRACT. Demuth tests generalize Martin-Löf tests $(G_m)_{m\in\mathbb{N}}$ in that one can exchange the m-th component for a computably bounded number of times. A set $Z \subseteq \mathbb{N}$ fails a Demuth test if Z is in infinitely many final versions of the G_m. If we only allow Demuth tests such that $G_m \supseteq G_{m+1}$ for each m, we have weak Demuth randomness.

We show that a weakly Demuth random set can be high, yet not superhigh. Next, any c.e. set Turing below a Demuth random set is strongly jump-traceable.

We also prove a basis theorem for non-empty Π^0_1 classes P. It extends the Jockusch-Soare basis theorem that some member of P is computably dominated. We use the result to show that some weakly 2-random set does not compute a 2-fixed point free function.

1. INTRODUCTION

The notion of Demuth randomness is stronger than Martin-Löf-randomness yet compatible with being Δ^0_2. Demuth tests generalize Martin-Löf tests $(G_m)_{m\in\mathbb{N}}$ in that one can exchange the m-th component (a Σ^0_1 set in Cantor space of measure at most 2^{-m}) for a computably bounded number of times. A set $Z \subseteq \mathbb{N}$ fails a Demuth test if Z is in infinitely many final versions of the G_m. If we only allow Demuth tests such that $G_m \supseteq G_{m+1}$ for each m, we have weak Demuth randomness. The implications are

$$\text{Demuth random} \Rightarrow \text{weak Demuth random} \Rightarrow \text{ML-random}.$$

These randomness notions, introduced and studied by Demuth [3, 4], remained obscure for a long time, but now begin to stand out for their rich interaction with the computational complexity aspect of sets. We study two examples of such an interaction.

(a) A highness property of a set determines a sense in which the set is close to being Turing complete. We study to what extent highness depends on the degree of randomness of a set. Using this we show that the implications above are strict.

(b) A lowness property of a set specifies a sense in which the set is close to being computable. We show that each c.e. set Turing below a Demuth random set satisfies an extreme lowness property: it is strongly jump-traceable. There is multiple evidence [8] that the

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strongly jump-traceable c.e. sets, introduced in [6], form a very small subclass of the c.e. K-trivials.

1.1. **The results in more detail.**

(a) Recall that a set Y is called *high* if $\emptyset'' \leq_T Y'$, and Y is *superhigh* if even $\emptyset'' \leq_{\mathrm{tt}} Y'$. We show that a weakly Demuth random Δ^0_2 set can be high. In contrast, every Demuth random Δ^0_2 set is known to be low. Next, a ML-random such as Ω is Turing complete. We show that no weakly Demuth random set is Turing complete. In fact, such a set is not even superhigh. The intuition is that the more random Y, the further it must be from computing $\emptyset'$.

(b) The first author proved in [14] that every Δ^0_2 random set Y Turing bounds some noncomputable c.e. set A. In [10] it is shown that if Y is Turing incomplete then A must be a base for randomness, and hence K-trivial. Greenberg, Hirschfeldt and Nies, in a preliminary version of [8], showed that there is a Δ^0_2 Martin-Löf-random set Y such that every c.e. set computable from Y is strongly jump-traceable. (For the definition, recall that a *c.e. trace* for a partial function ψ is a uniformly c.e. sequence $(T_x)_{x\in\mathbb{N}}$ of finite sets such that for all $x \in \mathrm{dom}(\psi)$ we have $\psi(x) \in T_x$; that an *order function* is a computable, nondecreasing, and unbounded function $h\colon \mathbb{N} \to \mathbb{N} \setminus \{0\}$; that a c.e. trace $(T_x)_{x\in\mathbb{N}}$ is *bounded* by an order function h if for all x, $|T_x| \leq h(x)$; and finally, that a set A is strongly jump-traceable if for every order function h, every partial function $\psi\colon \mathbb{N} \to \mathbb{N}$ that is partial computable in A has a c.e. trace that is bounded by h.) We prove here that any Demuth random Δ^0_2 set Y serves this purpose. The intuition is that the more random Y, the closer to being computable must be a c.e. set bounded by Y.

In a final section we prove a basis theorem for non-empty Π^0_1 classes P. It extends the Jockusch-Soare basis theorem [11] that some member of P is computably dominated. The extension is that, if $B >_T \emptyset'$ is Σ^0_2, then there is a computably dominated set $Y \in P$ such that $Y' \leq_T B$.

In the applications, one takes P to be a class of ML-random sets. Note that each computably dominated ML-random set is already weakly 2-random. Recall that a function g is 2-*fixed point free* if $W_{g(x)} \neq^* W_x$ for each x. We use the result to show that, unlike the case of 2-randomness, some weakly 2-random set does not compute a 2-fixed point free function. Further, in [2], the basis theorem was used to show that some weakly 2-random Y is K-trivial in $\emptyset'$. It suffices to take B K-trivial in $\emptyset'$ but not Δ^0_2, and let $Y \leq_T B$ be ML-random and computably dominated.

2. The randomness notions

We will formulate test via sequences of open classes in Cantor space. However, via the binary representation, co-infinite sets can be identified with the reals in $[0, 1)$. In fact, Demuth tests were introduced originally for real numbers. In [3] only the arithmetical real numbers were considered. Later on [4], tests were generalized to all real numbers. Sets which fail some test of this type were called $\mathcal{A}_\alpha$ numbers in [3], or WAP-sets, where WAP stands for weakly approximable in measure.

Demuth was primarily interested in various kinds of effective null classes because of their role in constructive mathematical analysis. For instance, he studied differentiability of constructive (in the Russian sense, mapping computable reals to computable reals) functions f defined on the unit interval. He proved that for each Demuth random real $x \in [0, 1)$ the “Denjoy alternative” holds: either $f'(x)$ is defined, or $+\infty = \limsup_{h\to 0}[f(x + h) - f(x)]/h$ and $-\infty = \liminf_{h\to 0}[f(x + h) - f(x)]/h$.
He also showed that mere Martin-Löf-randomness of x does not imply the Denjoy alternative for every constructive f.

For more background on Demuth randomness see Section 3.6 of [19].

2.1. **Formal definition and basics on Demuth randomness.**

Definition 2.1. *A* Demuth test *is a sequence of c.e. open sets* $(S_m)_{m\in\mathbb{N}}$ *such that* $\forall m\, \lambda S_m \le 2^{-m}$, *and there is a function* $f \le_{\mathrm{wtt}} \emptyset'$ *such that* $S_m = [W_{f(m)}]^{\prec}$.

A set Z passes *the test if* $Z \notin S_m$ *for almost every* m. *We say that* Z *is* Demuth random *if* Z *passes each Demuth test.*

Recall that for a function f, $f \le_{\mathrm{wtt}} \emptyset'$ if and only if f is an ω-c.e. function. Hence, as already mentioned, the intuition is that we can change the m-component S_m a computably bounded number of times. We will sometimes denote by $S_m[t]$ the version of the component S_m that we have at stage t.

We cannot allow an arbitrary effective null sequence α_m as upper bounds in tests: at least we need $\sum_m \alpha_m < \infty$. For instance, consider the example of $\alpha_m = 1/m$. Let $(k_i)_{i\in\mathbb{N}}$ be an increasing computable sequence such that $k_0 = 1$, $\sum_{m=k_i}^{k_{i+1}-1} \alpha_m \ge 1$. Then it is easy to find strings σ_j such that $\bigcup_{m=k_i}^{k_{i+1}-1}[\sigma_m] = 2^\omega$ and such that $\lambda[\sigma_m] \le \alpha_m$. This yields a modified test in an obvious sense. No set Z passes this test since Z belongs to infinitely many $[\sigma_m]$.

Given that, the choice of 2^{-m} as an upper bound for λS_m is still less arbitrary here than for Martin-Löf tests. However, we could replace the condition $\forall m\, \lambda S_m \le 2^{-m}$ by the more general condition that there is a computable function $\alpha\colon \mathbb{N} \to \mathbb{Q}_0^+$ such that $\sum_m \alpha(m) < \infty$, the sequence of tail sums converges to 0 effectively, and $\forall m\, \lambda S_m \le \alpha(m)$. Given a test in this more general sense, define a computable sequence by

$$k_{-1} = 0 \text{ and } k_{i+1} = \mu k > k_i.\ \textstyle\sum_{j=k}^{\infty} \alpha(j) \le 2^{-i}.$$

Let $\widehat{S}_i = \bigcup_{m=k_i}^{k_{i+1}-1} S_m$. Then $(\widehat{S}_i)_{i\in\mathbb{N}}$ is a Demuth test. Further, if $Z \in S_m$ for infinitely many m, then Z fails this Demuth test.

Demuth proved several interesting results concerning Turing and truth-table degrees of sets at various levels of randomness. We mention a few that are relevant for the rest of the paper.

Proposition 2.2.

(i) *Each Demuth random set* A *is* GL_1, *i.e.,* $A' \equiv_T A \oplus \emptyset'$
(ii) *If* A *is a set such that* $\emptyset' \le_T A$, *there is a Demuth random set* B *such that* $B' \equiv_T A$.

Proof. The first part is stated in [4, Remark 10, part 3b] with a sketch of a proof. A full proof can be found in [19], Theorem 3.6.26. The second part is in [4, Theorem 12]. □

As a consequence of the foregoing theorem, a Demuth random set can be Δ^0_2 (and hence low). A proof of this special case is also given in [19, Theorem 3.6.25].

2.2. Weak Demuth randomness.

Definition 2.3. *In the context of Definition 2.1, if we also have $S_m \supseteq S_{m+1}$ for each m, we say that $(S_m)_{m\in\mathbb{N}}$ is a* monotonic Demuth test. *In this case the passing condition is equivalent to $Z \notin \bigcap_m S_m$. If Z passes all monotonic Demuth tests we say that Z is* weakly Demuth random.

This type of tests was introduced by Demuth [3], in a slightly different, but equivalent, form. (He called sets that fail some test of this type $\mathcal{A}^*_\alpha$ numbers.) Note that we would define the same randomness notion if we retained the test concept of Definition 2.1 and only changed the passing condition to $Z \notin \bigcap_m S_m$. For in that case, an equivalent monotonic Demuth test $(\widetilde{S}_i)_{i\in\mathbb{N}}$ is given by $\widetilde{S}_i = \bigcap_{m\le i} S_m$.

Recall that a set A is ω-c.e. if and only if $A \le_{\mathrm{wtt}} \emptyset'$. Clearly no ω-c.e. set is weakly Demuth random.

Fact 2.4. *Each weakly 2-random set is weakly Demuth random.*

Proof. Suppose $(G_m)_{m\in\mathbb{N}}$ is a monotonic Demuth test. Then $\bigcap_m G_m$ is a null Π^0_2 class, because $Z \in \bigcap_m G_m \leftrightarrow \forall m\, \forall s\, \exists t \ge s\; Z \in G_{m,t}[t]$. □

In fact, we didn't need here that the number of changes to the c.e. open set G_m is computably bounded.

Proposition 2.5. *Both Demuth randomness and weak Demuth randomness are closed downward under Turing reducibility within the ML-random sets.*

Proof. The case for Demuth randomness is stated as Theorem 11 in [4], and is an immediate corollary of Theorem 18 in [5]. The case of weak Demuth randomness can be derived from the same theorem in a similar way. For the convenience of the reader we give proofs in more standard terminology. This appeared as the solution to Exercise 5.1.16 in [19].

Given a set A, and a Turing functional Φ, for $n > 0$ let

$$S^A_{\Phi,n} = [\{\sigma\colon\ A\restriction_n \preceq \Phi^\sigma\}]^\prec.$$

By a result of Miller and Yu (see [19, 5.1.14]) if A is ML-random, then for each Turing functional Φ there is a constant c such that $\forall n\, \lambda S^A_{\Phi,n} \le 2^{-n+c}$. This result of Miller and Yue plays a similar role here as Theorem 18 of [5].

Given a c.e. open set R, we will effectively obtain a c.e. open set $\widehat{R}$ such that $\lambda\widehat{R} \le 2^c \lambda R$. Suppose

$$A = \Phi(Y).$$

If A fails a Demuth test $(G_m)_{m\in\mathbb{N}}$, then Y fails the Demuth test $(\widehat{G}_{m+c})_{m\in\mathbb{N}}$.

For $x \in 2^{<\omega}$, let S_x be the effectively given c.e. set which follows the canonical computable enumeration of $\{\sigma\colon\ x \preceq \Phi^\sigma\}$ as long as the measure of the open set generated does not exceed $2^{-|x|+c}$. From a c.e. open set R we can effectively obtain a (finite or infinite) c.e. antichain $\{x_0, x_1, \ldots\}$ such that $R = \bigcup_i [x_i]$. Let $\widehat{R} = \bigcup_i [S_{x_i}]^\prec$. Since $[S_{x_i}]^\prec \cap [S_{x_j}]^\prec = \emptyset$ for $i \neq j$, we have $\lambda\widehat{R} = \sum_i \lambda S_{x_i} \leq 2^c \lambda R$. Moreover, $A \in R$ implies $x_i \prec A$ for some i and hence $Y \in \widehat{R}$ by the hypothesis on c. Clearly $(\widehat{G}_{m+c})_{m\in\mathbb{N}}$ is a Demuth test which Y fails.

For weak Demuth randomness, suppose $(G_m)_{m\in\mathbb{N}}$ is a monotonic Demuth test failed by A, namely, $A \in \bigcap_m G_m$. Then $Y \in \bigcap_m \widehat{G}_{m+c}$. If the Demuth test $(\widehat{G}_{m+c})_{m\in\mathbb{N}}$ is not monotonic, we replace its m-th component by the intersection of its first $m+1$ components to ensure monotonicity. □

Chaitin's halting probability Ω, viewed as a set, is Turing complete and ML-random. Being ω-c.e., it is not weakly Demuth random. So we have an immediate corollary.

Corollary 2.6. *Every weakly Demuth random set is Turing incomplete.*

Remark 2.7 (The number of changes)**.** Note that by the argument above, every $Y \geq_T \Omega$ fails a Demuth test $(S_m)_{m\in\mathbb{N}}$ with bound 2^m on the number of times a version of S_m changes.

Suppose $(S_m)_{m\in\mathbb{N}}$ is a Demuth test and h is a function such that $h(m)$ bounds the number of times a version of S_m changes. If $\sum_m h(m)2^{-m} < \infty$ then we can take the effective sequence of all versions and obtain a Solovay test failed by any set that fails the Demuth test $(S_m)_{m\in\mathbb{N}}$. Thus, if some ML-random set fails the Demuth test $(S_m)_{m\in\mathbb{N}}$ then $\sum_m h(m)2^{-m} = \infty$. For instance, this means that $h(m) \geq 2^{m/2}$ for infinitely many m.

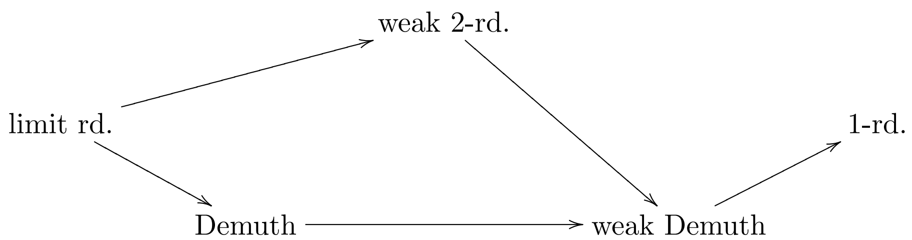
It is not hard to see that the class of Demuth random sets, the weakly Demuth random sets and the weakly 2-random sets form Π^0_4 classes.

For instance, in the case of Demuth randomness, observe that the sets which pass a given Demuth test $(S_m)_{m\in\mathbb{N}}$ form a Σ^0_3 class, namely,

$$\{Z \colon \exists m_0 \forall m \geq m_0 \forall n\, \exists s \geq n\, [Z\restriction_n] \not\subseteq S_{m,s}[s]\}.$$

A Demuth test $(S_m)_{m\in\mathbb{N}}$ is given by a pair of computable functions f, g, where $f(m,s)$ is the index for the Σ^0_1 class which is the version of S_m at stage s, and $g(m)$ bounds the number of changes. As totality of indices for partial computable functions is Π^0_2, we can universally quantify over all Demuth tests and obtain a Π^0_4 expression for the class of Demuth random sets.

The following diagram gives an overview of the randomness notions discussed and their implications.



The leftmost notion is limit randomness, which is defined similar to Demuth randomness in 2.1, but with the weaker requirement that $f \le_T \emptyset'$. Thus, a version can change any (finite) number of times. Stronger notions than limit randomness have been studied:

$$\text{2-random} \Rightarrow \text{Schnorr random relative to } \emptyset' \Rightarrow \text{limit random.}$$

See [2] for more on Schnorr randomness relative to $\emptyset'$.

3. Complexity of weakly Demuth random sets

In this section we construct a weakly Demuth random high Δ^0_2 set. Since each Demuth random set is generalized low, this shows that some weakly Demuth random set is not Demuth random. We will also show that no weakly Demuth random set is superhigh. In particular, it cannot be LR-complete.

Theorem 3.1. *Each Π^0_1 class P of positive measure contains a weakly Demuth random set B which is Δ^0_2 and high.*

Proof. We combine two strategies. The first strategy is used to construct a weakly Demuth random Δ^0_2 set. The second strategy is used for jump inversion.

The *first strategy* is a straightforward modification of the proof of [19, Theorem 3.6.25]. Let H_e be $[W_e]^{\prec}$. We use an auxiliary type of tests: a *special test* is a sequence of c.e. open sets $(V_m)_{m\in\mathbb{N}}$ such that $\forall m\, \lambda V_m \le 2^{-m}$, and there is a function $g \le_T \emptyset'$ such that $V_m = H_{g(m)}$. A set Z *passes* this test if $Z \notin V_m$ for almost every m. (Special tests are more general than Demuth tests in that the function g is merely Δ^0_2, not ω-c.e.)

By Fact 1.4.9 from [19] there is a binary function $\widetilde{g} \le_T \emptyset'$ that emulates all unary ω-c.e. functions f in the sense that there is i such that $f(n) = \widetilde{g}(i,n)$ for each n. We can stop the enumeration of $H_{\widetilde{g}(e,m)}$ whenever it attempts to exceed the measure 2^{-m} . Hence there is a function $g \le_T \emptyset'$ such that for all $e, m,$ $\forall m\, \lambda H_{g(e,m)} \le 2^{-m}$ and $H_{g(e,m)} = H_{\widetilde{g}(e,m)}$ if already $\lambda H_{\widetilde{g}(e,m)} \le 2^{-m}$.

Now let $V_m = \bigcup_{e \le m} H_{g(e,e+m+1)}$. Then $\lambda V_m \le \sum_{e\le m} 2^{-(e+m+1)} = 2^{-m} \cdot \sum_{e\le m} 2^{-(e+1)} \le 2^{-m}$.

Clearly, $(V_m)_{m\in\mathbb{N}}$ is a special test. Observe also that if $(S_m)_{m\in\mathbb{N}}$ is a Demuth test then $S_m \subseteq V_m$ for almost every m. Thus, each set passing this test is Demuth random.

We will use an additional property of this test. Suppose we merely have $Z \notin V_m$ for *infinitely many* m. Then $Z \notin \bigcap_m S_m$ for each monotonic Demuth test $(S_m)_{m\in\mathbb{N}}$. Thus we have proved:

Claim. *There is a special test $(V_m)_{m\in\mathbb{N}}$ such that any set Z for which $\exists^\infty m\, Z \notin V_m$ is weakly Demuth random.*

This strategy can be used to construct various weakly Demuth random sets (such as Δ^0_2 sets), similar to Theorem 3.6.25 in [19]. Here we will combine it with a further method.

The *second strategy.* The method of jump inversion is based on coding a set into members of Π^0_1 classes of positive measure. This technique was first used for the so called Kučera/Gács theorem [13, 7] (see Theorem 3.3.2 in [19]). It can be combined with a cone avoidance technique for members of

Π^0_1 classes and with an injury technique at a construction relative to $\emptyset'$ to construct a high, but incomplete ML-random Δ^0_2 set [15].

We use a standard computable enumeration of all Π^0_1 classes. Let Q_e be the Π^0_1 class with index e (see [19, Section 1.8]).

A Π^0_1 class P is called *rich* if $\lambda P > 0$ and there exists a computable function h such that for all e, if $\emptyset \neq Q_e \subseteq P$ then $\lambda Q_e > 2^{-h(e)}$. Each Π^0_1 class P of positive measure contains a rich Π^0_1 class. (To prove this one can use the original method of [13], or a more direct way described in the proof [21, Theorem 5.1].) Thus we may assume that the given Π^0_1 class P is rich, with computable function h as above.

Since P is rich, given a string σ and a Π^0_1 class $Q \subseteq P$ we can compute k such that if $Q \cap [\sigma] \neq \emptyset$ then $\lambda(Q \cap [\sigma]) > 2^{-k}$. So, there are at least two distinct strings ρ extending σ of length k such that if $Q \cap [\sigma] \neq \emptyset$, then also $Q \cap [\rho] \neq \emptyset$. Thus, it is easy to construct a computable function g such that

- $g(0, e) = 0$ for all e
- $g(-, e)$ is increasing for all e
- for each k, e, σ with $|\sigma| = g(k, e)$, if $Q_e \subseteq P$, then there are at least two distinct strings ρ extending σ of length $g(k+1, e)$ such that $Q_e \cap [\sigma] \neq \emptyset$ implies $Q_e \cap [\rho] \neq \emptyset$.

To build a weakly Demuth random Δ^0_2 set B in P which is high, we first describe two strategies in isolation.

Isolated strategy of jump inversion. We will code one bit $\emptyset''(m)$ into required set B in a way which B' can decode. Let m and a Π^0_1 class $Q = Q_e$ such that $\emptyset \neq Q \subseteq P$ be given. We first define a nonempty Π^0_1 class $(Q)^0$, by $X \in (Q)^0 \leftrightarrow X \in Q \wedge$

$$\forall k \exists \tau (X \restriction_{g(k,e)} \prec \tau <_L X \restriction_{g(k+1,e)} \wedge |\tau| = g(k+1, e) \wedge Q \cap [\tau] \neq \emptyset).$$

The idea is that $(Q)^0$ consists of those X's from Q for which for all k, $X \restriction_{g(k+1.e)}$ is not the beginning of the leftmost member of Q extending $X \restriction_{g(k.e)}$.

Secondly, we define a nonempty Π^0_1 class $(Q)^{1,s}$, as follows. Let $\tau_0, \dots, \tau_n$ be all strings τ of length $g(s+1, e)$ such that they are the leftmost extension of $\tau \restriction_{g(s,e)}$ for which $Q \cap [\tau] \neq \emptyset$. Note, that we can find these strings using the oracle $\emptyset'$. Now let

$$(Q)^{1,s} = \{X : X \in Q \wedge \exists j \leq n(\tau_j \prec X)\}.$$

Here the idea is that $(Q)^{1,s}$ consists of those X's from Q such that $X \restriction_{g(s+1,e)}$ is the beginning of the leftmost member of Q extending $X \restriction_{g(s,e)}$.

We will ensure that

- if $m \notin \emptyset''$ then $B \in (Q)^0$
- if $m \in \emptyset''$ and m enters $\emptyset''$ at step s (in a standard enumeration of $\emptyset''$ relatively to $\emptyset'$), then $B \in (Q)^{1,j}$ for some j.

For any set X, membership of X in a Π^0_1 class is always Π^0_1 relative to X, and, therefore, computable from X'. So we can compute a value $\emptyset''(m)$ from B' by asking whether $B \in (Q)^0$.

During our construction, which is relative to $\emptyset'$, we cannot decide which case applies ($m \in \emptyset''$ or $m \notin \emptyset''$). Thus, if m enters $\emptyset''$ at step s it may not be possible to take any of $\tau_0, \ldots, \tau_n$ mentioned above, due to actions of other strategies. Instead, we take a properly chosen n (as explained later) and choose some string of length $g(n+1, e)$, say ρ, which is the leftmost extension of $\rho\restriction_{g(n,e)}$ for which $Q_e \cap [\rho] \neq \emptyset$. Then we define

$$(Q)^1(\rho) = \{X : X \in Q \wedge \rho \prec X)\}$$

and we ensure that $B \in (Q)^1(\rho)$. Note, that $(Q)^1(\rho) \cap (Q)^0 = \emptyset$.

Isolated strategy to make B weakly Demuth random - called wD strategy. To guarantee that our constructed set B is weakly Demuth random we will have to ensure that $B \notin V_m$ for infinitely many m.

Given a Π^0_1 class Q_e, $\emptyset \neq Q_e \subseteq P$ we can compute k such that $\lambda Q_e > 2^{-k}$. Then $Q_e \setminus V_{k+1}$ is a nonempty Π^0_1 class. Provided that Q_e was already a restriction on B, to which class to belong to, the next restriction will be $Q_e \setminus V_{k+1}$. Let us denote this class by $wD(Q_e)$.

The construction. We build, computably in $\emptyset'$, a sequence of strings $(\sigma_s)_{s\in\mathbb{N}}$ such that $\sigma_s \preceq \sigma_{s+1}$ for all s, where $B = \bigcup_s \sigma_s$. We will also build, not computably in $\emptyset'$ but only in $\emptyset''$, a sequence of Π^0_1 classes $(B_m)_{m\in\mathbb{N}}$ together with their indices $(e_m)_{m\in\mathbb{N}}$. To adapt it to our construction we define computably in $\emptyset'$ their approximations, which at step s we denote by $B_m[s]$ and $e_m[s]$. For each m there will be only finitely many changes in these sequences and they settle down eventually to their limit values.

Let $\sigma_{-1} = \varnothing$, $B_{-1} = P$ and e_{-1} be an index of P (here all approximations equal to these final values).

Step s. Look whether there is $m \leq s$ which enters $\emptyset''$ at step s (in a standard enumeration of $\emptyset''$ relatively to $\emptyset'$).

Case 1. If yes, let m be the least such. For all $j < m$ approximations to B_j and e_j remain at this step the same as at step $s-1$. Further, let $n, n \geq s$, be the least number for which $g(n, e_{m-1}[s-1]) \geq |\sigma_{s-1}|$. Define a Π^0_1 class $A_m = (B_{m-1}[s-1])^1(\rho)$, where ρ is the leftmost string of length $g(n+1, e_{m-1}[s-1])$ extending σ_{s-1} for which $B_{m-1}[s-1] \cap [\rho] \neq \emptyset$. Let τ_m be ρ. To the class A_m apply one more wD strategy to get $wD(A_m)$, and let $B_m[s]$ be $wD(A_m)$ and $e_m[s]$ its index. It remains to redefine classes $B_j[s]$ for all j, $m < j \leq s$. This is done inductively. Suppose $B_{j-1}[s]$ (and its index $e_{j-1}[s]$) and a string τ_{j-1} are already defined for j, $m < j \leq s$.

If $j \notin \emptyset''[s]$, then define $A_j = (B_{j-1}[s-1])^0$ and apply one more wD strategy to A_j to get $B_j[s]$, together with its index $e_j[s]$. Also let $\tau_j = \tau_{j-1}$.

If $j \in \emptyset''[s]$, then let ρ be the leftmost string of length $g(1, e_{j-1}[s])$ extending τ_{j-1} for which $B_{j-1}[s] \cap [\rho] \neq \emptyset$. Define $A_j = B_{j-1}[s] \cap [\rho]$, $\tau_j = \rho$ and, further, apply one more wD strategy to A_j to get $B_j[s]$ together with its index $e_j[s]$.

Finally (at the end of this process), let $\sigma_s = \tau_s$.

Case 2. If there is no such m, then for all $j, j < s$ approximations to B_j and e_j remain at this step the same as at step $s-1$. Further, let $A_s = (B_{s-1}[s])^0$, apply one more wD strategy to A_s to get $B_s[s]$ together with its index $e_s[s]$. Let $\sigma_s = \sigma_{s-1}$. This ends the construction.

Obviously, B is Δ^0_2. By a standard induction argument it is straightforward to show that B' can find, for all m, limit values e_m of Π^0_1 classes B_m. Since each B_m arises by an application of a wD startegy, B is weakly

Demuth random. It remains to show that $\emptyset'' \leq_T B'$. As pointed out before, $m \notin \emptyset''$ if and only if $B \in (B_{m-1})^0$. Since membership of any set X in a Π^0_1 class is computable from X', we can computably in B' decide whether $m \in \emptyset''$. ☐

The preceding result can be generalized.

Theorem 3.2. *Let P be a Π^0_1 class of positive measure. For any set $A \geq_T \emptyset'$ that is c.e. in $\emptyset'$, and any set C such that $\emptyset <_T C \leq_T \emptyset'$, we can find a weakly Demuth random Δ^0_2 set $B \in P$ such that $B' \equiv_T A$ and $C \not\leq_T B$.*

Proof of Theorem 3.2. The above proof can be easily modified as follows.
1) Jump inversion method is applied not to $\emptyset''$ but rather to a given set A which c.e. in $\emptyset'$ and $\geq_T \emptyset'$.
2) The method of the proof is well compatible with the method of

- the proof of the Low Basis Theorem, introduced by Jockusch and Soare [12],which is used to control the jump of B, i.e. to ensure that $B' \leq_T A$
- avoiding an upper cone above a given noncomputable Δ^0_2 set,

since the latter methods are forcing by Π^0_1 classes and require only an oracle $\emptyset'$.

☐

Before we proceed, we need to review some definitions from [19, Section 5.3].

Definition 3.3. (i) A *monotonic cost function* is a computable function

$$c : \mathbb{N} \times \mathbb{N} \to \{x \in \mathbb{Q}_2 \colon\ x \geq 0\}$$

that is nonincreasing in the first, and nondecreasing in the second argument.

Definition 3.4. (i) A *computable approximation* of a set A is an effective sequence $(A_s)_{s\in\mathbb{N}}$ of strong indices for finite sets such that $A(x) = \lim_s A_s(x)$ for each x.
(ii) Given a computable approximation $(A_s)_{s\in\mathbb{N}}$ and a cost function c, the total cost of A-changes is

$$\textstyle\sum_{x,s} c(x, s)\, [x \text{ is least s.t. } A_{s-1}(x) \neq A_s(x)].$$

We say $(A_s)_{s\in\mathbb{N}}$ *obeys* c if this quantity is finite.
(iii) We say that a set A *obeys* c, written $A \models c$, if some computable approximation of A obeys c.

In [9] (also see [19, 8.5.3]) a monotonic cost function c is called *benign* if there is a computable function g such that

$$x_0 < x_1 < \ldots < x_k \ \&\ \forall i < k\, [c(x_i, x_{i+1}) \geq 2^{-n}] \text{ implies } k \leq g(n).$$

In the following we show that no weakly Demuth random set is superhigh. We obtain this result as a corollary to the Theorem 3.5 below that there is a c.e. set which obeys a given benign cost function, and is not below any weakly Demuth random. This is interesting on its own right because of the persistent open question [17] whether each K-trivial set A is below an incomplete ML-random Y. Since K-triviality is equivalent to obeying a certain benign cost function $c_{\mathcal{K}}$, we know that, at least, such a Y cannot always be weakly Demuth random.

Theorem 3.5. *Let c be a benign cost function. Then there is a c.e. set $A \models c$ such that $A \not\leq_T Y$ for each weakly Demuth random set Y.*

Proof. Let Θ be the Turing functional such that $\Theta^{0^e1X} = \Phi_e^X$ for each oracle X. If $A = \Phi_e^X$ for some weakly Demuth random X, then $Y = 0^e1X$ is also weakly Demuth random and $A = \Theta^Y$. So it suffices to build a c.e. set $A \models c$ and a Demuth test $(G_m)_{m\in\mathbb{N}}$ such that for each Y we have

$$A = \Theta^Y \rightarrow Y \in \bigcap_m G_m.$$

Given the cost function c we define numbers $v_k[s]$ for $k \leq s$. Let $v_0[0] = 0$. At stage $s > 0$, let j be least such that $j = s$ or $c(v_j[s-1], s) \geq 2^{-j}$.

- For $k < j$ let $v_k[s] = v_k[s-1]$.
- For $k \geq j$ (re)define values $v_k[s]$ in an increasing fashion and larger than all numbers previously mentioned,
  and such that $c(v_k[s], s) < 2^{-k}$.

Suppose c is benign via a computable function g. Note that the value of v_k changes for at most $\widehat{g}(k) = \sum_{j\leq k} g(j)$ times.

Construction of a c.e. set A and a Demuth test $(G_m)_{m\in\mathbb{N}}$.
Stage s.

(a) The version of G_m at stage s is
$$G_m[s] = \{Z\colon\ \Theta^Z \succeq A_s\restriction_{v_{\langle m,i\rangle}[s]+1}\},$$
where i is the number of times a number of the type $v_{\langle m,j\rangle}$ has so far been enumerated into A.

(b) If $\lambda G_{m,s}[s] > 2^{-m}$ put $v_{\langle m,i\rangle}$ into A_{s+1}.

Verification. Since we have $c(v_k[s], s) \leq 2^{-k}$, the total cost of A-changes is at most 2.

Given m, as long as we are at (a), the version $G_m[s]$ can change at most $\widehat{g}(\langle m,i\rangle)$ times. If we pass (b), all the later versions are disjoint from the previous versions because we chose the v_k in an increasing fashion at each stage. Hence we pass (b) for at most 2^m times. The total number of times the version of G_m can change is thereby bounded by $2^m \cdot \widehat{g}(\langle m, 2^m\rangle)$.

Clearly, if $A = \Theta^Y$ then Y is in the final version of G_m for each m. □

Corollary 3.6. *No weakly Demuth random set is superhigh.*

Proof. For each ML-random superhigh set Y, [8, Theorem 4.2] define a benign cost function c such that $A \models c$ implies $A \leq_T Y$ for each c.e. set A. (In fact c only depends on the truth table reduction procedure showing that $\emptyset'' \leq_{tt} Y'$.) If we let A be the c.e. set obeying c given by the foregoing theorem, this shows that Y cannot be weakly Demuth random.

It is also possible to prove this result directly, without relying on Theorem 3.5. Rather, one only uses the methods of [8, Theorem 4.2]: given a truth table reduction procedure Γ one builds a weak Demuth test such that each set Z with $\emptyset'' = \Gamma(Z')$ fails the test. □

4. Demuth randomness and strong jump-traceability

We begin with some preliminaries. As in [8], we define a *Turing functional* to be a partial computable function $\Gamma\colon 2^{<\omega} \times \omega \rightarrow \omega$, such that for all

$x < \omega$, the domain of $\Gamma(-, x)$ is an antichain of $2^{<\omega}$ (in other words, that domain is prefix-free). The idea is that the functional is the collection of minimal oracle computations of an oracle Turing machine. For any set A and number x, we let $\Gamma^A(x) = y$ if there is some initial segment τ of A such that $\Gamma(\tau, x) = y$. Then Γ^A is an A-partial computable function, and every A-partial computable function is of the form Γ^A for some Turing functional Γ. We write $\Gamma^A(x)\downarrow$ if x is in the domain of Γ^A; otherwise we write $\Gamma^A(x)\uparrow$. The *use* of a computation $\Gamma^A(x) = y$ is the length of the unique initial segment τ of A such that $\Gamma(\tau, x) = y$.

If $(A_s)_{s\in\mathbb{N}}$ is a computable approximation for a Δ^0_2 set A, and $(\Gamma_s)_{s\in\mathbb{N}}$ is an effective enumeration of (the graph of) a Turing functional, then we let $\Gamma^A[s] = \Gamma_s^{A_s}$.

The following is a special case of a lemma in [8].

Lemma 4.1. *Suppose the c.e. set A is superlow. Then for each Turing functional Γ there is a computable enumeration $(A_s)_{s\in\mathbb{N}}$ of A and a computable function g such that $g(x)$ bounds the number of stages s such that $\Gamma^A(x)[s-1]$ is defined with use u and $A_s\restriction_u \neq A_{s-1}\restriction_u$.*

In the situation of the lemma we say the computation $\Gamma^A(x)[s-1]$ is destroyed at stage s.

Proof. Let $(\widetilde{A}_s)_{s\in\mathbb{N}}$ be some computable enumeration of A. There is a Turing functional Δ such that for each x and each stage s such that $\Gamma^{\widetilde{A}}(x)[s]\downarrow$, the output of $\Delta^{\widetilde{A}}(x)[s]$ is the stage $t \le s$ when this computation became defined. Clearly the defined distinct values $\Delta^{\widetilde{A}}(x)[s]$ are increasing in s.

By [18] A is jump-traceable. Thus, there is a c.e. trace $(T_x)_{x\in\mathbb{N}}$ with computable bound g for Δ^A. Define a computable sequence of stages as follows. Let $s_0 = 0$. For $i \ge 0$, let

$$s_{i+1} = \mu s > s_i. \forall x < s_i\, [\Gamma^{\widetilde{A}}(x)[s]\downarrow \longrightarrow \Delta^{\widetilde{A}}(x)[s] \in T_{x,s}].$$

Define a computable enumeration $(A_s)_{s\in\mathbb{N}}$ of A by $A_s(x) = \widetilde{A}_{s_i}(x)$ for $s_i \le s < s_{i+1}$. For each s such that $\Gamma^A(x)[s]$ is newly defined, a further element must enter T_x. Thus $(A_s)_{s\in\mathbb{N}}$ is a as required. □

Theorem 4.2. *Suppose the c.e. set A is Turing below a Demuth random set. Then A is strongly jump-traceable.*

Proof. Since a Demuth random set is Turing incomplete, A is a basis for ML-randomness. Hence A is low for K and therefore superlow. See [19, 5.1.23] for more detail.

Fix a Turing functional Φ. For each order function h we will build a c.e. trace $(T_x)_{x\in\mathbb{N}}$ such that $\#T_x \le h(x)$; we will also define a Demuth test $(G_m)_{m\in\mathbb{N}}$ such that, whenever $A = \Phi^Y$, we have

(1) $$\exists^\infty x\, J^A(x) \notin T_x \;\Rightarrow\; Y \text{ fails } (G_m)_{m\in\mathbb{N}}.$$

Thus, if $A = \Phi^Y$ for some Demuth random set Y, then A is strongly jump-traceable.

Fix an order function h. For $m \in \mathbb{N}$ let

$$I_m = \{x\colon\ 2^m \le h(x) < 2^{m+1}\}.$$

Let $(A_s)_{s\in\mathbb{N}}$ be a computable enumeration of A such that the conclusion of Lemma 4.1 holds for the jump functional J via a computable bound g.
Construction of the c.e. trace $(T_x)_{x\in\mathbb{N}}$.

For each m we run a procedure for m which defines T_x for each $x \in I_m$. The actions of these procedures will be exploited later to define the Demuth test $(G_m)_{m\in\mathbb{N}}$. Namely, if $J^A(x) \notin T_x$ for some $x \in I_m$, then $Y \in G_m$ for each Y such that $A = \Phi^Y$.

The procedures for different m act independently. In the following fix m. The procedure for m has a parameter v which is nondecreasing over stages. Initially $v = 0$. At stage s we have a description of a c.e. open set

$$G = \{Z\colon\ \Phi^Z \succeq A_s\restriction_v\}. \tag{2}$$

Let G_s be the clopen set approximating G at stage s, namely,
$G_s = \{Z\colon\ \Phi^Z[s] \succeq A_s\restriction_v\}$.
Procedure for m.

(a) WHILE $\lambda G \le 2^{-m}$ DO:
  IF there is a new convergence $J^A(x)\downarrow$ for $x \in I_m$,
  raise v to the stage number.
(b) Enumerate $J^A(x)[s]$ into T_x for each $x \in I_m$ such that this computation is defined.
(c) WAIT for a stage s such that $A_s\restriction_v \ne A_{s-1}\restriction_v$.
(d) Let $v = s$ and GOTO (a).

Claim 1. *For each x we have $\#T_x \le h(x)$.*
Let m be the number such that $x \in I_m$. Thus $2^m \le h(x)$. Each time the procedure for m goes back to (a), $A\restriction_v$ has changed. Because the parameter v is non-decreasing over stages, this means that the next set G defined in (2) will be disjoint from the previous versions. Since λG exceeds 2^{-m} when the procedure enters (b), the procedure enters (b) for at most 2^m times. This proves Claim 1.

We now wish to define the Demuth test $(G_m)_{m\in\mathbb{N}}$. We cannot let G_m copy all the versions of G the procedure for m goes through. For, since we have to keep the values of v nondecreasing, typically v is much larger than the maximum of the uses of the computations $J^A(x)$ for $x \in I_m$. This means that even if we have applied Lemma 4.1 to J, there may be too many changes of $A\restriction_v$ for the computable enumeration $(A_s)_{s\in\mathbb{N}}$ used in the construction.

As a remedy, we introduce a new enumeration $(\widehat{A}_s)_{s\in\mathbb{N}}$ of A. For this, we define an auxiliary functional Γ which always has output 0. Given m, initialize a counter i with value -1. When v is raised at a stage s in (a) of the procedure for m, increment i and define $\Gamma^A(\langle m, i\rangle)$ with use v. From now on, each time $A\restriction_v$ changes, redefine $\Gamma^A(\langle m, i\rangle)$ with the same use.

Recall that $g(x)$ bounds the number of times $J^A(x)$ can become destroyed with the given computable enumeration of A. Then the maximum value of i is bounded by $r(m) = 2^m \sum_{x\in I_m} g(x)$.

Now, by Lemma 4.1, there is a computable enumeration $(\widehat{A}_s)_{s\in\mathbb{N}}$ of A and an increasing computable function f such that $\Gamma^{\widehat{A}}(w)$ gets destroyed for at most $f(w)$ times.

At any stage s, for each m, if v is the parameter of procedure m, let G_m copy the c.e. open set $\{Z\colon \Phi^Z \succeq \widehat{A}_s\restriction_v\}$, as long as its measure does not exceed 2^{-m}. (This is similar to (2) but with the new enumeration of A.)

Clearly, G_m can only change at a stage s if $\Gamma^{\widehat{A}_{s-1}}(\langle m, i\rangle)$ is destroyed for the current $i < r(m)$. Hence the number of times G_m changes is bounded by $\sum_{i<r(m)} f(\langle m, i\rangle)$. This shows that $(G_m)_{m\in\mathbb{N}}$ is a Demuth test.

Claim 2. *The property (1) is satisfied.*
Suppose that $A = \Phi^Y$, and that there are infinitely many m such that $J^A(x) \notin T_x$ for some $x \in I_m$. For such an m, whenever the procedure for m reaches (c) it will after some waiting go back to (a), because the use of $J^A(x)$ is at most v for each $x \in I_m$. This can happen at most 2^m times, so eventually the procedure stays permanently at (a).

Recall that the number of times the parameter v is raised is bounded by $r(m)$. For the final value of this parameter, since $\Phi^Y \succeq A\restriction_v$, we put Y into the final version of G_m by a stage s when $\widehat{A}\restriction_v = A\restriction_v = A_s\restriction_v$. □

We give an application. Recall that each ML-random Δ^0_2 set Turing bounds an incomputable c.e. set (Kučera; see [19, Thm. 4.2.1]). However, a stronger statement fails: $Y_0 \not\leq_T Y_1$ for ML-random Δ^0_2 sets does *not* imply that some c.e. set A is below Y_0 but not below Y_1. Still better would be to find ML-random Δ^0_2 sets $Y_0 \not\equiv_T Y_1$ that bound the same c.e. sets. This remains open.

Note that if a set $Y = Y_0 \oplus Y_1$ is ML-random then Y_0, Y_1 are ML-random and $Y_0 \mid_T Y_1$.

Corollary 4.3. *There is a ML-random Δ^0_2 set of the form $Y_0 \oplus Y_1$ such that each c.e. set Turing below Y_0 is Turing below Y_1.*

Proof. Let Y_1 be a ML-random superlow set. Let Y_0 be a Δ^0_2 set that is Demuth random relative to Y_1. By van Lambalgen's theorem, Y is ML-random.

If A is c.e. and $A \leq_T Y_0$, then A is s.j.t., whence $A \leq_T Y_1$ by [8]. □

5. A basis theorem and its application to weak 2-randomness

Theorem 5.1. *Let P be a non-empty Π^0_1 class. Suppose that $B >_T \emptyset'$ is Σ^0_2. Then there is a computably dominated set $Y \in P$ such that $Y' \leq_T B$.*

Proof. We combine the techniques of the Low Basis Theorem of Jockusch and Soare [12] and the basis theorem for computabaly dominated sets of Martin and Miller [16], see Theorem 1.8.42 from [19], with permitting below B relative to $\emptyset'$. Fix an enumeration $(B_s)_{s\in\mathbb{N}}$ of B relative to $\emptyset'$. We use the function $c_B \leq_T B$ given by $c_B(i) = \mu t > i.\, B_t\restriction_i = B\restriction_i$ for the permitting. Note also that $c_B \oplus \emptyset' \equiv_T B$.

Construction relative to $\emptyset'$ of Π^0_1 classes $(P^i)_{i\in\mathbb{N}}$. Let $P^0 = P$.
Stage $2i+1$. If

$$P^{2i} \cap \{X\colon J^X(i)\uparrow\} \neq \emptyset,$$

then let P^{2i+1} be this class. Otherwise, let $P^{2i+1} = P^{2i}$.
Stage $2i+2$. See whether there is $e \leq i$ which has not been active so far such that for some $m \leq c_B(i)$ we have

$$Q^i_{e,m} := P^{2i+1} \cap \{X\colon\ \Phi^X_e(m)\uparrow\} \neq \emptyset.$$

If so let e be the least such number, let m be the least such number for e, and let $P^{2i+2} = Q^i_{e,m}$. Say that e is *active*. Otherwise, let $P^{2i+2} = P^{2i+1}$.

A standard argument shows that there is a unique set Y such that $Y \in \bigcap_r P^r$, i.e. $\bigcap_r P^r = \{Y\}$.
Verification. Since B can determine an index for each P^r, we have $Y' \leq_T B$ by the usual argument of the Low Basis Theorem. Each e is active at most once, and if so then Φ^Y_e is partial. Suppose now that Φ^Y_e is total. We claim that there is r such that Φ^Z_e is total for each $Y \in P^r$, and therefore Φ^Z_e is computably dominated by the argument in the proof of the basis theorem for computably dominated sets (Theorem 1.8.42 from [19]). If the claim fails then $B \leq_T \emptyset'$, as follows. Let s_0 be a stage such that no $j < e$ is active from s_0 on. Given $i \geq s_0$, using the oracle $\emptyset'$ find the least m such that $Q^i_{e,m} \neq \emptyset$. Then $c_B(i) \leq m$ (otherwise we would now ensure $\Phi^Y_e(m)$ is undefined), so that $B_m\restriction_i = B\restriction_i$. □

Corollary 5.2. *There is a weakly 2-random set Y that does not compute a 2-f.p.f. function.*

Proof. Let $B >_T \emptyset'$ be a Σ^0_2 set such that $B' \equiv_T \emptyset''$. By 5.1 there is a computably dominated ML-random set Y such that $Y \leq_T B$. Thus Y is weakly 2-random. If $g \leq_T Y$ is 2-f.p.f then there is 2-d.n.c. function $f \leq_T Y$, whence $\emptyset'' \leq_T B \oplus \emptyset'$ by completeness criterion of Arslanov relativized to $\emptyset'$, [1], (see Theorem 4.1.11 from [19]), contradiction. □

An alternative proof can be obtained from a result in the literature. By [20] relative to $\emptyset'$, there is a set $Y <_T \emptyset''$ such that Y is Schnorr random relative $\emptyset'$ and left-Σ^0_2. Then Y is weakly 2-random and does not compute a 2-f.p.f. function again by [1] relative to $\emptyset'$.

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