# Category theory

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## 1 Introduction

The contrast between set theory and categories is instructive for the different motivations they provide. Set theory says encouragingly that it's what's inside that counts. By contrast, category theory proclaims that it's what you do that really matters.

Category theory is the study of mathematical structures by means of the relationships between them. It provides a framework for considering the diverse contents of mathematics from the logical to the topological. Consequently we present numerous examples to explicate and contextualize the abstract notions in category theory.

Category theory was developed by Saunders Mac Lane and Samuel Eilenberg in the 1940s. Initially it was associated with algebraic topology and geometry and proved particularly fertile for the Grothendieck school. In recent years category theory has been associated with areas as diverse as computability, algebra and quantum mechanics.

Sections 2 to 3 outline the basic grammar of category theory. The major reference is the expository essay [Sch01]. The basic material was studied and written jointly with Helen Broome.

In Sections 4 to 6, the focus turns to the *abelian categories*, a broad family of categories which satisfy a set of axioms similar to the category of modules. The aim of these sections is to illustrate the use of categorical methods in algebra.

The example explored is the classical Krull-Schmidt theorem for modules satisfying mild finiteness conditions, which guarantees the existence of a unique decomposition as a sum of indecomposable modules. We gradually develop the tools needed to re-formulate the Krull-Schmidt theorem and its proof in the language of abelian categories, its natural, more general setting.

The exposition of the elementary theory of abelian categories follows [Fre64]. The treatment of Grothendieck categories and the categorical Krull-Schmidt theorem is based on those in [Par70] and [BD68]. Lang's algebra text [Lan80] was used as a reference for the classical Krull-Schmidt theorem.

# 2 Fundamental concepts

## 2.1 Categories

**Definition 1.** *A* category $\mathfrak{A}$ *comprises*

*(i)* *a class of* objects $\mathrm{Ob}(\mathfrak{A})$*;*

*(ii)* *for any pair* $A, A' \in \mathrm{Ob}(\mathfrak{A})$*, a set* $\mathbf{Hom}(A, A')$ *of* morphisms*;*

*(iii)* *for any triple* $A, A', A'' \in \mathrm{Ob}(\mathfrak{A})$*, a* composition *map*

$$\mathbf{Hom}(A', A'') \times \mathbf{Hom}(A, A') \to \mathbf{Hom}(A, A''),$$

*denoted* $(f, g) \mapsto f \circ g$*,*

*which satisfy the following axioms:*

1. *The sets* $\mathbf{Hom}(A, A')$ *are disjoint.*
2. *Composition is associative where defined. That is, for any quadruple* $A, A', A'', A''' \in \mathrm{Ob}(\mathfrak{A})$*, and any triple*

$$(f, g, h) \in \mathbf{Hom}(A, A') \times \mathbf{Hom}(A', A'') \times \mathbf{Hom}(A'', A'''),$$

*we demand that*

$$(h \circ g) \circ f = h \circ (g \circ f).$$

3. *For each object* $A \in \mathrm{Ob}(\mathfrak{A})$*, there exists an* identity *morphism*

$$id_A \in \mathbf{Hom}(A, A)$$

*with the obvious composition properties.*

We often write $A \in \mathfrak{A}$ for objects of $\mathfrak{A}$ rather than $A \in Ob(\mathfrak{A})$, and $f : A \to A'$ for morphisms rather than $f \in \mathbf{Hom}(A, A')$.

The categories typically studied are standard classes of mathematical structure, together with the appropriate structure-preserving functions.

**Example 2.** *The following are categories.*

1. *The category* **Set** *has sets as objects and functions as morphisms.*
2. *Define a* pointed set *to be an ordered pair* $(X, p)$*, where* $X$ *is a set and* $p \in X$ *– as objects. Define a* point-preserving function $f : (X, p) \to (Y, q)$ *to be a function* $f : X \to Y$ *such that* $f(p) = q$*. The category* **pSet** *has pointed sets as objects and point-preserving functions as morphisms.*
3. *Let* $G$ *be a group. Define a (left) group action on a set* $X$ *to be a function* $G \times X \to X$ *given by* $(g, x) \longmapsto g \cdot x$ *satisfying the obvious composition and identity properties. A set* $X$ *equipped with an action of* $G$ *on* $X$ *is said to be a* $G$-set*. Given two* $G$*-sets* $X$ *and* $Y$*, call a function* $f : X \to Y$ *a* $G$-map*, if* $f(g \cdot x) = g \cdot f(x)$*. The category* ***G-Set*** *has* $G$*-sets as objects and* $G$*-maps as morphisms.*

4. *The categories* **Grp**, **Ring**, *and (for any fixed ring unitary $R$)* ***R-Mod*** *have, respectively, groups, rings, and unitary $R$-modules as objects, and group homomorphisms, ring homomorphisms, and module homomorphisms as morphisms.*

5. *The category* **Top** *has topological spaces as objects and continuous maps as morphisms. The category* **Met** *has metric spaces as objects and uniformly continuous maps as morphisms.*

**Definition 3.** *The category $\mathfrak{B}$ is a* subcategory *of the category $\mathfrak{A}$, if the following conditions are satisfied:*

1. $\mathrm{Ob}(\mathfrak{B})$ *is a subclass of* $\mathrm{Ob}(\mathfrak{A})$.

2. *For each pair $B, B' \in \mathrm{Ob}(\mathfrak{B})$, the set $\mathbf{Hom}_{\mathfrak{B}}(B, B')$ of $\mathfrak{B}$-morphisms from $B$ to $B'$ is a subset of the set $\mathbf{Hom}_{\mathfrak{A}}(B, B')$ of $\mathfrak{A}$-morphisms from $B$ to $B'$.*

3. *The composition operation in $\mathfrak{B}$, where defined, is the restriction of the composition operation in $\mathfrak{A}$.*

*A subcategory $\mathfrak{B}$ of $\mathfrak{A}$ is* full, *if for each pair $B$, $B'$ of objects in $\mathfrak{B}$, the set $\mathbf{Hom}_{\mathfrak{B}}(B, B')$ is equal to all of the set $\mathbf{Hom}_{\mathfrak{A}}(B, B')$.*

**Example 4.** *The category* **Ab** *of abelian groups is a full subcategory of* **Grp**. *The category* **Met** *of metric spaces is a subcategory, not full, of* **Top**.

It must be stressed that a category depends as much on its objects as its morphisms; morphisms usually form the basis of definitions of categorical notions. Quite different categories can therefore be constructed whose underlying classes of objects are the same. For instance, the category **hTop**, with topological spaces as objects and homotopy equivalence classes of continuous maps as morphisms, differs from the category **Top**.

By contrast, families of categories ‘extremal’ in some way often reduce to well-studied classes of mathematical object.

**Example 5.** *A category with one object is essentially a monoid. The elements of the monoid are the morphisms from the unique object to itself, and monoid multiplication is composition of morphisms.*

**Example 6.** *Let $\mathfrak{J}$ be a* small category*; that is, a category whose class of objects is a set. Suppose that there is at most one morphism $f : J \to J'$ between any pair of objects in $\mathfrak{J}$, and that $\mathbf{Hom}(J, J')$ and $\mathbf{Hom}(J', J)$ are both nonempty only if $A = A'$.*

*Such a category is essentially a partially ordered set. The elements of the poset are the objects of $\mathfrak{J}$. The order relation on $\mathfrak{J}$ is defined by*

$$J \leq J' \quad \textit{iff} \quad \mathbf{Hom}(J, J') \textit{ is nonempty.}$$

*Reflexivity and transitivity are proved respectively by the existence of identity morphisms and of morphism composition.*

**Example 7.** *There is a unique category* 1 *with one object and one morphism.*

**Example 8.** *A directed graph $G$ gives rise to a small category, whose objects are $G$'s vertices and whose morphisms are $G$'s paths.*

Duality pervades category theory and for any concept there is a dual notion. Given a category $\mathfrak{A}$ the dual category $\mathfrak{A}^{op}$ consists of the same objects as $\mathfrak{A}$ but $Hom_{\mathfrak{A}^{op}}(A, A') = Hom_{\mathfrak{A}}(A', A)$. The dual notion is always found by simply reversing all the morphisms.

## 2.2 Morphisms

**Definition 9.** *A morphism $m : A \to A'$ in $\mathfrak{A}$ is monic if for any object $K$ and any pair of morphisms $f, g : K \to A$, the equality $m \circ f = m \circ g$ implies that $f = g$.*

$$K \underset{g}{\overset{f}{\rightrightarrows}} A \xrightarrow{m} A'$$

Dual to monic morphisms are the epic morphisms.

**Definition 10.** *A morphism $e : A' \to A$ in $\mathfrak{A}$ is epic if for any object $K$ and any pair of morphisms $f, g : A \to K$, the equality $f \circ e = g \circ e$ implies that $f = g$.*

$$A' \xrightarrow{e} A \underset{g}{\overset{f}{\rightrightarrows}} K$$

**Example 11.** *In* **Set** *and in* **Grp**, *the monic and epic morphisms are the injective and surjective functions respectively.*

*For* **Grp** *this is not entirely obvious. For instance, in the epic case, the proof consists of the construction, for each subgroup $H$ of a group $G$, of a group $K$ and nontrivial homomorphism $f : G \to K$, such that $f$ is trivial outside $H$.*

**Example 12.** *In the category* **Haus** *of Hausdorff spaces and continuous maps, a morphism is epic precisely if its image is dense in its range.*

**Definition 13.** *A morphism $f : A \to A'$ is an* isomorphism *if there exists a morphism $f' : A' \to A$ such that $f \circ f' = id_{A'}$ and $f' \circ f = id_A$.*

In many common categories, such as **Set**, **Grp** and $R$-**Mod**, all morphisms which are monic and epic are isomorphisms. A general explanation for this phenomenon develops in Section 5. On the other hand, the example of the category **Haus** shows that this need not always be the case.

## 2.3 Functors

In a given category a morphism acts between objects of that category. This concept of a morphism can be expanded beyond the context of one category to talk about an operation between categories.

**Definition 14.** *Let $\mathfrak{A}$ and $\mathfrak{B}$ be arbitrary categories. A* functor $F : \mathfrak{A} \to \mathfrak{B}$ *is an operation which acts as follows;*

1. *An object $A \in \mathfrak{A}$ is assigned to $FA \in \mathfrak{B}$;*
2. *If $f : A \to A'$ is a morphism in $\mathfrak{A}$ then $F(f) : FA \to FA'$ is a morphism in $\mathfrak{B}$;*
3. *For any $A \in \mathfrak{A}$, $F(id_A) = id_{FA}$;*
4. *If $g \circ f$ is defined in $\mathfrak{A}$ then $F(g \circ f) = F(g) \circ F(f)$ in $\mathfrak{B}$.*

**Example 15.** *The diagonal functor $\Delta : \mathfrak{A} \to \mathfrak{A} \times \mathfrak{A}$ sends each object $A \in \mathfrak{A}$ to $(A, A) \in \mathfrak{A} \times \mathfrak{A}$.*

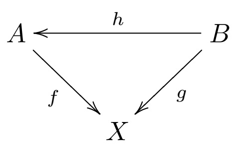
**Example 16.** *The forgetful functor* **For** : **Grp** $\to$ **Set** *sends each group to the underlying set. By comparison the free functor* **Free** : **Set** $\to$ **Grp** *sends each set to the free group on that set.*

## 2.4 Subobjects

The useful concept of a subset or subgroup is traditionally defined in terms of element membership. As category theory is based on morphisms rather than elements, the categorical definition of a subobject is instead based on the idea of an inclusion morphism.

For instance, consider the category **Set** and objects $X$ and $S$ where $S \subseteq X$. The image of the inclusion function $\iota : S \to X$ is $S$. While $\iota$ is monic, there are many other monics $m : S' \to X$ whose image is $S$. All such monics define the subset $S$ in a way equivalent to $\iota$. We can define the subobjects of $X$ by formalising the notion of equivalent monics.

**Definition 17.** *A monomorphisms $f : A \to X$* dominates *a monomorphism $g : B \to X$, if there exists a morphism $h : B \to A$ such that $g = f \circ h$.*




*Two monomorphisms are* equivalent*, if each dominates the other.*

**Example 18.** *The above analysis shows that for an inclusion function $\iota : S \to X$ in* **Set***, the equivalence class $[\iota]$ is precisely the set of all injective functions into the set $X$ whose image is the subset $S$.*

**Definition 19.** *A* subobject *of an object $A$ in $\mathfrak{A}$ is an equivalence class of monomorphisms.*

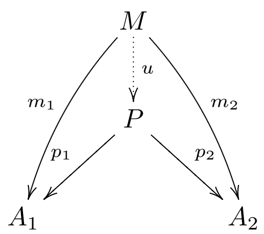
The domination relation bestows a natural partial order on an object's class of subobjects.

Dually, a *quotient object* of an object $A$ in $\mathfrak{A}$ is defined to be an equivalence class of epimorphisms out of $A$, under the appropriate equivalence relation.

## 2.5 Products, equalizers, limits

Let $A_1$ and $A_2$ be objects in a category $\mathfrak{A}$.

**Definition 20.** *A* product *of $A_1$ and $A_2$ is an object $P$ in $\mathfrak{A}$ along with a pair of morphisms $p_i : P \to A_i$ for $i = 1, 2$ such that for any object $M$ and pair of morphisms $m_i : M \to A_i$ for $i = 1, 2$, there exists a unique morphism $u : M \to A_1 \times A_2$ making $p_i \circ u = m_i$.*



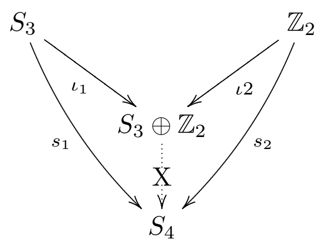

A product of $A_1$ and $A_2$, should it exist, is unique up to isomorphism. We denote the product of $A_1$ and $A_2$ by $A_1 \times A_2$.

A categorical product captures the notion of cartesian products in **Set** and direct products in **Grp**.

**Example 21.** *Consider a partially ordered set as a category with the order relation as the morphisms, then the product of two elements is their greatest lower bound.*

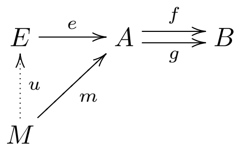
The dual notion is that of *coproduct.* In **Set** the coproduct is the disjoint union with the usual inclusion maps. The coproduct is sometimes referred to as the sum but while a coproduct in **Ab** is a direct sum, a coproduct in **Grp** is not.

**Example 22.** *Given $S_3$ and $\mathbb{Z}_2$ in* **Grp***, the direct sum $S_3 \oplus \mathbb{Z}_2$ with inclusion maps $\iota_1, \iota_2$ is not a coproduct. This can be seen by looking at the group $S_4$ and maps $s_1 : S_3 \to S_4$ and $s_2 : \mathbb{Z}_2 \to S_4$.*




*If we assume the existence of a unique map $u : S_3 \oplus \mathbb{Z}_2 \to S_4$ such that $u \circ \iota_i = s_i$ for i=1,2 then we get the contradiction that the image of u has order 24 while the domain of u has order 12. Hence the direct sum is not a coproduct in* **Grp***.*

**Definition 23.** *Given two morphisms $f, g : A \to B$ an* equalizer *is an object $E$ together with a morphism $e : E \to A$ such that $f \circ e = g \circ e$ with the property that for any other morphism $m : M \to A$ where $f \circ m = g \circ m$ there exists a* *unique morphism $u : M \to E$ such that $e \circ u = m$.*



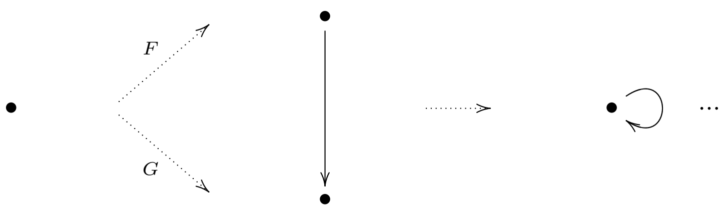

**Example 24.** *In* **Set** *an equalizer for $f, g$ is the set $E = \{x : f(x) = g(x)\}$ with inclusion map.*

For algebraic categories such as groups, rings and vector spaces the equalizer is constructed in the same way as in **Set**.

The dual notion, *coequalizer*, is not as simple.

**Example 25.** *In* **Set** *a coequalizer of $f, g$ is a quotient of the set $B$ by the smallest equivalence relation $\sim$ such that $f(a) \sim g(a)$ for all $a \in A$.*

**Example 26.** *In the category of small categories let* **1** *be the one object category with only the identity morphism and let* **2** *be the two object category with exactly one non identity morphism. The coequalizer of the only two unique functors $F, G : \mathbf{1} \to \mathbf{2}$ is the monoid of natural numbers under addition, $\mathbb{N}$.*




**Lemma 27.** *Equalizers are monic. Coequalizers are epic.*

There are obvious analogies in the definitions of products and equalizers. We can generalize to the notion of a *limit* as a suitable object and family of morphisms defined over a collection of objects and morphisms.

To make this precise, we first formalise the notion of a collection of objects and morphisms, secondly we give the criteria for being a suitable object with morphisms over this collection and lastly the conditions that make this suitable object and morphism a limit.

We call a collection of objects and morphisms a diagram which is like an indexed set that also accounts for the arrangement of the morphisms.

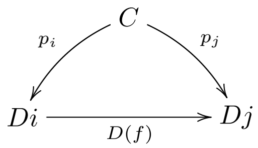
**Definition 28.** *A* diagram *into the category $\mathfrak{A}$ is a functor $D : \mathfrak{J} \to \mathfrak{A}$, such that the* index category $\mathfrak{J}$ *is small.*

**Example 29.** *An ordered pair in $\mathfrak{A}$ is a diagram into $\mathfrak{A}$ from the index category*

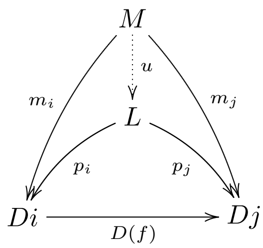


Let $D : \mathfrak{J} \to \mathfrak{A}$ be a diagram into $\mathfrak{A}$.

**Definition 30.** *A* cone *over $D$ is an object $C \in \mathfrak{A}$, together with morphisms $p_i : C \to Di$ for each object $i$ in the index category $\mathfrak{J}$, such that for any morphism $D(f) : Di \to Dj$ in the diagram $D(f) \circ p_i = p_j$.*




**Definition 31.** *A* limit *of $D$ is a cone $L$ over $D$ with morphisms $(p_i)_{i \in \mathfrak{J}}$, such that for any other cone $M$ with morphisms $(m_i)_{i \in \mathfrak{J}}$, there exists a unique morphism $u : M \to L$ such that for all $i \in \mathfrak{J}$, we have $m_i = p_i \circ u$.*




A limit of $D$, should it exist, is unique up to isomorphism.

**Example 32.** *An equalizer is a limit over a diagram whose index category is*

$$\bullet \rightrightarrows \bullet$$

**Example 33.** *A product is a limit over a diagram whose index category is*

$$\bullet \quad \bullet$$

Like any category the index category of a diagram must have identiy morphisms. For the purpose of establishing a limit over a diagram the identity morphisms are irrelevant as they commute with every morphism. For simplicity we ignore the identity morphisms in an index category.

A *terminal object* is a limit over the empty diagram. Suppose $\mathfrak{A}$ has a terminal object $T$. Then any other object $A \in \mathfrak{A}$ is a cone over the empty diagram, and therefore there exists a unique map $u : A \to T$.

**Example 34.** *In* **Set** *the terminal objects are the singleton sets. There is only one function from any set $A$ into a singleton set $\{x\}$.*

*(On the other hand, there may be many functions $f : \{x\} \to A$; these have the useful property of picking out elements of $A$.)*

# 3 Universal properties

## 3.1 Natural transformations

Just as a functor is like a morphism between two categories, a natural transformation is like a morphism between two functors. Let $\mathfrak{A}, \mathfrak{B}$ be two categories and $F, G : \mathfrak{A} \to \mathfrak{B}$ be two functors between the categories.

If $h : X \to Y$ is a morphism in $\mathfrak{A}$ then $F(h)$ and $G(h)$ are morphisms in $\mathfrak{B}$.

$$\begin{array}{ccc} FX & & GX \\ {\scriptstyle F(h)}\downarrow & & \downarrow{\scriptstyle G(h)} \\ FY & & GY \end{array}$$

A natural transformation between the functors $F, G$ guarantees a way to connect the two $\mathfrak{B}$ morphisms $F(h)$ and $G(h)$.

**Definition 35.** *A natural transformation $\eta$ between the functors $F$ and $G$ is a family of $\mathfrak{B}$ morphism $\eta_X : FX \to GX$, one for each $X$ in $\mathfrak{A}$ called the component at $X$. The family of morphisms $\eta_X$ satisfy the condition that for any $h : X \to Y$ in $\mathfrak{A}$ the following square commutes,*

$$\begin{array}{ccc} FX & \xrightarrow{\eta_X} & GX \\ {\scriptstyle F(h)}\downarrow & & \downarrow{\scriptstyle G(h)} \\ FY & \xrightarrow[\eta_Y]{} & GY \end{array}$$

Let $V$ be a vector space over the field $\mathbb{K}$, and $V^{**}$ its double dual. Both $V$ and $V^{**}$ are objects in the category $\mathfrak{V}_{\mathbb{K}}$ of vector spaces over $\mathbb{K}$.

**Example 36.** *There is a natural transformation $\eta$ from the identity functor $I : \mathfrak{V}_{\mathbb{K}} \to \mathfrak{V}_{\mathbb{K}}$ to the double dual functor $-^{**} : \mathfrak{V}_{\mathbb{K}} \to \mathfrak{V}_{\mathbb{K}}$. The components of the natural transformation are defined as $\eta_U : U \to U^{**}$ for each $U \in \mathfrak{V}_{\mathbb{K}}$.*

Let $\eta_U(x) = \sigma_x$ for any $x \in U$. As $\sigma_x$ is in $U^{**}$ it is a linear function from $U^*$ to $\mathbb{K}$ which we define as $\sigma_x(\varphi) = \varphi(x)$ for any $\varphi \in U^*$.

To show that $\eta$ is a natural transformation from $I$ to $-^{**}$ we will show that for vector spaces $U, V$ and a linear function $f : U \to V$ the following square commutes,

$$\begin{array}{ccc} U & \xrightarrow{\eta_U} & U^{**} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{**}} \\ V & \xrightarrow[\eta_V]{} & V^{**} \end{array}$$

Given $f : U \to V$ define $f^* : V^* \to U^*$ then $f^{**} : U^{**} \to V^{**}$ as follows,

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ & {\scriptstyle \varphi\circ f}\searrow & \downarrow{\scriptstyle \varphi} \\ & & \mathbb{K} \end{array} \qquad \text{for } \varphi \in V^* \quad f^*(\varphi) = \varphi \circ f$$

$$\begin{array}{ccc} U^* & \xleftarrow{f^*} & V^* \\ {\scriptstyle \phi}\downarrow & \swarrow{\scriptstyle \phi\circ f^*} & \\ \mathbb{K} & & \end{array} \qquad \text{for } \phi \in U^{**} \quad f^{**}(\phi) = \phi \circ f^*$$

To check that the natural transformation square commutes for all $x \in U$ we need $f^{**} \circ \eta_U(x) = \eta_V \circ f(x)$.

$$
\begin{aligned}
LHS &= f^{**} \circ \eta_U(x) \\
&= f^{**} \circ \sigma_x && \text{by definition of } \eta \\
&= \sigma_x \circ f^* && \text{by definition of } f^{**} \\
RHS &= \eta_V \circ f(x) \\
&= \sigma_{f(x)} && \text{by definition of } \eta
\end{aligned}
$$

To see that $\sigma_x \circ f^*$ is the same as $\sigma_{f(x)}$ notice that both are linear functions from $V^*$ to $\mathbb{K}$. Hence we need to check that they act the same on any $\varphi \in V^*$.

$$
\begin{aligned}
\sigma_x \circ f^*(\varphi) &= \sigma_x \circ \varphi \circ f && \text{by definition of } f^* \\
&= \varphi \circ f(x) && \text{by definition of } \sigma_x \\
&= \sigma_{f(x)}(\varphi) && \text{by definition of } \sigma_{f(x)}
\end{aligned}
$$

Let $\mathfrak{A}, \mathfrak{B}$ be two categories with functors $F, G : \mathfrak{A} \to \mathfrak{B}$ and a natural transformation $\eta$ from $F$ to $G$.

**Definition 37.** *If for all $X \in \mathfrak{A}$ the component of that natural transformation $\eta_X$ is an isomorphism then $\eta$ is a natural isomorphism from $F$ to $G$.*

In the category $\mathbf{Set}^{\mathfrak{A}}$, where $\mathfrak{A}$ is some small category, the objects are functors $F : \mathfrak{A} \to \mathbf{Set}$ and the morphisms are natural transformations. A natural isomorphism in $\mathbf{Set}^{\mathfrak{A}}$ is an isomorphism in that category.

## 3.2 Adjoint functors

Let $\mathfrak{A}$ and $\mathfrak{B}$ be categories with functors $F : \mathfrak{A} \to \mathfrak{B}$ and $G : \mathfrak{B} \to \mathfrak{A}$. We define the functor $\mathbf{Hom}_{\mathfrak{A}} G : \mathfrak{A}^{op} \times \mathfrak{B} \to \mathbf{Set}$ as follows:

1. Let $(A, B)$ be an object in $\mathfrak{A}^{op} \times \mathfrak{B}$. Then

$$\mathbf{Hom}_{\mathfrak{A}} G\ (A, B) = \mathbf{Hom}_{\mathfrak{A}}(A, GB).$$

2. Let $(f, g) : (A, B) \to (A', B')$ be a morphism in $\mathfrak{A}^{op} \times \mathfrak{B}$, where $f : A' \to A$ and $g : B \to B'$. Then $\mathbf{Hom}_{\mathfrak{A}} G$ sends $(f, g)$ to the morphism

$$\mathbf{Hom}_{\mathfrak{A}} G\ (f, g) \quad : \quad \mathbf{Hom}_{\mathfrak{A}}(A, GB) \to \mathbf{Hom}_{\mathfrak{A}}(A', GB')$$

in $\mathbf{Set}$ defined by, for each $a \in \mathbf{Hom}_{\mathfrak{A}}(A, GB)$,

$$[\mathbf{Hom}_{\mathfrak{A}} G\ (f, g)]\ (a) = Gg \circ a \circ f.$$

Then define a second functor $\mathbf{Hom}_{\mathfrak{B}} F : \mathfrak{A}^{op} \times \mathfrak{B} \to \mathbf{Set}$ similarly, so that for any object $(A, B)$ in $\mathfrak{A}^{op} \times \mathfrak{B}$,

$$\mathbf{Hom}_{\mathfrak{B}} F\ (A, B) = \mathbf{Hom}_{\mathfrak{B}}(FA, B).$$

**Definition 38.** *The functor $G$ is* adjoint *to $F$, and the functor $F$ is* coadjoint *to $G$, if the functors $\mathbf{Hom}_{\mathfrak{A}}G$ and $\mathbf{Hom}_{\mathfrak{B}}F$ are naturally isomorphic.*

**Lemma 39.** *The adjoint of a functor, if it exists, is unique up to natural isomorphism.*

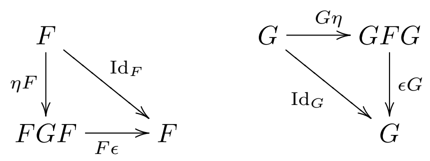
The same is true of the coadjoint.

**Lemma 40.** *The functor $F$ is adjoint to $G$ precisely if there are natural transformations*

$$\eta : \mathrm{Id}_{\mathfrak{B}} \to FG$$

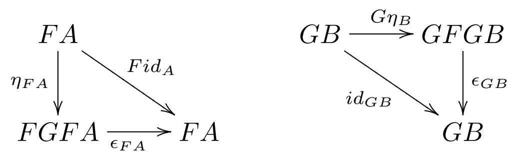
$$\epsilon : GF \to \mathrm{Id}_{\mathfrak{A}}$$

*which satisfy the two* triangle conditions $F\epsilon \circ \eta F = \mathrm{Id}_F$ *and* $\epsilon G \circ G\eta = \mathrm{Id}_G$*,*




We call $\eta$ the *unit*, and $\epsilon$ the *co-unit*, of the adjunction.

The triangle conditions, as stated in natural transformations, are shorthand for two more concrete families of identities in morphisms between objects. For any objects $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$, we must have $F\epsilon_A \circ \eta_{FA} = id_{FA}$ and $\epsilon_{GB} \circ G\eta_B = id_{GB}$:




**Example 41.** *Let* Free : **Set** $\to$ **Grp** *be the functor mapping a set to the free group on that set. Then* Free *is coadjoint to the forgetful functor* For : **Grp** $\to$ **Set**.

*That is, there is a natural isomorphism between the functors*

$$\mathbf{Hom}_{\mathbf{Set}}(\mathrm{For}) \quad \textit{and} \quad \mathbf{Hom}_{\mathbf{Grp}}(\mathrm{Free}).$$

*The component morphisms of this natural isomorphism are, for each set $X$ and each group $G$, the canonical bijection*

$$\mathbf{Hom}(X, \mathrm{For}(G)) \to \mathbf{Hom}(\mathrm{Free}(X), G),$$

*which arises since a homomorphism into $G$ from* $\mathrm{Free}(X)$ *is uniquely determined by a function into $G$ from* $\mathrm{Free}(X)$*'s generating set $X$.*

**Example 42.** *An isomorphism is adjoint to its inverse.*

**Example 43.** *Let* Poly : **Ring** $\to$ **pRing** *be the functor mapping a commutative ring $R$ with unity to the pointed commutative ring with unity $(R[X], X)$*

*of polynomials over R. Then* Poly *is coadjoint to the forgetful functor* For : $\mathbf{Ring} \to \mathbf{pRing}$.

*That is, there is a natural isomorphism between the functors*

$$\mathbf{Hom}_{\mathbf{Ring}}(\mathrm{For}) \quad \textit{and} \quad \mathbf{Hom}_{\mathbf{pRing}}(\mathrm{Poly}).$$

*The component morphisms of this natural isomorphism are, for each ring $R$ and each pointed ring $(S, s)$, the canonical bijection*

$$\mathbf{Hom}(R, \mathrm{For}(S, s)) \to \mathbf{Hom}(\mathrm{Poly}(R), (S, s)),$$

*which arises since a point-preserving ring homomorphism from $(R[X], X)$ into $(S, s)$ is uniquely determined by a ring homomorphism from $R$ into $S$.*

**Example 44.** *Let* **CMet** *be the category of complete metric spaces. (It is a subcategory of the category* **Met** *of metric spaces.) Let* Fill : $\mathbf{Met} \to \mathbf{CMet}$ *be the functor mapping a metric space $X$ to its completion $\overline{X}$. Then* Fill *is coadjoint to the inclusion functor* $\iota : \mathbf{CMet} \to \mathbf{Met}$.

*The component morphisms of this natural isomorphism between the functors*

$$\mathbf{Hom}_{\mathbf{Met}}(\iota) \quad \textit{and} \quad \mathbf{Hom}_{\mathbf{CMet}}(\mathrm{Fill}).$$

*are, for each metric space $X$ and each complete metric space $Y$, the canonical bijection*

$$\mathbf{Hom}(X, Y) \to \mathbf{Hom}(\overline{X}, Y),$$

*which arises since a uniformly continuous function from $X$ into $Y$ is uniquely determined by its (also uniformly continuous) restriction to $X$.*

**Example 45.** *Let $\mathfrak{A}$ be a category. Let $\Delta : \mathfrak{A} \to \mathfrak{A} \times \mathfrak{A}$ be the* diagonal functor *mapping an object $A$ to its self-pairing $(A, A)$. Then $\Delta$ is coadjoint precisely if $\mathfrak{A}$ has finite products; if it does, the product functor* Prod : $\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ *is $\Delta$'s adjoint.*

*The component morphisms of this natural isomorphism between the functors*

$$\mathbf{Hom}_{\mathfrak{A}}(\mathrm{Prod}) \quad \textit{and} \quad \mathbf{Hom}_{\mathfrak{A} \times \mathfrak{A}}(\Delta).$$

*are, for each triple $A$, $A_1$, $A_2$ of objects in $\mathfrak{A}$, the canonical bijection*

$$\mathbf{Hom}(A, A_1 \times A_2) \to \mathbf{Hom}((A, A), (A_1, A_2)),$$

*which arises since a morphism from $A$ into $A_1 \times A_2$ is uniquely determined by its two component morphisms from $A$ into $A_1$ and $A_2$ respectively.*

**Example 46.** *Let $\mathfrak{A}$ be a category, and let* **1** *be the category with one object and one morphism. Consider the unique functor $! : \mathfrak{A} \to \mathbf{1}$. The functor ! is coadjoint precisely if $\mathfrak{A}$ has a terminal object; if it does, then the adjoint of ! is the functor $T : \mathbf{1} \to \mathfrak{A}$ which maps* **1***'s one object onto $\mathfrak{A}$'s terminal object.*

**Example 47.** *The preceding two examples can be generalized to a statement for limits generally. Let $\mathfrak{A}$ be a category, $\mathfrak{J}$ be a small category, and $\mathbf{Diag}_{\mathfrak{J}}\mathfrak{A}$ the category of diagrams from $\mathfrak{J}$ into $\mathfrak{A}$. Consider the* diagonal functor

$$\Delta : \mathfrak{A} \to \mathbf{Diag}_{\mathfrak{J}}\mathfrak{A},$$

*where $\Delta$ sends an object $A \in \mathfrak{A}$ to the diagram $\Delta A : \mathfrak{J} \to \mathfrak{A}$, which sends each object in $\mathfrak{J}$ to $A$ and each morphism in $\mathfrak{J}$ to $id_A$.*

*Then $\Delta$ is coadjoint precisely if $\mathfrak{A}$ has finite $\mathfrak{J}$-limits. If it does, the $\mathfrak{J}$-limit functor $\mathfrak{J} - \mathrm{Lim} : \mathbf{Diag}_{\mathfrak{J}}\mathfrak{A} \to \mathfrak{A}$ is $\Delta$'s adjoint.*

**Lemma 48.** *Adjoint functors preserve limits. Coadjoint functors preserve colimits.*

**Example 49.** *From Example 41, the free functor $F : \mathbf{Set} \to \mathbf{Grp}$ is coadjoint. It therefore preserves colimits. For this reason the coproduct of two free groups $\mathrm{Free}(X), \mathrm{Free}(Y) \in \mathbf{Grp}$ is their* free product*; that is, the free group generated by the disjoint union (coproduct in* **Set***) of their generating sets $X$ and $Y$.*

## 3.3 Reflections

The adjointness of the completion functor from Example 44 can also be considerably generalized.

**Definition 50.** *A subcategory $\mathfrak{A}$ of a category $\mathfrak{B}$ is* reflective*, if the inclusion functor $\iota : \mathfrak{A} \to \mathfrak{B}$ is adjoint. It is* coreflective*, if the inclusion functor is coadjoint.*

Let $\mathfrak{A}$ be a reflective subcategory of $\mathfrak{B}$. We call the functor $R : \mathfrak{B} \to \mathfrak{A}$ coadjoint to $\iota$ the *reflector*, and the image of an object $B \in \mathfrak{B}$ under $R$ its *reflection* in $\mathfrak{A}$.

**Example 51.** *The category* **Ab** *of abelian groups is a reflective subcategory of the category* **Grp** *of groups. The reflector is the functor $Q : \mathbf{Grp} \to \mathbf{Ab}$ which sends a group to its quotient by its commutator subgroup.*

*To see this, observe that there is a canonical bijection, for each group $G$ and abelian group $A$, between the sets of group homomorphisms*

$$\mathbf{Hom}(G, A) \quad \text{and} \quad \mathbf{Hom}(G/[G, G], A),$$

*given since a homomorphism of $G$ into the abelian group $A$ must send each of $G$'s commutators to $1_A$.*

**Example 52.** *The fields (with morphisms the field embeddings) are a reflective subcategory of the category of integral domains. The reflector sends each integral domain to its field of fractions.*

# 4 Zero objects

**Definition 53.** *A* zero object *of a category $\mathfrak{G}$ is an object $O \in \mathfrak{G}$ such that, for each object $A \in \mathfrak{G}$, there is precisely one morphism in $\mathbf{Hom}(A, 0)$ and precisely one morphism in $\mathbf{Hom}(0, A)$.*

Equivalently, a zero object of $\mathfrak{G}$ is an object that is both terminal and initial. A zero object of $\mathfrak{G}$ is unique up to isomorphism.

Suppose that $\mathfrak{C}$ has a zero object $O$. For any pair of objects $A$ and $B$ in $\mathfrak{C}$, composing the unique morphisms $f : A \to O$ and $g : O \to B$ yields a distinguished morphism $g \circ f : A \to B$. We call this the *zero morphism* from $A$ to $B$, and denote it $0_{AB}$. It is clear that the zero morphism is well-defined, independent of the choice of zero object.

For the rest of this section, we work in a category $\mathfrak{C}$ with zero object $O$ (and hence with zero morphisms).

**Lemma 54** (Composition with zero gives zero)**.** *Let $A$ and $B$ be objects in $\mathfrak{C}$. Then for any morphism $f : B \to D$,*

$$f \circ 0_{AB} = 0_{AD}.$$

*Likewise, for any morphism $f : D \to A$,*

$$0_{AB} \circ g = 0_{BD}.$$

**Definition 55.** *Let $f : A \to B$ be a morphism in $\mathfrak{C}$. A* kernel *(respectively,* cokernel*) of $f$ is an equalizer (respectively, coequalizer) of $f$ with $0_{AB}$.*

From our earlier work on equalizers, a morphism's kernel, if it exists, is unique up to equivalence of subobjects. Likewise a cokernel is unique up to equivalence of quotient objects.

**Lemma 56.** *A morphism $f : A \to B$ is monic, precisely if it has kernel the unique morphism $0 : O \to A$. It is epic, precisely if it has cokernel the unique morphism $0 : B \to O$.*

**Lemma 57.** *Let $A$ be an object in $\mathfrak{C}$. Then $id_A : A \to A$ is a kernel of the unique morphism $0 : A \to O$, and a cokernel of the unique morphism $0 : O \to A$.*

**Lemma 58.** *Let $f : A \to B$ be a morphism in $\mathfrak{C}$. Suppose that $k : K \to A$ is a kernel of $f$, $l : A \to L$ is a cokernel of $k$, and $m : M \to A$ is a kernel of $l$. Then $m$ is a kernel of $f$.*

*Proof.* Repeated applications of the definition of kernel and cokernel. □

# 5 Abelian categories

## 5.1 Axioms

**Definition 59.** *An* abelian category *is a category $\mathfrak{C}$ satisfying the following conditions.*

1. *$\mathfrak{C}$ has a zero object.*
2. (i) *Each pair of objects in $\mathfrak{C}$ has a product.*
   (ii) *Each pair of objects in $\mathfrak{C}$ has a coproduct.*
3. (i) *Each morphism in $\mathfrak{C}$ has a kernel.*

(ii) *Each morphism in $\mathfrak{G}$ has a cokernel.*

*4.* (i) *Each monomorphism in $\mathfrak{G}$ is some morphism's kernel.*

(ii) *Each epimorphism in $\mathfrak{G}$ is some morphism's cokernel.*

**Example 60.** *The category* **Ab** *of abelian groups and group homomorphisms is abelian. More generally, for any unitary ring $R$, the category $R$-***Mod** *of unitary $R$-modules and module homomorphisms is abelian. The zero object is the module* $(0)$.

**Example 61.** *The category* **Grp** *of groups and group homomorphisms is not abelian. It satisfies all axioms except for 4(i): Not all its monomorphisms are kernels, since not all subgroups are normal.*

**Example 62.** *The category* **pCompHaus** *of pointed compact Hausdorff spaces and point-preserving continuous maps is not abelian. It satisfies all axioms except for 4(ii).*

**Example 63.** *Consider the category whose objects are the smooth vector bundles on a manifold $M$, and whose morphisms are smoothly varying families of linear maps between fibres. This category is not abelian; it satisfies all axioms except for 4(i).*

*Indeed, the monomorphisms of this category are the morphisms which restrict to injective linear maps on a dense subset of $M$. The kernels are a strict subclass of this: they are the morphisms which restrict everywhere to injective linear maps.*

**Example 64.** *The category* **CompHausAb** *of compact Hausdorff abelian topological groups and continuous group homomorphisms is an abelian category. Objects in this category include, for instance,*

- *The finite abelian groups, equipped with the discrete topology.*
- *The torus groups. That is, the circle $S = \mathbb{R}/\mathbb{Z}$, and more generally the products of arbitrarily many copies of $S$.*
- *For each prime $p$, the $p$-adic integers.*

**Example 65.** *The category of sheaves of abelian groups on a topological space $X$ is an abelian category.*

## 5.2 Elementary properties

In this section we explore the properties of a fixed abelian category $\mathfrak{G}$.

**Theorem 66.** *A monic, epic morphism is an isomorphism.*

*Proof.* Let $f : A \to B$ be both monic and epic. Since $f$ is a monomorphism, Axiom 4(i) implies that it is the kernel of some morphism $g : B \to C$. Since $f$ is an epimorphism, it has cokernel the unique morphism $0 : B \to O$, and, as observed in the section on zero objects, the morphism $0 : B \to O$ has kernel $id_B : B \to B$.

Since $id_B$ is a kernel of 0 is a cokernel of $f$ is a kernel of $g$, it follows (by Lemma 58) that $id_B$ is a kernel of $g$.

The uniqueness of kernels therefore implies that $id_B$ and $f$ are equivalent subobjects of $B$. That is, there exists a pair of mutually inverse isomorphisms $x : A \to B$ and $y : B \to A$, such that $id_B \circ x = f$ and $f \circ y = id_B$. From the former we conclude that $x = f$. Hence $f$ is an isomorphism. □

This proof uses only Axioms 1 and 4(i). (A dual proof using only Axioms 1 and 4(ii) works equally well.) Theorem 66 is therefore also valid in, for instance, the categories **Grp** and **pCompHaus**.

Let us single out one idea from the above proof for future use.

**Lemma 67** (Ker-Coker duality)**.** *$g : B \to C$ is a cokernel of a monomorphism $f : A \to B$, then $f$ is a kernel of $g$. If $f : A \to B$ is a kernel of an epimorphism $g : B \to C$, then $g$ is a cokernel of $f$.*

*Proof.* Suppose $g : B \to C$ is a cokernel of a monomorphism $f : A \to B$. By Axiom 4(i), there is some morphism $r : B \to R$ of which $f$ is a kernel. By Axiom 3(i), the morphism $g$ has some kernel $k : K \to B$.

We therefore have that $k$ is a kernel of $g$ is a cokernel of $f$ is a kernel of $r$. Hence (Lemma 58) $k$ is a kernel of $r$. So $k$ and $f$ are equivalent as subobjects of $B$, and so $f$ is a kernel of $g$.

The proof of the dual statement is similar. □

Rather suggestively, we call a pair $(f, g)$ of morphisms satisfying either of the two (equivalent) conditions in Lemma 67 an *exact sequence.* (However, the concept will not be pursued any further in this report.)

**Definition 68.** *An* intersection *(respectively,* union*) of two subobjects of an object $A$, is a greatest lower bound (respectively,* least upper bound*) on them in the canonical partial order on the subobjects of $A$.*

*An* intersection *(respectively,* union*) of two quotient objects of an object $A$, is a greatest lower bound (respectively,* least upper bound*) on them in the canonical partial order on the quotient objects of $A$.*

Intersections and unions of subobjects (respectively, quotient objects) are unique up to equivalence of subobjects (respectively, quotient objects).

**Lemma 69.** *Let $A$ be an object in $\mathfrak{G}$. Then each pair of subobjects of $A$, and each pair of quotient objects of $A$, has an intersection.*

*Proof.* Let $b_1 : B_1 \to A$ and $b_2 : B_2 \to A$ be monomorphisms. By Axiom 4(i), the monomorphism $b_1$ is the kernel of some morphism $f : A \to F$. By Axiom 3(i), the morphism $f \circ b_1$ has some kernel $k : K \to B_2$. Applying repeatedly the definition of a kernel, together with the fact that $b_2$ is monic, we find that the subobject $b_2 \circ k : K \to A$ of $A$ is an intersection of $B_1$ and $B_2$.

A dual proof establishes the statement about quotient objects. □

**Corollary 70.** *Let $A$ be an object in $\mathfrak{G}$. Then each pair of subobjects of $A$, and each pair of quotient objects of $A$, has a union.*

*Proof.* Let $b_1 : B_1 \to A$ and $b_2 : B_2 \to A$ be monomorphisms. Axiom 3(ii) ensures the existence of cokernels of $B_1$ and $B_2$. By the lemma just proved, these cokernels – treated as a pair of quotient objects of $A$ – have an intersection. The kernel of this intersection (which exists by Axiom 3(i)) is a union of the kernels of the cokernels of $B_1$ and $B_2$. But by Lemma 67 on Ker-Coker duality, the kernels of the cokernels of $B_1$ and $B_2$ are just $B_1$ and $B_2$ themselves.

The proof of the dual statement is similar. □

## 5.3 Direct sums

Let $\mathfrak{G}$ be a category with zero objects and with finite products and coproducts. That is, $\mathfrak{G}$ satisfies the first two of the axioms for an abelian category. As well as the abelian categories, the categories **Grp** and **pCompHaus** are for instance of this kind.

We explore a number of constructions we can make for this family of categories. Let $A_1$ and $A_2$ be objects of $\mathfrak{G}$. For each pair $f_1 : F \to A_1$, $f_2 : F \to A_2$ we denote by $(f_1\ f_2)$ the canonical morphism induced from $F$ to $A_1 \times A_2$. Likewise, for each pair $g_1 : A_1 \to G$, $g_2 : A_2 \to G$, we denote by $\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ the canonical morphism induced from $A_1 + A_2$ to $G$.

In particular, along with the canonical injections and projections

$$\begin{aligned} \iota_1 &: A_1 \to A_1 + A_2, & \iota_2 &: A_2 \to A_1 + A_2, \\ \pi_1 &: A_1 \times A_2 \to A_1, & \pi_2 &: A_1 \times A_2 \to A_2, \end{aligned}$$

we have distinguished morphisms

$$\begin{aligned} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &: A_1 + A_2 \to A_1, & \begin{pmatrix} 0 \\ 1 \end{pmatrix} &: A_1 + A_2 \to A_2, \\ (1\ 0) &: A_1 \to A_1 \times A_2, & (0\ 1) &: A_2 \to A_1 \times A_2, \end{aligned}$$

as well as a canonical morphism

$$\begin{pmatrix} (1\ 0) \\ (0\ 1) \end{pmatrix} = \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix}\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) : A_1 + A_2 \to A_1 \times A_2,$$

which we henceforth denote by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

**Lemma 71.** *The morphism $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a cokernel of the canonical injection $\iota_1$.*

*Proof.* By construction $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \circ \iota_1 = 0$. On the other hand, suppose that $g : A_1 + A_2 \to G$ is such that $g \circ \iota_1 = 0$. Then for some $g_2 : A_2 \to G$, we have that $g = \begin{pmatrix} 0 \\ g_2 \end{pmatrix} = g_2 \circ \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. □

For the rest of this section, we further suppose that $\mathfrak{G}$ is abelian.

**Corollary 72.** *The canonical injection $\iota_1$ is a kernel of the morphism $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.*

*Proof.* Certainly $\iota_1$ is a monomorphism, and by the preceding lemma, the morphism $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is $\iota_1$'s cokernel. The result follows by Lemma 67 on Ker-Coker duality. □

Corollary 72 (as well as the results to be established in the rest of this section) can certainly fail in categories that satisfy only the first two abelian category axioms. It is instructive, before going further, to consider a counterexample.

**Example 73.** *In the category* **Grp** *of groups, the coproduct $A_1 + A_2$ of two groups $A_1$ and $A_2$ is their free product. Here the kernel of the morphism*

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} : A_1 + A_2 \to A_2$$

*is much larger than $A_1 + A_2$'s subgroup $A_1$. Indeed, it is the set of all reduced words on the elements of $A_1$ and $A_2$, for which the overall product of all the elements on $A_1$ involved is $A_1$'s identity.*

**Lemma 74.** *The subobjects $\iota_1 : A_1 \to A_1 + A_2$ and $\iota_2 : A_2 \to A_1 + A_2$ of $A_1 + A_2$ have intersection $0 : O \to A_1 + A_2$.*

*Proof.* By the previous corollary, we know that $\iota_1$ is a kernel of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Observe that the unique morphism $0 : O \to A_2$ is the kernel of

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \circ \iota_2 = id_{A_2} : A_2 \to A_2.$$

From the construction in the proof of existence of intersections in Lemma 69, we deduce that the intersection of $\iota_1$ and $\iota_2$ is $A_1 + A_2$'s subobject

$$0 = \iota_2 \circ 0 : O \to A_1 + A_2.$$

□

**Theorem 75.** *Suppose that $\mathfrak{C}$ is abelian. Then the morphism*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : A_1 + A_2 \to A_1 \times A_2$$

*is an isomorphism.*

*Proof.* We will prove that it is a monomorphism. It will follow by a dual argument that it is an epimorphism, and therefore by Theorem 66 that it is an isomorphism.

Let $k : K \to A_1 \to A_2$ be a kernel of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$0_{KA_2} = \pi_2 \circ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \circ k = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \circ k.$$

So in the canonical partial order on the subobjects of $A_1 + A_2$, the subobject $k$ is at most the kernel of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, which by Lemma 72 is $\iota_1$.

Likewise $k$ is at most $\iota_2$. So $k$ is at most the intersection of $\iota_1$ and $\iota_2$, which by Lemma 74 is the unique morphism $0 : O \to A_1 + A_2$.

The morphism $0 : O \to A_1 + A_2$ is therefore a kernel of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and so $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a monomorphism. □

In an abelian category, we can therefore define the *direct sum* (unique up to isomorphism) of two objects $A_1$ and $A_2$ to be the object $A_1 + A_2 \simeq A_1 \times A_2$. We denote the direct sum of $A_1$ and $A_2$ by $A_1 \oplus A_2$.

We are therefore justified in using our previously-established notation as follows. With the direct sum of $A_1$ and $A_2$ are associated canonical injections and projections

$$\begin{array}{ll} \iota_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : A_1 \to A_1 \oplus A_2, & \iota_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} : A_2 \to A_1 \oplus A_2, \\ \pi_1 = (1\ 0) : A_1 \oplus A_2 \to A_1, & \pi_2 = (0\ 1) : A_1 \oplus A_2 \to A_2. \end{array}$$

For each pair $f_1 : F \to A_1$, $f_2 : F \to A_2$ we have a canonical morphism

$$(f_1\ f_2) : F \to A_1 \oplus A_2.$$

For each pair $g_1 : A_1 \to G$, $g_2 : A_2 \to G$, we have a canonical morphism

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} : A_1 \oplus A_2 \to G.$$

## 5.4 The abelian group structure on homomorphisms

Let $\mathfrak{G}$ be an abelian category, and $A$ and $B$ two objects of $\mathfrak{G}$.

We define a binary operation $+$ on $\mathbf{Hom}(A, B)$ as follows: if $x, y \in \mathbf{Hom}(A, B)$, then

$$x + y = (x\ y) \circ \begin{pmatrix} 1 \\ 1 \end{pmatrix} : A \to B.$$

The following striking and fundamental theorem is then straightforward to deduce.

**Theorem 76.** *The binary operation $+$ defines an abelian group structure on $\mathbf{Hom}(A, B)$, with the morphism $0_{AB}$ as identity.*

We will make considerable use of this abelian group structure on each **Hom**-set. The following two consequences will be particularly useful.

**Corollary 77.** *Let $A$ be an object of $\mathfrak{G}$. The binary operations $+$ and $\circ$ define a (not necessarily commutative) ring structure on $\mathbf{Hom}(A, A)$, known as the* endomorphism ring *of $A$ and denoted* $\mathbf{End}(A)$. *This ring has additive identity $0_{AA}$ and multiplicative identity $id_A$.*

**Theorem 78** (Direct sum systems)**.** *Let $A_1$ and $A_2$ be objects of $\mathfrak{G}$. The canonical injections and projections of $S = A_1 \oplus A_2$ satisfy*

$$\begin{array}{ll} \pi_1 \circ \iota_1 = id_{A_1}, & \pi_2 \circ \iota_1 = 0_{A_1 A_2} \\ \pi_1 \circ \iota_2 = 0_{A_2 A_1} & \pi_2 \circ \iota_2 = id_{A_2} \\ \iota_1 \pi_1 + \iota_2 \pi_2 = 1_S. & \end{array}$$

*Conversely, if for three objects $A_1$, $A_2$, $S$ and four morphisms $\iota_1$, $\iota_2$, $\pi_1$, $\pi_2$ the five conditions above are satisfied, then $S$ is a direct sum of $A_1$ and $A_2$, with those four morphisms as its canonical injections and projections.*

All the remarks on direct sums thus far made extend directly to products and coproducts of any finite number of objects.

# 6 Krull-Schmidt decomposition theorem

The aim of this section is to prove an analogue, in a general fixed abelian category $\mathfrak{G}$, of the classical Krull-Scmidt decomposition theorem for modules.

**Definition 79.** *An object $A \in \mathfrak{G}$ is* indecomposable*, if, for any direct sum*

$$A = B \oplus C,$$

*either $B$ or $C$ is the subobject $id_A : A \to A$.*

**Definition 80.** *An object $A \in \mathfrak{G}$ has* finite length*, if for each set of subobjects of $A$ there is a maximal and a minimal subobject.*

**Example 81.** *Finite-dimensional vector spaces over a field $k$ are finite-length objects in the abelian category $k$-**Mod** of vector spaces over $k$.*

**Example 82.** *Finite abelian groups are finite-length objects in the category **Ab** of abelian groups. The additive group $\mathbb{Z}$ of the integers, as an object of **Ab**, does not have finite length: the set*

$$n \mapsto n, n \mapsto 2n, n \mapsto 4n, n \mapsto 8n, \ldots$$

*of subobjects of $\mathbb{Z}$ each contains the last, and hence has no minimum.*

Our work will concern the existence and uniqueness of decompositions of finite-length objects $A \in \mathfrak{G}$ as a direct sum of indecomposable modules. Our proof will be to use the additive structure on the endomorphism ring of each object involved. Why this is particularly promising as a strategy is shown by the following lemma.

**Lemma 83.** *Let $A \in \mathfrak{G}$. If $\mathbf{End}(A)$ is local, then $A$ is indecomposable. Conversely, if $A \in \mathfrak{G}$ is indecomposable and has finite length, then $\mathbf{End}(A)$ is local.*

*Proof.* We prove here only the former direction. Suppose $A = B \oplus B'$. Let $\iota : B \to A$ and $\pi : A \to B$ be the natural injection and projection respectively. Consider the morphism $f = \iota \circ \pi : A \to A$. Then $f^2 = f$, so, since $\mathbf{End}(A)$ is local, either $f = 1_A$ or $f = 0_A$. If the former, then $B = A$ and $B' = 0$. If the latter, then $B = 0$ and $B' = A$. □

**Theorem 84** (Krull-Schmidt)**.** *Suppose that $A \in \mathfrak{G}$ has finite length. Then it has a decomposition*

$$A = \bigoplus_{i \in I} A_i,$$

*with $I$ also finite, and such that each $A_i$ is indecomposable. Moreover, if*

$$A = \bigoplus_{j\in J} B_j,$$

*is any other such decomposition, then $|I| = |J|$, and there exists a bijection $\varphi : I \to J$, such that for each $i \in I$, we have the isomorphism $A_i \simeq B_{\varphi(i)}$.*

*Proof. Existence.*

Take a maximal decomposition. Then each part is indecomposable.

*Uniqueness*

Since $A$ has finite length, so does each of the $A_i$. Using, as guaranteed by Lemma 83, that each ring $\mathbf{End}(A_i)$ is local, we find by induction on $|I|$ that:

**Lemma 85.** *Let $f \in \mathbf{End}(A)$. Then there exist subobjects $(\iota'_i : B_i \to A)_{i\in I}$, and isomorphisms $(h_i : A_i \to B_i)_{i\in I}$, such that for each $i \in I$ either $\iota'_i \circ h_i = f \circ \iota_i$ or $\iota'_i \circ h_i = (id_A - f) \circ \iota_i$. Moreover,*

$$A = \bigoplus_{i\in I} B_i.$$

□

With some further finiteness assumptions (and some messy set theory), the uniqueness part of theorem extends from direct products more generally to coproducts, holding even when $I$ and $J$ are infinite.

**Definition 86.** *An abelian category $\mathfrak{G}$ is a* Grothendieck category, *if it has colimits, and satisfies the following conditions:*

1. *The category $\mathfrak{G}$ is* locally small*; that is, for each object $A \in \mathfrak{G}$, the class of subobjects of $A$ is a set.*

2. *Let $A \in \mathfrak{G}$. Let $(I, \leq)$ be an ordered set; let $\{\iota_i : A_i \to A : i \in I\}$ be a set of subobjects of $A$, such that for each pair $i, j \in I$, we have $i \leq j$ precisely if there is some morphism $f : A_j \to A_i$ such that $\iota_i \circ f = \iota_j$. Let $\iota : B \to A$ be any subobject of $A$. Then*

$$\bigcup_{i\in I}(A_i \cap B) = \left(\bigcup_{i\in I} A_i\right) \cap B.$$

**Theorem 87.** *Let $\mathfrak{G}$ be a Grothendieck category, and let $A \in \mathfrak{G}$. Suppose that $A$ decomposes into two coproducts*

$$A = \coprod_{i\in I} A_i = \coprod_{j\in J} B_j,$$

*where all $B_i$ are indecomposable, and the endomorphism rings of all $A_i$ are local. Then there exists a bijection $\varphi : I \to J$, such that for all $i \in I$ we have $A_i = B_{\varphi(i)}$.*

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