DENSITY RANDOMNESS

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ABSTRACT. We investigate a new randomness notion. A point is density random in Cantor space if it is Martin-Löf random and each effectively closed set or Π^0_1-class in Cantor space has density one at the point. Aside from the original definition, we also give equivalent characterizations of density random points, such as points where every left-c.e. martingales converge. Identifying a point in real line with a set in Cantor space via binary expansion, we are also able to characterize density randomness in methods of computable analysis. We show density random points are exactly Lebesgue points for each integral test. We also investigate density-one points in other arithmetical subsets of Cantor space.

CONTENTS

1. Introduction 1
2. Preliminaries 2
2.1. Algorithmic Randomness 2
2.2. Overview of Results 3
3. Upper density 3
4. Equivalent definitions of density randomness 4
5. Lowness for density randomness 8
6. Density-one points for Π^0_n-classes and Σ^1_1 classes 9
7. Computable Analysis 11
7.1. Background on computable Analysis 11
7.2. Integral tests 11
8. Questions 17
References 17

1. INTRODUCTION

In the classical setting of analysis, given a measurable set $A \subseteq \mathbb{R}$, the density of $z \in A$ is defined as $\rho(A|z) = \lim_{\epsilon\to 0} \frac{\mu(A\cap B_\epsilon(z))}{B_\epsilon(z)}$ where $B_\epsilon(z) = \{y \in \mathbb{R} : d(y,z) < \epsilon\}$ d is the usual metric on $\mathbb{R}$, which exists and equals one almost everywhere by the well-known Lebesgue Density Theorem (see for example, [3]). It has been proven fruitful to take the effective version of theorems in analysis of "almost everywhere"-type and transform them into notions of randomness. These analytic characterizations further elaborate and enrich the hierarchy of known randomness notions. Two open problems in algorithmic randomness are recently solved relying on the analytic aspects of randomness, the Martin-Löf cupping [4] and covering problem

[5]. The randomness notion we explore in this paper is also characterized by the effective version of Lebesgue Density Theorem. While in [2], computable randomness, Schnorr randomness and Martin-Löf randomness are characterized by the differentiability of special class of functions, we show a point is density random is equivalent to being a Lebesgue point of each integral test. This expands our understanding of randomness notion in effective analysis terms.

The following classes of functions that appear have interesting connections with each other.

1 Function g of the form $g(z) = \int_{[0,z]} f d\mu$ where f is an lower semi-computable integrable function.
2 Function g is interval-c.e. and absolutely continuous.
3 Function g is interval-c.e. and continuous.

In [1] and [8], it is shown that density random points are exactly those points where all interval-c.e. functions are differentiable. We will see later that density randomness is characterized via Lebesgue points of the class of functions in (1). Thus it is interesting to investigate the interaction among the those classes of functions. We will show that (1) ⇒ (2) ⇒ (3) and each implication is strict.

2. Preliminaries

2.1. **Algorithmic Randomness.**

The idea of algorithmic randomness is to think of a real as random if no *effective* null set captures it. Different criteria for effectiveness might result in different randomness notions. For general background we refer the readers to [14] and [6]. Next we give the central definition of density randomness which we will use throughout the paper.

We first need the following notions of density.

Definition 2.1 (Lower density of points in Cantor space and real line)**.** For any measurable set $P \subset 2^\omega$ and $Z \in 2^\omega$, the lower density of $Z \in 2^\omega$ in P is defined to be

$$\rho_2(P|Z) = \liminf_{\sigma \prec Z} \frac{\mu(P \cap [\sigma])}{\mu([\sigma])}$$

Similarly, for measurable set $Q \subset \mathbb{R}$ and $z \in \mathbb{R}$, the lower density of z in Q is defined to be

$$\rho(Q|z) = \liminf_{z \in I \text{ is open and } |I| \to 0} \frac{\mu(I \cap Q)}{\mu(I)}$$

Definition 2.2 (Density random sets)**.** A set $Z \in 2^\omega$ is density random if it is Martin-Löf random and for any Π^0_1-class $P \subseteq 2^\omega$, $Z \in P \Rightarrow \rho_2(P|Z) = 1$. Then Z is called a *dyadic density-one point.*

Given a real number $z \in \mathbb{R}$, we identify it with a set in 2^ω via its binary expansion, i.e. z is identified with Z if $z = 0.Z$.

Definition 2.3 (Density random points)**.** A point $z \in \mathbb{R}$ is density random if it is Martin-Löf random and for any effectively closed set P, $z \in P \Rightarrow \rho(P|Z) = 1$. Then z is called a *(full) density-one point.*

We recall the following result.

Theorem 2.4 (Mushfeq Khan and Joseph Miller [8])**.** *Let z be a ML-random dyadic density-one point. Then z is a full density-one point.*

Definition 2.3 and Definition 2.2 result in the same notion via the aforementioned identification of real numbers with elements in Cantor space.

Martingales are important tools to calibrate the randomness of a set.

Definition 2.5. A martingale is a function $d : 2^{<\omega} \to \mathbb{Q}^+$ such that $2d(\sigma) = d(\sigma^\frown 0) + d(\sigma^\frown 1)$. A martingale succeeds on a set $Z \in 2^\omega$ if $\limsup_n d(Z\upharpoonright_n) = \infty$.

It is well-known fact that a set is Martin-Löf random if and only if no left-c.e. martingale succeeds on the set. Based on this perspective, computably random sets are defined as those no computable martingale could succeed on. In this paper, we will show that density random sets are exactly those on which every left-c.e. martingale converges on. This is the c.e. version of computable random sets since each computable martingale converges on any computably random set.

2.2. **Overview of Results.** In Section 4, we establish the result that density random sets are exactly those on which each left-c.e. martingale converges.

In Section 5 we show lowness for density randomness coincides with K-triviality.

In Section 7, we prove density random points are exactly Lebesgue points for each integral test.

In Section 6, we extend the analysis to Π^0_n-class and investigate how much randomness it requires for a point to have density one in every Π^0_n-class.

3. Upper density

The *upper* (Cantor-space) density of a set $\mathcal{C} \subseteq 2^{\mathbb{N}}$ at a point Z is:

$$\overline{\varrho}_2(\mathcal{C}|Z) := \limsup_{\sigma \prec Z \,\wedge\, |\sigma| \to \infty} \frac{\lambda(I \cap \mathcal{C})}{|I|},$$

where I ranges over basic dyadic intervals containing z. Bienvenu et al. [1, Prop. 5.4] showed that for any effectively closed set $\mathcal{P}$ and ML-random $Z \in \mathcal{P}$, we have $\overline{\varrho}_2(\mathcal{P} \mid Z) = 1$; this implies of course that the upper density in $\mathbb{R}$ also equals 1.

The following shows that ML-randomness was actually too strong an assumption. The right level seems to be partial computable randomness. See [14, Ch. 7] for background.

Proposition 3.1. *Let $\mathcal{P} \subseteq 2^{\mathbb{N}}$ be effectively closed. Let $Z \in \mathcal{P}$ be partial computably random. Then $\overline{\varrho}_2(\mathcal{P} \mid Z) = 1$.*

Proof. Suppose there is $q < 1$ and n^* such that $\lambda_\sigma(\mathcal{P}) < q$ for each $\eta \prec Z$ with $|\eta| \geq n^*$. We will define a partial computable martingale M that succeeds on Z. Let $M(\eta) = 1$ for all strings η with $|\eta| \leq n^*$. Now suppose that $M(\eta)$ has been defined for a string η of length at least n^*, but M is as yet undefined on extensions of η. Search for $t > |\eta|$ such that

$$2^{-(t-|\eta|)}\#\{\tau \mid |\tau| = t \wedge [\tau] \cap \mathcal{P}_t = \emptyset\} > 1 - q.$$

If t is found, bet all the capital existing at η on the strings $\sigma \succ \eta$ with $|\sigma| = t$ that are not τ's as above, thereby multiplying the capital by $1/q$. Now repeat with all such strings $\sigma \succ \eta$ of length t.

The formal definition of M is as follows. For all $|\tau| \leq n^*$, $M(\tau) = 1$. Next we define M inductively on $2^{<\omega}$. Suppose M has been defined on α and $M(\alpha) = \beta$, let $t \in \omega$ such that $t > |\alpha|$ and let $S = \{\tau \in 2^t : [\tau] \cap P_t\}$ and $r = |S| > 2^{t-|\tau|}(1-q)$. For each $\sigma \in 2^t \backslash S$, define $M(\sigma) = \frac{1}{q}\alpha$, and let $\tau^* \in S$ be the leftmost element and define

$$M(\tau^*) = 2^{t-|\tau|}\alpha - \frac{1}{q}\alpha(2^{t-|\tau|} - r)$$

For any $\sigma \succcurlyeq \alpha$ and $|\sigma| < t$, define M accordingly to make M a martingale.

Next is the verification. First we check that $\forall \tau \preccurlyeq Z$, $M(\tau)$ is defined. We verify this inductively. Suppose $\eta \preccurlyeq Z$ is already defined. Then by assumption, $\lambda_\eta(\bar{P}) > (1-q)$. Therefore, there exists a stage $t \in \omega$ such that $\lambda_\eta(\bar{P}_t) > (1-q)$. Thus we have $\Sigma_{\tau \in 2^t, \eta \preccurlyeq \tau}\lambda_\tau(\bar{P}_t) = 2^{t-|\eta|}\lambda_\eta(\bar{P}_t) > 2^{t-|\eta|}(1-q)$; here we use the fact that for any measurable class $Q \subset 2^\omega$, the function $\sigma \mapsto \lambda_\sigma(Q)$ is a martingale. Therefore, we have found such a t to define a proper extension of η. By induction, M is defined on Z. It is easy to see Z succeeds on M since every time a new string is defined, the capital becomes $\frac{1}{q} > 1$ times of the original capital.

Note that M succeeds on Z because *all* strings $\sigma \prec Z$ of length $\geq n^*$ qualify as possible η's where t exists. On the other hand, if η is off Z then there may be no t, so M can be partial. □

Question 3.2. *Is there a computably random Z in some Π^0_1 class $\mathcal{P}$ so that $\overline{\varrho}_2(\mathcal{P} \mid Z) < 1$?*

Proposition 3.3. *Let $\mathcal{P} \subseteq 2^{\mathbb{N}}$ be effectively closed with $\lambda\mathcal{P}$ computable. Let $Z \in \mathcal{P}$ be Schnorr random. Then $\varrho_2(\mathcal{P} \mid Z) = 1$.*

Proof. [**?**] and [9]. The characteristic function 1_P is L_1-computable because there is a sequence $\left\langle 1_{P_{g(n)}} \right\rangle_{n\in\mathbb{N}}$, where g is a computable function such that $\lambda(P_{g(n)} - P) \leq 2^{-n}$. Now use e.g. [**?**, Theorem 3.15]. □

4. Equivalent definitions of density randomness

The goal of this section is to give the martingale characterization of density randomness.

A martingale $L\colon 2^{<\omega} \to \mathbb{R}^+_0$ is called *left-c.e.* if $L(\sigma)$ is a left-c.e. real uniformly in σ. We focus on convergence of such a martingale along a real Z, which means that $\lim_n L(Z\upharpoonright_n)$ exists in $\mathbb{R}$. Unlike the case of computable martingales, convergence requires more randomness than boundedness. For instance, let $\mathcal{U} = [0, \Omega)$, and let $L(\sigma) = \lambda(\mathcal{U} \mid [\sigma])$ (as a shorthand we use $\lambda_\sigma(\mathcal{U})$ for this conditional measure); then the left-c.e. martingale L is bounded by 1 but diverges on Ω because Ω is Borel normal.

The following theorem characterizing dyadic density-points is due to Madison Group consisting of Andrews, Cai, Diamondstone, Lempp and Miller. We give a proof here. Note a slight generalization could drop the "dyadic" in the theorem.

Theorem 4.1. *The following are equivalent for a ML-random real $z \in [0,1]$.*
(i) *z is a dyadic density-one point.*
(ii) *Every left-c.e. martingale converges along Z, where $0.Z$ is the binary expansion of z.*

By Theorem 2.4, z is a full density-one point iff z is a dyadic density-one point. Thus we have the following consequence.

Theorem 4.2. *The following are equivalent for a ML-random real $z \in [0,1]$.*
(i) *z is a density-one point.*
(ii) *Every left-c.e. martingale converges on z.*

A ML-random satisfying one of these conditions will be called *density random.*

Proof of Theorem 4.1: (ii) $\to$ (i) is [1, Corollary 5.5].
(i) $\to$ (ii). We can work within Cantor space because dyadic density is the same in Cantor space as in $[0,1]$. For $X \subseteq 2^{\mathbb{N}}$ we define the weight $\mathsf{wt}(X) = \sum_{\sigma \in X} 2^{-|\sigma|}$. Let $\sigma^{\prec} = \{\tau \in 2^{<\omega} \colon \sigma \prec \tau\}$. We use a technical test concept.

Definition 4.3. A *Madison test* is a computable sequence $\langle U_s \rangle_{s\in\mathbb{N}}$ of computable subsets of $2^{<\omega}$ such that $U_0 = \emptyset$, for there is a constant c such that for each stage s we have $\mathsf{wt}(U_s) \leq c$, and for all strings σ, τ,
(a) $\tau \in U_s - U_{s+1} \to \exists \sigma \prec \tau \, [\sigma \in U_{s+1} - U_s]$
(b) $\mathsf{wt}(\sigma^{\prec} \cap U_s) > 2^{-|\sigma|} \to \sigma \in U_s$.

Note that by (a), $U(\sigma) := \lim_s U_s(\sigma)$ exists for each σ; in fact, $U_s(\sigma)$ changes at most $2^{|\sigma|}$ times. We say that Z *fails* $\langle U_s \rangle_{s\in\mathbb{N}}$ if $Z \upharpoonright_n \in U$ for infinitely many n; otherwise Z *passes* $\langle U_s \rangle_{s\in\mathbb{N}}$.

Notice from the definition that $\mathsf{wt}(U_s) \leq \mathsf{wt}(U_{s+1}) \leq c$, and $\mathsf{wt}(U) = \sup_s \mathsf{wt}(U_s)$. Thus, $\mathsf{wt}(U)$ is a left-c.e. real. To see this, we show that $\mathsf{wt}(U_s) \leq \mathsf{wt}(U_{s+1})$. For each $\tau \in U_s - U_{s+1}$, there exists $\sigma \prec \tau$ such that $\sigma \in U_{s+1} - U_s$. Further, since $\mathsf{wt}(\sigma^{\prec} \cap U_s) \leq 2^{-|\sigma|}$ due to the fact that $\sigma \notin U_s$ and (b), the weight of strings removed in U_s above σ is at most $2^{-|\sigma|}$ which is less than or equal to the weight added to U_{s+1}. Hence we have $\mathsf{wt}(U_s) \leq \mathsf{wt}(U_{s+1})$.

Lemma 4.4. *Let Z be a ML-random dyadic density-one point. Then Z passes each Madison test.*

Proof. To see this, suppose that Z fails a Madison test $\langle U_s \rangle_{s\in\mathbb{N}}$. We build a ML-test $\langle \mathcal{S}^k \rangle_{k\in\mathbb{N}}$ such that $\forall \sigma \in U \, [\lambda_\sigma(\mathcal{S}^k) \geq 2^{-k}]$, and therefore $\overline{\rho}(2^{\mathbb{N}} - \mathcal{S}^k \mid Z) \leq 1 - 2^{-k}$. Since Z is ML-random we have $Z \notin \mathcal{S}^k$ for some k. So Z is not a density-one point.

To define the $\mathcal{S}^k$ we introduce for each $k, s \in \omega$ and each string $\sigma \in U_s$ clopen sets $\mathcal{A}^k_{\sigma,s} \subseteq [\sigma]$ given by uniformly computable strong indices, such that $\lambda(\mathcal{A}^k_{\sigma,s}) = 2^{-|\sigma|-k}$ for each $\sigma \in U_s$. We update these clopen sets at stages s when $\sigma \in U_{s+1} - U_s$. For each $\tau \succ \sigma$ with $\tau \in U_s - U_{s+1}$, put $\mathcal{A}^k_{\tau,s}$ into an auxiliary clopen set $\widetilde{\mathcal{A}}^k_{\sigma,s+1}$. Since $\sigma \notin U_s$, by (b) we have $\mathsf{wt}(\sigma^{\prec} \cap U_s) \leq 2^{-|\sigma|}$, and so inductively $\lambda(\widetilde{\mathcal{A}}^k_{\sigma,s+1}) \leq 2^{-|\sigma|-k}$. Now to obtain $\mathcal{A}^k_{\sigma,s+1}$ simply add mass from $[\sigma]$ to $\widetilde{\mathcal{A}}^k_{\sigma,s+1}$ in order to ensure equality as required.

Let $\mathcal{S}^k_t = \bigcup_{\sigma \in U_t} \mathcal{A}^k_{\sigma,t}$. Then $\mathcal{S}^k_t \subseteq \mathcal{S}^k_{t+1}$ by property (a) of Madison tests. Clearly $\lambda \mathcal{S}^k_t \leq 2^{-k}\mathsf{wt}(U_t) \leq 2^{-k}$. So $\mathcal{S}^k = \bigcup_t \mathcal{S}^k_t$ determines a ML-test. So $Z \notin \mathcal{S}^k$ for some k. If $\sigma \in U$ then by construction $\mathcal{A}^k_{\sigma,s}$ has measure $2^{-|\sigma|-k}$ for almost all s. Thus $\lambda_\sigma(\mathcal{S}^k) \geq 2^{-k}$ as required. This shows the lemma. □

Next we turn to the other direction. We first show if a set passes all Madison tests, it is already computably random.

Lemma 4.5. *Suppose that Z passes each Madison test, then Z is computably random.*

Proof. We build computable sets V_n by levels, such that each V_n is prefix-free and for each $\tau \in V_{n+1}$ there exists some $\sigma \in V_n$ such that $\sigma \prec \tau$. Aside from this, we maintain the following invariant during the construction:

Invariant: *For each n, if $\eta \notin \bigcup_{i \leq n} V_n$, then $\mathsf{wt}(\eta^{\prec} \cap \bigcup_{i \leq n} V_n) < 2^{-|\eta|}$. Notice the value of $\mathsf{wt}(\eta^{\prec} \cap \bigcup_{i \leq n} V_n)$ is computable.*

Let $V_0 = \{\lambda\}$ where λ is the empty string. It trivially satisfies the invariants. Suppose we have defined up to V_n, we describe the procedure on how to define V_{n+1}. First of all, pick $q_0, q_1 \in \mathbb{Q}$ such that $\mathsf{wt}(\eta^{\prec} \cap \bigcup_{i \leq n} V_n) < q_0 < q_1 < 2^{-|\eta|}$. This could be done effectively since the value of $\mathsf{wt}(\eta^{\prec} \cap \bigcup_{i \leq n} V_n)$ is computable. For each $\sigma \in V_n$, let $h = \max\{2^{|\gamma|} : \gamma \prec \sigma\}$ and define $V_{n+1,\sigma}$ to be a maximal prefix-free subset of $\{\tau : \sigma \prec \tau \wedge M(\tau) > \max\{\frac{2^{|\sigma|}}{q_1 - q_0}, h \cdot 4 \cdot 2^{|\sigma|}\}\}$. Then $V_{n+1} = \bigcup_{\sigma \in V_n} V_{n+1,\sigma}$. It is easy to see that V_{n+1} is computable and prefix-free. We are left to check that the invariant is satisfied. Given $\eta \notin \bigcup_{i \leq n+1} V_i$, assume $\eta^{\prec} \cap \bigcup_{i \leq n+1} V_i \neq \emptyset$ (otherwise we are done), we consider the following two cases.

Case 1: $\eta \prec \sigma$ for some $\sigma \in V_n$. By Kolmogorov inequality, for each such string, the measure added above σ is at most $\frac{q_1 - q_0}{2^{|\sigma|}}$. Therefore, $\mathsf{wt}(\eta^{\prec} \cap \bigcup_{i \leq n+1} V_i) \leq \mathsf{wt}(\eta^{\prec} \cap \bigcup_{i \leq n} V_i) + \Sigma_{\sigma \in V_n} 2^{-|\sigma|}(q_1 - q_0) \leq 2^{|\eta|}$, the last inequality uses the fact that V_n is prefix-free.

Case 2: $\eta \prec \beta$ for some $\beta \in V_{n+1}$ and $\eta \succ \sigma$ for some $\sigma \in V_n$. Let $k = \max\{\frac{2^{|\sigma|}}{q_1 - q_0}, h \cdot 4 \cdot 2^{|\sigma|}\}$. Since $\eta \notin V_{n+1}$, we know $M(\eta) \leq k$. Suppose $\mathsf{wt}(\bigcup_{j \leq n+1} V_j \cap \eta^{\prec}) = \mathsf{wt}(V_{n+1} \cap \eta^{\prec}) \geq 2^{-|\eta|}$. Then we have $k 2^{-|\eta|} \geq 2^{-|\eta|} M(\eta) \geq \Sigma_{\eta \prec \tau \wedge \tau \in V_{n+1}} 2^{-|\tau|} M(\tau) > k \Sigma_{\eta \prec \tau \wedge \tau \in V_{n+1}} 2^{-|\tau|} = k\mathsf{wt}(V_{n+1} \cap \eta^{\prec}) \geq k 2^{-|\eta|}$, contradiction.

The Madison test would be defined as $U_n = \bigcup_{i \leq n} V_n$, such that no string ever leaves, so the first requirement of Madison test is satisfied trivially. The second requirement is ensured by the construction invariant. Z fails the test since $\limsup_n M(Z \upharpoonright_n) = \infty$.

□

Lemma 4.6. *Suppose that Z passes each Madison test. Then every left-c.e. martingale L converges along Z.*

Proof. Now let $L(\sigma) = \sup_s L_s(\sigma)$ where $\langle L_s \rangle$ is a uniformly computable sequence of martingales and $L_0 = 0$. Since Z is computably random, for each s the limit $\lim_n L_s(Z\upharpoonright_n)$ exists. If L diverges along Z, there is $\varepsilon < L(\langle\rangle)$ such that for each s

$$\limsup_n L(Z\upharpoonright_n) - \lim_n L_s(Z\upharpoonright_n) > \varepsilon.$$

Based on this insight we define a Madison test which Z fails. Along with the U_s we define a uniformly computable labelling function $\gamma_s \colon U_s \to \{0, \dots, s\}$.

> *Let $U_0 = \emptyset$. For $s > 0$ we put the empty string $\langle\rangle$ into U_s and let $\gamma_s(\langle\rangle) = 0$. If already $\sigma \in U_s$ with $\gamma_s(\sigma) = t$, then we also put into U_s all strings $\tau \succ \sigma$ that are minimal under the prefix ordering $\prec$ with $L_s(\tau) - L_t(\tau) > \varepsilon$. Let $\gamma_s(\tau)$ be the least r with $L_r(\tau) - L_t(\tau) > \varepsilon$.*

Note that $\gamma_s(\tau)$ simply records the greatest stage $r \le s$ at which τ entered U_r. We verify that $\langle U_s \rangle$ is a Madison test. For (a), suppose that $\tau \in U_s - U_{s+1}$. Let $\sigma_0 \prec \sigma_1 \prec \dots \prec \sigma_n = \tau$ be the prefixes of τ in U_s. We can choose a least $i < n$ such that σ_{i+1} is no longer the minimal extension of σ_i at stage $s + 1$. Thus there is η with $\sigma_i \prec \eta \prec \sigma_{i+1}$ and $L_{s+1}(\eta) - L_{\gamma_s(\sigma_i)}(\eta) > \varepsilon$. Then $\eta \in U_{s+1}$ and $\eta \prec \tau$, as required.

To verify (b) requires more work.

We fix s and write $M_t(\eta)$ for $L_s(\eta) - L_t(\eta)$.

Claim 4.7. *For each $\rho \in U_s$, where $\gamma_s(\rho) = r$, we have*

$$2^{-|\rho|} M_r(\rho) \ge \varepsilon \cdot \mathsf{wt}(U_s \cap \rho^{\prec}).$$

In particular, for $\rho = \langle\rangle$, we obtain that $\mathsf{wt}(U_s)$ is bounded by a constant $c = L_s(\langle\rangle)/\varepsilon$ as required.

For $\sigma \in U_s$ and $n \in \mathbb{N}$ let $U_s^{\sigma,n}$ be the strings strictly above σ and at a distance to σ of at most n, that is, the set of strings τ such that there is $\sigma = \sigma_0 \prec \dots \prec \sigma_m = \tau$ on U_s with $m \le n$ and σ_{i+1} a child of σ_i for each $i < m$. To establish the claim, we show by induction on n that

$$2^{-|\rho|} M_r(\rho) \ge \varepsilon \cdot \mathsf{wt}(U_s^{\rho,n}).$$

If $n = 0$ then $U_s^{\rho,n}$ is empty so the right hand side is 0. Now suppose that $n > 0$. Let F be the set of immediate successors of ρ on U_s. Let $r_\tau = \gamma_s(\tau)$. By the inductive hypothesis, we have for each $\tau \in F$

$$\begin{aligned} 2^{-|\tau|} M_r(\tau) &= 2^{-|\tau|}[(L_{r_\tau}(\tau) - L_r(\tau)) + M_{r_\tau}(\tau)] \\ &\ge 2^{-|\tau|} \cdot \varepsilon + \varepsilon \cdot \mathsf{wt}(U_s^{\tau,n-1}). \end{aligned} \tag{1}$$

Then, taking the sum over all $\tau \in F$,

$$2^{-|\rho|} M_r(\rho) \ge \sum_{\tau \in F} 2^{-|\tau|} M_r(\tau) \ge \varepsilon \cdot \mathsf{wt}(U_s^{\rho,n}).$$

The first inequality is Kolmogorov's inequality for martingales, using that the τ form an antichain. For the second inequality we have used (1) and that $U_s^{\rho,n} = F \cup \bigcup_{\tau \in F} U_s^{\tau,n-1}$. This completes the induction and shows the claim.

Now, to obtain (b), suppose that $\mathsf{wt}(U_s \cap \sigma^{\prec}) > 2^{-|\sigma|}$. We use Claim 4.7 to show that $\sigma \in U_s$. Assume otherwise. Let $\rho \prec \sigma$ be in U_s with $|\rho|$ maximal, and let $r = \gamma_s(\rho)$. As before, let F be the prefix minimal extensions of σ in U_s, and $r_\tau = \gamma_s(\tau)$. Then $L_{r_\tau}(\tau) - L_r(\tau) > \varepsilon$ for $\tau \in F$. Since $\tau \in U_s$, we can apply the claim to τ, so (1) is valid.

Arguing as before, but with σ instead of ρ, we have

$$2^{-|\sigma|} M_r(\sigma) \geq \sum_{\tau \in F} 2^{-|\tau|} M_r(\tau) \geq \varepsilon \cdot \mathsf{wt}(U_s \cap \sigma^{\prec})$$

(that part of the argument did not use that $\rho \in U_s$). Since $\mathsf{wt}(U_s \cap \sigma^{\prec}) > 2^{-|\sigma|}$, this implies that $M_r(\sigma) > \epsilon$. Hence some σ' with $\rho \prec \sigma' \preceq \sigma$ is in U_s, contrary to the maximality of ρ.

This concludes the verification that $\langle U_s \rangle$ is a Madison test. As mentioned already, for each r there are infinitely many n with $L(Z\upharpoonright_n) - L_r(Z\upharpoonright_n) > \varepsilon$. This shows that Z fails this test: suppose inductively that we have $\sigma \prec Z$ and r is least such that $\sigma \in U_t$ for all $t \geq r$ (so that $\gamma_t(\sigma) = r$ for all such t). Choose $n > |\sigma|$ for this r. Then $\tau = Z\upharpoonright_n$ is a viable extension of σ, so τ, or some prefix of it that is longer that σ, is in U. □

5. Lowness for density randomness

We say A is *low for density randomness* if whenever Z is density random, Z is already density random relative to A. Although density randomness is a stronger notion than Martin-Löf randomness, the class of sets low for density randomness coincides with the class of sets low for Martin-Löf randomness. To establish the equivalence, we need the following lemmas. Denote $\mathsf{W2R}$ as the class of all weakly-2-random sets, i.e. sets that do not lie in any Π^0_2-null class of sets.

Lemma 5.1 (Downey, Nies, Weber and Yu [7]). *Low(*$\mathsf{W2R}$*,* MLR*)=Low(*MLR*).*

Lemma 5.2 (Day and Miller [4]). *Suppose Z is Martin-Löf random and A is K-trivial and $\mathcal{P}$ is a $\Pi^{0,A}_1$-class containing Z. Then there exists Π^0_1-class $\mathcal{Q} \subseteq \mathcal{P}$ such that $A \in \mathcal{Q}$.*

Theorem 5.3. *Given $A \in 2^\omega$, A is K-trivial if and only if A is low for density randomness.*

Proof. ($\Leftarrow$): Let DR denote the class of density random sets. By 5.1, and the fact the $\mathsf{W2R} \subseteq \mathsf{DR} \subseteq \mathsf{MLR}$, we know Low($\mathsf{DR}$) $\subseteq$ Low($\mathsf{W2R}$, MLR)=Low(MLR). It is known that the class K-trivial sets coincide with the class of sets that are low for Martin-Löf randomness.

($\Rightarrow$): Suppose A is not low for density randomness, i.e., there exists a set Z such that Z is density random but not density random relative to A. Since Z is Martin-Löf random, so it is Martin-Löf random relative to A by the K-triviality of A. Hence, there exists a $\Pi^{0,A}_1$-class $\mathcal{P}$ containing Z such that $\rho_2(\mathcal{P}|Z) < 1$. By Lemma 5.2, have Π^0_1-class $\mathcal{Q} \subseteq \mathcal{P}$ such that $Z \in \mathcal{Q}$, since Z is density random thus Martin-Löf random. Therefore, $\rho_2(\mathcal{Q}|Z) \leq \rho_2(\mathcal{P}|Z) < 1$, contradicting with the fact the Z is density random. □

6. Density-one points for Π^0_n-classes and Σ^1_1 classes

The discussion above focuses on density-one points of effectively closed sets. It is natural to ask how much randomness is required for a point to be a density-one point for Π^0_n-classes. The following analysis shows that these are not really different from density-one points in $\Pi_1^{0,\emptyset^{(n-1)}}$-classes. We rely on a technical lemma about the approximation of Π^0_n-classes with $\Pi_1^{0,\emptyset^{(n-1)}}$-classes.

Lemma 6.1 (Kurtz [11], Kautz [10])**.** *From an index of a Π^0_n-class P and $q \in \mathbb{Q}^+$, $\emptyset^{(n-1)}$ can compute an index of a $\Pi_1^{0,\emptyset^{(n-1)}}$-class $V \subseteq P$ such that $\lambda(P) - \lambda(V) < q$.*

Theorem 6.2. *Suppose $n \geq 1$ and $z \in 2^\omega$ is density random relative to $\emptyset^{(n-1)}$. Let P be Π^0_n-class such that $z \in P$. Then $\rho_2(P|z) = 1$.*

Proof. Suppose $P = \bigcap_s U_s$ where $\langle U_s : s \in \omega\rangle$ is a uniform nested Σ^0_{n-1}-class. If we could show that there exists a $\Pi_1^{0,\emptyset^{(n-1)}}$-class $Q \subseteq P$ such that $z \in Q$ then we are done.

We inductively define the following test. Given n, by Lemma 6.1 we $\emptyset^{(n-1)}$-effectively obtain the index of $\Pi_1^{0,\emptyset^{(n-1)}}$-class $Q_n \subseteq U_n$ such that

$$\lambda(U_n) - \lambda(Q_n) < 2^{-n}.$$

Then consider the test

$$\langle U_n \backslash Q_n : n \in \omega\rangle,$$

which is a uniform sequence of $\Sigma_1^{0,\emptyset^{(n-1)}}$-classes.

This is a Solovay test relative to $\emptyset^{(n-1)}$ since $\lambda(U_n\backslash Q_n) \leq 2^{-n}$. Notice $z \in P \subseteq U_n$ for each $n \in \omega$. Since z is Martin-Löf random relative to $\emptyset^{(n-1)}$, there exists $k \in \omega$ such that for all $j \geq k$, $z \in Q_j$. Since $\langle Q_j : j \geq k\rangle$ is a uniform sequence of $\Pi_1^{0,\emptyset^{(n-1)}}$-classes, the set $V = \bigcap_{j\geq k} Q_j$ is itself a $\Pi_1^{0,\emptyset^{(n-1)}}$-class. Also $V \subseteq \bigcap_{i\in\omega} U_i = P$ because $Q_j \subseteq U_j$. Thus we have found a $\Pi_1^{0,\emptyset^{(n-1)}}$-class $V \subseteq P$ that contains z. □

Corollary 6.3. *A Martin-Löf random set Z is a density one point for Π^0_n-classes if and only if every left-$\emptyset^{(n-1)}$-c.e. martingale converges along Z.*

The work of the Madison group described in Section **??** can be lifted to the domain of higher randomness. Interestingly, density one now can be equivalently required for any Σ^1_1 class containing the real, not necessarily closed.

We use the following fact due to Greenberg. It is a higher analog of the original weaker version of Prop. 3.1.

Proposition 6.4 (N. Greenberg, 2013)**.** *Let $\mathcal{C} \subseteq 2^{\mathbb{N}}$ be Σ^1_1. Let $Z \in \mathcal{C}$ be Π^1_1-ML-random. Then $\underline{\varrho}_2(\mathcal{C} \mid Z) = 1$.*

Proof. If $\underline{\varrho}_2(\mathcal{C} \mid Z) < 1$ then there is a positive rational $q < 1$ and n^* such that for all $n \geq n^*$ we have $\lambda_{Z\upharpoonright_n}(\mathcal{C}) < q$. Choose a rational r with $q < r < 1$. We define Π^1_1-anti chains in $2^{<\omega}$ U_n, uniformly in n. Let $U_0 = \{\langle Z\upharpoonright_{n^*}\rangle\}$. Suppose U_n has been defined. For each $\sigma \in U_n$, at a stage

α such that $\lambda_\sigma(\mathcal{C}_\alpha) < q$, we obtain effectively a hyper-arithmetical antichain V of extensions of σ such that $\mathcal{C}_\alpha \cap [\sigma] \subseteq [V]^{\prec}$ and $\lambda_\sigma([V]^{\prec}) < r$. Put V into U_{n+1}.

Clearly $\lambda U_n \le r^n$ for each n. Also, $Z \in \bigcap_n U_n$, so Z is not Π^1_1-ML-random. □

A martingale $M\colon 2^{<\omega} \to \mathbb{R}$ is called left-Π^1_1 if $M(\sigma)$ is a left-Π^1_1 real uniformly in σ.

Theorem 6.5. *Let Z be Π^1_1-ML-random. The following are equivalent.*

(i) *$\rho(\mathcal{C} \mid Z) = 1$ for each Σ^1_1-class $\mathcal{C}$ containing Z.*
(ii) *$\rho(\mathcal{C} \mid Z) = 1$ for each* closed *Σ^1_1-class $\mathcal{C}$ containing Z.*
(iii) *each left-Π^1_1 martingale converges on Z to a finite value.*

Proof. (iii) → (i): The measure of a Σ^1_1 set is left-Σ^1_1 in a uniform way (see e.g. [14, Ch. 9]). Therefore $M(\sigma) = 1 - \lambda_\sigma(\mathcal{C})$ is a left-Π^1_1 martingale. Since M converges along Z, and since by Prop. 6.4 $\liminf_n M(Z\restriction_n) = 0$, it converges along Z to 0. This shows that $\rho(\mathcal{C} \mid Z) = 1$.
(ii) → (iii). We follow the proof of the Madison Theorem 4.1 given above. All stages s are now interpreted as computable ordinals. Computable functions/ constructions, are now functions $\omega_1^{CK} \to L_{\omega_1^{CK}}$ with Σ_1 graph/ assignments of recursive ordinals to instructions.

Definition 6.6. A Π^1_1-*Madison test* is a Σ_1 over $L_{\omega_1^{CK}}$ function $\langle U_s\rangle_{s<\omega_1^{CK}}$ mapping ordinals to (hyperarithmetical) subsets of $2^{<\omega}$ such that $U_0 = \emptyset$, for each stage s we have $\mathsf{wt}(U_s) \le c$ for some constant c, and for all strings σ, τ,

(a) $\tau \in U_s - U_{s+1} \to \exists \sigma \prec \tau\, [\sigma \in U_{s+1} - U_s]$
(b) $\mathsf{wt}(\sigma^{\prec} \cap U_s) > 2^{-|\sigma|} \to \sigma \in U_s$.

Also $U_\gamma(\sigma) = \lim_{\alpha<\gamma} U_\alpha(\sigma)$ for each limit ordinal γ.

The following well-known fact can be proved similar to [14, 1.9.19].

Lemma 6.7. *Let $\mathcal{A} \subseteq 2^{\mathbb{N}}$ be a hyperarithmetical open. Given a rational q with $q > \lambda A$, we can effectively determine from $\mathcal{A}, q$ a hyperarithmetical open $\mathcal{S} \supseteq \mathcal{A}$ with $\lambda \mathcal{S} = q$.*

Lemma 6.8. *Let Z be a Π^1_1 ML-random such that $\rho(\mathcal{C} \mid Z) = 1$ for each* closed *Σ^1_1-class $\mathcal{C}$ containing Z. Then Z passes each Π^1_1-Madison test.*

The proof follows the proof of the analogous Lemma 4.4. The sets $\mathcal{A}^k_{\sigma,s}$ are now hyperarithmetical open sets computed from k, σ, s. Suppose $\sigma \in U_{s+1} - U_s$. The set $\widetilde{\mathcal{A}}^k_{\sigma,s}$ is defined as before. To effectively obtain $\mathcal{A}^k_{\sigma,s+1}$, we apply Lemma 6.7 to add mass from $[\sigma]$ to $\widetilde{\mathcal{A}}^k_{\sigma,s+1}$ in order to ensure $\lambda(\mathcal{A}^k_{\sigma,s+1}) = 2^{-|\sigma|-k}$ as required.

As before let $\mathcal{S}^k_t = \bigcup_{\sigma \in U_t} \mathcal{A}^k_{\sigma,t}$. Then $\mathcal{S}^k_t \subseteq \mathcal{S}^k_{t+1}$ by property (a) of Π^1_1 Madison tests. Clearly $\lambda \mathcal{S}^k_t \le 2^{-k}\mathsf{wt}(U_t) \le 2^{-k}$. So $\mathcal{S}^k = \bigcup_{t<\omega_1^{CK}} \mathcal{S}^k_t$ determines a Π^1_1 ML-test.

By construction $\overline{\rho}(2^{\mathbb{N}} - \mathcal{S}^k \mid Z) \le 1 - 2^{-k}$. Since Z is ML-random we have $Z \notin \mathcal{S}^k$ for some k. So $\overline{\rho}(\mathcal{C} \mid Z) < 1$ for the closed Σ^1_1-class $\mathcal{C} = 2^{\mathbb{N}} - \mathcal{S}_k$ containing Z.

The analog of Lemma 4.6 also holds.

Lemma 6.9. *Suppose that Z passes each Π^1_1-Madison test. Then every left-Π^1_1 martingale L converges along Z.*

The proof of 4.6 was already set up so that this works. The uniformly hyp labelling functions γ_s now map U_s to ω_1^{CK}. Note that the antichains F can now be infinite. □

7. Computable Analysis

We give a characterization of density randomness via the Lebesgue differentiation theorem.

7.1. Background on computable Analysis.

Computable Reals. A sequence of rational numbers $\langle q_k : k \in \omega \rangle$ is call *Cauchy name* if $|q_k - q_n| \leq 2^{-n}$ for each $n \geq k$. If $\lim_k q_k = x \in \mathbb{R}$ then we say $\langle q_k : k \in \omega \rangle$ is a *Cauchy name* for x. A real is computable if it has computable Cauchy name.

Computable Functions. Let $f : [0, 1] \to \mathbb{R}$, then f is computable if there exists a oracle Turing functional Φ such that given any Cauchy name $\langle q_k : k \in \omega \rangle$ for x, $\Phi^{\langle q_k : k \in \omega \rangle}$ outputs a Cauchy name for $f(x)$.

Lower/upper semi-computable functions. A function $f : [0, 1] \to \mathbb{R}$ is lower semi-computable if $f^{-1}(q, \infty)$ is effectively open, or equivalently, if it has an computable approximation from below by increasing rational-valued step functions $\langle f_s : s \in \omega \rangle$ defined on dyadic intervals such that $f = \sup_s f_s$ pointwise. Upper semi-computable functions are defined analogously.

For a more detailed background, we refer the readers to [16].

7.2. Integral tests.

Definition 7.1 (Integral test)**.** A function $f : [0, 1] \to \bar{\mathbb{R}}^+ = \mathbb{R}^+ \cup \{\infty\}$ is an integral test if it is lower semi-computable and $\int_{[0,1]} f d\mu < \infty$. Here the measure is the standard Lebesgue measure on real numbers.

The following fact is well known.

Theorem 7.2 (Li, Vitanyi [12])**.** *A real z is Martin-Löf random if and only if for each integral test f, $f(z) < \infty$.*

Definition 7.3. Given a Lebesgue integrable non-negative function f on $[0, 1]$, a point z in the domain of f is a weak Lebesgue point if

$$\lim_{\epsilon \to 0} \frac{\int_{B_\epsilon(x)} f d\mu}{\mu(B_\epsilon(x))}$$

exists. Further if this value equals $f(z)$, then z is called a Lebesgue point.

The following lemma shows that Martin-Löf randomness is strong enough to guarantee that weak Lebesgue points coincide with Lebesgue points. The general idea of the following proof is, given a point z, by assumption the derivative of the integration at z exists and if it does not equal the value of the function at z, then we are able to define an integral test whose value is infinity at z. By Theorem 7.2, we conclude z is not Martin-Löf random.

Lemma 7.4. *Suppose a Martin-Löf random set z is a dyadic weak Lebesgue point for an integral test f. Then z is a dyadic Lebesgue point for f.*

Proof. As a notation, for a function $f :\subseteq [0,1] \to \mathbb{R}$ and $z \in [0,1]$, let

$$E(f,\sigma) = \frac{\int_{[\sigma]} f \, d\mu}{2^{-n}}.$$

Then, z is a dyadic Lebesgue point iff $\lim_n E(f, z \upharpoonright n) = f(z)$. If f is a integral test, then $E(f,-)$ is a left-c.e. martingale.

Suppose that z is not a dyadic Lebesgue point for an integral test f and z is a dyadic weak Lebesgue point for f. Then $\lim_n E(f, z \upharpoonright n) =: r$ exists and $f(z) \neq r$.

By Lemma 4.6, 4.8 in [13], f could be approximated by a computable sequence of rational step functions up to Kurtz equivalence, i.e. their values coincide at each Kurtz random points. Let

$$f =_{\mathrm{Kurtz}} \sup_s f_s$$

where $\{f_s\}$ is a computable sequence of rational step functions.

Then, there is a computable order u such that $E(f_s,\sigma) = E(f_s,\sigma 0) = E(f_s,\sigma 1)$ for each σ satisfying $|\sigma| \geq u(s)$. Unless z is a dyadic rational, we have

$$\lim_n E(f_s, z \upharpoonright n) = f_s(z).$$

Firstly, suppose that $r < f(z)$. Since $\lim_s f_s(z) = f(z)$ for each Kurtz random point z, there is t such that

$$r < f_t(z) \leq f(z).$$

Then

$$r < f_t(z) = \lim_n E(f_t, z \upharpoonright n) \leq \lim_n E(f, z \upharpoonright n).$$

This is a contradiction.

Secondly, suppose that $r > f(z)$. Let q be a rational number such that $f(z) < q < r$. We build a new integral test g such that $g(z) = \infty$.

We prepare auxiliary uniformly c.e. sets $\{S_n\}$ where $S_n \subseteq 2^{<\omega} \times \omega$ for each n. Let $S_0 = \{(\lambda, 0)\}$ where λ is the empty string. For each $n \geq 1$ and $(\sigma, s) \in S_{n-1}$, computably enumerate (τ, t) into S_n^σ so that

- $\sigma \prec \tau$,
- $|\tau| \geq u(s)$,
- $E(f_t, \tau) > q$,
- $\{\tau \in 2^{<\omega} \ : \ (\tau, t) \in S_n^\sigma\}$ is prefix-free,

We can further assume that

$$\bigcup \{[\tau] \ : \ \sigma \prec \tau, \ |\tau| \geq u(s), \ E(f,\tau) > q\} = \bigcup \{[\tau] \ : \ (\tau, t) \in S_n^\sigma\}.$$

Let $S_n = \bigcup_{(\sigma,s) \in S_{n-1}} S_n^\sigma$.

For each $(\tau, t) \in S_n$, let

$$g_\tau = (q - E(f_s, \tau)) \mathbf{1}_{[\tau]}$$

where $(\sigma, s) \in S_{n-1}$ and $\sigma \prec \tau$. We define g by

$$g = \sum_n \sum_{(\tau,t) \in S_n} g_\tau.$$

Note that

$$\int g_\tau \, d\mu \leq (E(f_t,\tau) - E(f_s,\tau))2^{-|\tau|} = \int_{[\tau]} (f_t - f_s) d\mu,$$

thus $\int g \, d\mu \leq \int f \, d\mu < \infty$. Hence, g is an integral test.

Since $\lim_n E(f, z \upharpoonright n) = r > q$, there exists $(\tau_n, t_n) \in S_n$ such that $\tau_n \prec z$ for each n. Then,

$$g(z) = \sum_n (q - E(f_s,\tau)) \geq \sum_n (q - f(z)) = \infty.$$

□

Remark 7.5 (A special class of left-c.e. martingales)**.** *Let L be a left-c.e. martingale. It is called* `special` *if it has a non-decreasing approximations $\langle L_s : s \in \omega \rangle$ and there exists a computable function $u : \omega \to \omega$ such that whenever $\tau \in 2^{<\omega}$ and $|\tau| > u(s)$, for all $\alpha, \beta \succ \tau$, $L_s(\alpha) = L_s(\beta)$. The martingale we construct in the proof above is* `special`*. In fact we actually prove the following: for each Martin-Löf random set Z and* `special` *left-c.e. martingale $L = \sup L_s$, if $\lim_n L(Z \upharpoonright_n)$ exists, then $\lim_n L(Z \upharpoonright_n) = \lim_s \lim_n L_s(Z \upharpoonright_n)$.*

Theorem 7.6. *The following are equivalent for $z \in [0,1]$:*

(i) *z is density random.*
(ii) *z is a dyadic Lebesgue point for each integral test.*
(iii) *z is a Lebesgue point for each integral test.*

Note that one direction is easy.

Proof of (ii) ⇒ (i) of Theorem 7.6. Suppose that z is a Lebesgue point for each integral test. Then $f(z)$ is finite for each integral test f, whence z is ML-random.

Let C be a Π^0_1 class containing z. We define a function $f : [0,1] \to \overline{\mathbb{R}}^+$ by

$$f(x) = \begin{cases} 1 & \text{if } x \notin C \\ 0 & \text{if } x \in C. \end{cases}$$

Then, f is an integral test. Since z is a Lebesgue point for f, C has density-one at z. □

We next prove the converse.

Proof of (i) ⇒ (ii) of Theorem 4.2. Suppose that z is density random. Let f be an integral test. Then $E(f, -)$ as defined in Lemma 7.4 is a left-c.e. martingale. By Theorem 4.2, $\lim_n E(f, z \upharpoonright n)$ exists, whence z is a dyadic weak Lebesgue point for f. By Lemma 7.4, z is a dyadic Lebesgue point for f. □

Recall the definition of interval-c.e. functions.

Definition 7.7. A non-decreasing, lower semi-continuous function $f : [0,1] \to \mathbb{R}$ is call interval-c.e. if $f(0) = 0$ and $f(y) - f(x)$ is a left-c.e. real uniformly in rationals $x < y$.

To drop "dyadic", we recall the following result.

Theorem 7.8 (Nies [8])**.** *Let $f\colon [0,1] \to \mathbb{R}$ be interval-c.e. Let z be density random. Then $f'(z)$ exists.*

Proof of (ii) $\iff$ (iii) of Theorem 4.2. Note that (iii) $\Rightarrow$ (ii) holds by definition.

We prove (ii) $\Rightarrow$ (iii). Suppose that z is a dyadic Lebesgue point for each integral test. Then z is a convergence point for left-c.e. martingales.

Let f be an integral test. Then, $F(x) = \int_{[0,x]} f \ d\mu$ is interval-c.e. Thus, $F'(z)$ exists.

In particular, $\limsup_{Q \to z} \frac{\int_Q f \ d\mu}{\mu(Q)}$ exists and is equal to $\limsup_n \frac{\int_{[z\restriction n]} f \ d\mu}{2^{-n}} = f(z)$. Hence, z is a Lebesgue point for f. □

Our next goal is to find a special integral test such that it alone characterizes density random points. To achieve that, we need the following lemma, due to Madison Group, Andrews, Cai, Diamondstone, Lempp and Miller, which states the convergence of the sum of two left-c.e. martingales implies the convergence of each summand.

Lemma 7.9. *Let d_1, d_2 be left-c.e. martingales and $Z \in 2^\omega$ is a Martin-Löf random set. Suppose $\lim_n (d_1 + d_2)(Z\restriction_n)$ exists, then $\lim_n d_i(Z\restriction_n)$ exists for each $i = 1, 2$.*

Proof. Let $c \in \mathbb{R}$ such that $\lim_n (d_1 + d_2)(Z\restriction_n) = c$. If $c = 0$ we are done so assume $c > 0$. Suppose d_1 does not converge on Z, the other case being symmetric. The idea of the proof is as follows: since $d_1 + d_2$ converges, there exists a natural number N where for any $n > N$ the value of $(d_1 + d_2)(Z\restriction_n)$ is sufficiently close to c. Whenever the value of d_1 (d_2 respectively) is large, we know d_2 (d_1) is relatively small. We thus bet our capital according to d_2 (d_1). We give the details in the following.

Claim 7.10. *There are four rationals $a > b, e > d$ and a natural number N such that*

(i) *there are infinitely many i, j such that $d_1(Z\restriction_i) > a$ and $d_2(Z\restriction_j) > e$;*
(ii) *for all $n > N$, whenever $d_1(Z\restriction_n) > a$ then $d_2(Z\restriction_n) < d$; whenever $d_2(Z\restriction_n) > e$ then $d_1(Z\restriction_n) < \frac{a+b}{2}$.*

Proof of the claim. By the assumption of divergence of d_1, there exists $a, b \in \mathbb{Q}$ such that $0 < b < a < c$ with $\limsup_n d_1(Z\restriction_n) > a$ and $\liminf_n d_1(Z\restriction_n) < b$. Choose $q \in \mathbb{Q}$ with

$$q < \min\{\frac{a-b}{6}, \frac{a}{2}, \frac{b}{2}\}$$

and $N \in \omega$ such that

$$\forall n > N \ |(d_1 + d_2)(Z\restriction_n) - c| < q.$$

The existence of N follows from the convergence of $d_1 + d_2$ on Z.

Fix $n > N$. Since $d_1(Z\restriction_n) + d_2(Z\restriction_n) < c + q$, we know for infinitely $i > N$,

$$d_2(Z\restriction_i) < c + q - d_1(Z\restriction_i) < c + q - a < c + 2q - a < c.$$

Choose $d \in \mathbb{Q}$ such that

$$c + q - a < d < c + 2q - a.$$

Also for infinitely many $j > N$, we have $d_1(Z\restriction_j) + d_2(Z\restriction_j) > c - q$ and $d_1(Z\restriction_j) < b$ so $d_2(Z\restriction_j) > c - q - b$. Notice that $c - q - b > c + 2q - a$ since $q < \frac{a-b}{6}$. We choose $e \in \mathbb{Q}$ such that

$$\max\{c + 2q - a, c - 2q - b\} < e < c - q - b.$$

Thus there are infinitely $j > N$ such that $d_2(Z\restriction_j) > e$. For such j,

$$d_1(Z\restriction_j) + d_2(Z\restriction_j) < c + q \Rightarrow d_1(Z\restriction_j) < c + q - e < 3q + b < \frac{a+b}{2}.$$

□

Next we build a left-c.e. martingale M which succeeds on Z. We inductively associate each string with a strategy– {0,1,2}. For any $\tau \in 2^{<\omega}$ with $|\tau| \leq N$, $M(\tau) = 1$. Each such τ has 0-strategy.

Given τ,

Case 1: It is associated with 0-strategy. if $|\tau| > N$ and $d_1(\tau) > a$, τ switches to 2-strategy. If $|\tau| > N$ and $d_2(\tau) > e$, τ switches to 1-strategy. Otherwise $M(\tau^\frown 0) = M(\tau^\frown 1) = M(\tau)$.

Case 2: It is associated with 1-strategy. If $d_1(\tau) > a$, switch to 2-strategy. Otherwise if $d_1(\tau) = 0$, $M(\tau^\frown i) = 0, i = 0, 1$. If $d_1(\tau) \neq 0$, $M(\tau^\frown 0) = \dfrac{d_1(\tau^\frown 0)}{d_1(\tau)} \cdot M(\tau)$ and $M(\tau^\frown 1) = \dfrac{d_1(\tau^\frown 1)}{d_1(\tau)} \cdot M(\tau)$.

Case 3: It is associated with 2-strategy. If $d_2(\tau) > e$, switch to 1-strategy. Otherwise if $d_2(\tau) = 0$, $M(\tau^\frown i) = 0, i = 0, 1$. If $d_2(\tau) \neq 0$,$M(\tau^\frown 0) = \dfrac{d_2(\tau^\frown 0)}{d_2(\tau)} \cdot M(\tau)$ and $M(\tau^\frown 1) = \dfrac{d_2(\tau^\frown 1)}{d_2(\tau)} \cdot M(\tau)$.

It is easy to verify that the martingale M as defined is left-c.e..

To check M succeeds on Z, notice that every time we switch from 1-strategy to 2-strategy, the capital increases by a factor of $\dfrac{a}{(a+b)/2} > 1$, and every time we switch from 2-strategy to 1-strategy the capital increases by a factor of $\dfrac{e}{d} > 1$. Since by assumption, this happens infinitely often, M succeeds on Z. □

Corollary 7.11. *Let f, g be integral tests. If an ML-random set x is a dyadic weak Lebesgue point for $f + g$, then x is a dyadic weak Lebesgue point for f.*

Proof. Notice for any lower semi-computable function f, the function

$$E(f, \sigma) = \frac{\int_{[\sigma]} f d\mu}{2^{-|\sigma|}}$$

is a left-c.e. martingale. Apply the previous lemma, we are done. □

Actually one integral test characterizes density randomness.

Theorem 7.12. *Let f be a Solovay-complete integral test. Then x is a Lebesuge point for f if and only if x is density random.*

Proof. The "if" direction follows from Theorem 7.6.

Suppose x is not density random. We can assume that x is ML-random, because, otherwise, $f(x) = \infty$ and x is not a dyadic Lebesgue point for f. Then there is an integral test g such that x is not a dyadic Lebesgue point for g. Since f is Solovay-complete, there are a rational q and an integral test h such that

$$f = \frac{g}{q} + h.$$

Notice that x is not a dyadic Lebesgue point for $\frac{g}{q}$. By Lemma 7.4, x is not a dyadic weak Lebesgue point for $\frac{g}{q}$. By the lemmas above, x is not a dyadic weak Lebesgue point for f. Thus, x is not a Lebesgue point for f. □

This finishes the characterization of density random points using integral tests. Finally, compare different classes of functions as mentioned in the introduction.

Remark 7.13 (Compare different classes of functions)**.** *Recall the three classes of functions mentioned in the Introduction 1.*

Next we construct the following counter-examples to show each implication is strict.

Example 7.14 ([1], (2) $\not\Rightarrow$ (1))**.** *There is a non-decreasing computable (hence, interval-c.e.) Lipschitz function f that is not of the form $f(x) = \int_0^x g d\mu$ for any lower semi-computable function g.*

Proof. Let M be a computable martingale that succeeds on any $Z \in 2^{\mathbb{N}}$ failing the law of large numbers. By Theorem 4.2 of [9] (and its proof) there is a computable Lipschitz function f such that $f'(z)$ fails to exist whenever M succeeds on a binary expansion Z of z. Adding a linear term, we may assume that f is nondecreasing. Now suppose $f(x) = \int_0^x g \,\mathrm{d}\lambda$ for a lower semicontinuous function g. If C is a Lipschitz constant for f, then $\{x\colon g(x) > C\}$ is a null set. Since this set is also open, it is empty. Hence g is bounded.

If Z is 1-generic then Z is a density-one point by the following observation: an effectively closed class C contains a 1-generic z only if C contains an open interval around z. Hence by [1, Proposition 7.1], z is a Lebesgue point of g. Then $f'(z)$ exists.

On the other hand, each 1-generic Z fails the law of large numbers. So M succeeds on Z, and $f'(z)$ does not exist. Contradiction. □

Example 7.15 (See for example [15], (3) $\not\Rightarrow$ (2))**.** *There is an interval-c.e. continuous function which is not absolutely continuous.*

This example is the well-known Cantor's function. Formally, we define $f : [0, 1] \to [0, 1]$ as follows:

(i) *Express x in base 3;*
(ii) *If x contains a 1, replace every digit after the first 1 by 0;*
(iii) *Replace all 2's by 1's*
(iv) *Interpret the result as a binary expression of a real number, and output as $f(x)$.*

We could define the computable lower approximation of the function as we inductively define the cantor set. This function is not absolutely continuous since it maps the complement of Cantor set, which has measure 1, to a null set.

8. Questions

Question 8.1. *Does Density Randomness coincide with Oberwolfach Randomness.*

Question 8.2. *Does there exist an ω-c.e. Turing incomplete Martin-Löf random set which computes all K-trivial sets?*

Question 8.3. *Does there exist a Δ^0_2 random exact pair for the class of K-trivial sets?*

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