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Computable Component-wise Reducibility

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Abstract

We consider equivalence relations and preorders complete for various levels of the arithmetical hierarchy under computable, component-wise reducibility. We show that implication in first order logic is a complete preorder for Σ_1^0, the $\leq_m^P$ relation on EXPTIME sets for Σ_2^0 and the embeddability of computable subgroups of $(\mathbb{Q}, +)$ for Σ_3^0. In all cases, the symmetric fragment of the preorder is complete for equivalence relations on the same level. We present a characterisation of Π_1^0 equivalence relations which allows us to establish that equality of polynomial time functions and inclusion of polynomial time sets are complete for Π_1^0 equivalence relations and preorders respectively. We also show that this is the limit of the enquiry: for $n \geq 2$ there are no Π_n^0 nor Δ_n^0-complete equivalence relations.

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Contents

1 Introduction

LEPIDUS   What manner o' thing is your crocodile?
ANTONY   It is shaped, sir, like itself; and it is as broad as it hath breadth:
           it is just so high as it is, and moves with its own organs:
           it lives by that which nourisheth it, and the elements once out of it,
           it transmigrates.

<div align="right">

-William Shakespeare, *Antony and Cleopatra*

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The concept of reducibility plays a prominent role throughout the field of computer science, and for good reason; questions like "What is computable", "tractable", "complex" are notoriously difficult. Worse, they are not even mathematical: at best we find ourselves in empirical science, more probably in philosophy. And philosophical disputes have the unfortunate tendency to sustain themselves for centuries with no resolution.

Reducibility allows us to handle this issue with grace. While we cannot, with any certainty, say anything objective about the difficulty of boolean satisfiability in light of the yet unresolved relation regarding P and NP, and the more fundamental question as to whether P indeed characterises the tractable problems, we can be confident that this difficulty lies within a polynomial factor of graph colouring. Some may deny the always-halting Turing machine as the limit of computability, but the fact that if such a machine could solve first order provability it could also solve the halting problem is uncontroversial. The more determined constructivists among us may reject the axiom of choice as having a place in mathematics, but in doing so we can all agree that they reject Zorn's lemma as well.

Given the flexibility of the concept it is not surprising that it found its way into many mathematical disciplines. Any field dealing with structures and equipped with a notion of complexity can quite naturally seek to use the one to impose some hierarchy on the other. Over and above its utility in mathematics, the concept of reducibility has been described as a characteristic feature of Computer Science. In particular the notion of NP-completeness is seen as an important intellectual export of the field to other disciplines ([Pap97]).

The specific form of reducibility studied here is what we term computable componentwise reducibility: a computable function acts on each component of a tuple separately;

the objects under consideration are predominantly equivalence relations, although often we will obtain the results via the more general case of preorders.

As we will see in the literature review, the degree structure of equivalence relations under this form of reducibility is rich and sometimes baffling. An assault on the mysteries concealed within lies outside the scope of our enquiry. Instead, we content ourselves with the much more limited goal of studying the maximum elements of this structure: the complete relations for various levels of the arithmetical hierarchy.

Our goal is to find equivalence relations and preorders that are not merely complete, but in some sense natural, at least to a mathematician. Of course, mulling too much over what it means to be "natural" will pull us back into the deep, dark woods of philosophical debate, so instead we present relations from a variety of disciplines in the hope that the reader may find at least one palatable. For a logician we have implication in first order logic, a computer scientist we will serve polynomial time reducibility and to a mathematician we hope to sell the embeddability of computable groups. Ultimately, however, the reader should bear in mind that these are all just subsets of $\mathbb{N} \times \mathbb{N}$ in various fineries. That the natural numbers can display such richness and variety is perhaps the most fascinating and rewarding aspect of our discipline. If the rest of this work is lost in formalism and technicalities, we hope that at the very least the reader will find time to reflect on the most beautiful mathematical structure of them all.

1.1 Outline

There are five parts to the sequel. In Section 2 we define precisely what we mean by component-wise reducibility, and offer some motivation for the choice. We introduce the terminology and notation we will use and prove some basic properties about the subject matter. In Section 3 we present an overview of some previous results in the field, and compare the computable case to component-wise reducibilities in the related disciplines of complexity and descriptive set theory. In Section 4 we begin our search for Σ_n^0-complete equivalence relations. We present examples of complete relations and preorders for Σ_1^0, Σ_2^0 and Σ_3^0, based in logic, complexity theory and group theory respectively. In Section 5 we construct a Π_1^0-complete equivalence relation and preorder and prove that this is as high as we can go: we show that for $n \geq 2$, no Π_n^0-complete equivalence relations exist. We conclude in Section 6.

2 Preliminaries

The subject matter of this thesis is what we will term computable component-wise reducibility, and denote by $\leq$. To state the definition, given two binary relations A and B, we say that $A \leq B$ if there exists a computable function f such that:

$$(x, y) \in A \iff (f(x), f(y)) \in B. \tag{2.1}$$

For contrast, recall that the standard notion of m-reducibility extended to pairs would be:

$$(x, y) \in A \iff f(x, y) \in B. \tag{2.2}$$

Neither the name nor the notation for this form of reducibility is standard; we will mention some of the other names used in the literature in Section 3. While the definition given can be applied to any binary relation, our main focus will be on equivalence relations and preorders. Note also that we will also use $\leq$ to denote the standard less-than-or-equal relation on the integers. This should cause no ambiguity given the context.

This choice of reducibility and the focus on equivalence relations is not arbitrary. In Section 2.2 we will show how this definition is equivalent to concepts in other areas of mathematics. In Section 2.3 we will prove some basic results about this manner of reducibility, both to offer a glimpse as to how it differs from the standard m-reduction of computability theory and to establish some facts that will be useful in the later sections. First, however, we must establish notation and terminology.

2.1 Notation and terminology

Our model of computation, where needed, will be a one-way infinite Turing machine. Most results will not, however, explicitly invoke this and will be based on a tacit acceptance of the Church-Turing Thesis.

As usual we assume some enumeration of all such machines and use φ_e to denote the eth partial recursive (p.r.) function arising from this enumeration and W_e the eth recursively enumerable (r.e.) set. Likewise, $\varphi_{e,t}$ and $W_{e,t}$ represent the eth function and set when the computation time is restricted to t steps. As every machine induces a function we may sometimes use $M(w)$ to denote the output of M on input w. If M is a decider, then we will say $M(w) = 1$ to mean M accepts w and 0 if it rejects. For an oracle machine, $M(A; w)$ is used to mean Machine M on input w with access to A as an oracle.

As our subject matter is discrete, we will use $[a, b]$ to denote all integers (rather than real numbers) between a and b inclusive.

For an equivalence relation E, $[x]_E$ will be used to denote the equivalence class of x under E and $\min[x]_E$ the least element (with respect to the standard order on the natural numbers) of the equivalence class of x. If E is clear from context we may omit the subscript. Where we are dealing with a preorder P, we will reserve the words "larger" and "smaller" for the standard order on the natural numbers, instead using "above" and "below" for the order induced by P.

Angular brackets $\langle \cdot \rangle$ are used to denote an encoding function, mapping $\cdot$ injectively to some natural number. If we do not mention the requirements on the encoding explicitly, it is to be taken that any computable encoding suffices. Strictly speaking, as the sets W_e are subsets of $\mathbb{N}$ we should really use $\langle x, y \rangle$ for any binary relation considered here. However we will often tacitly extend the notion of a r.e. set to be a subset of $\mathbb{N} \times \mathbb{N}$, as such an extension does not affect any of our results. This allows us to reserve the angular brackets for instances where the encoding function plays some necessary role, or simply to reduce stacked parentheses.

Standard notation is used for the arithmetical hierarchy. We obtain a Σ_n^0 relation by making an existential query over a Π_{n-1}^0 relation, a Π_n^0 from the complement of a Σ_n^0 and $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$. The base case are the computable relations $\Sigma_0^0 = \Delta_1^0 = \Pi_0^0$. We do not introduce special notation to distinguish between the Σ_n^0 (Π_n^0, Δ_n^0) sets, equivalence relations or preorders. The specific meaning should be clear from context.

2.2   Intuition

Component-wise reducibility is a clear analogue of embeddability in model theory. Recall that given $A \subseteq U^2$ and $B \subseteq V^2$, we say that the structure (U, A) is embeddable in (V, B) if there exists a function $f : U \to V$ satisfying:

$$(x, y) \in A \iff (f(x), f(y)) \in B. \tag{2.3}$$

If we restrict the setting so that f is required to be computable, and the domains of the structures are $\mathbb{N}$ (or, given a suitable encoding, some countable set), we obtain (2.1): $(\mathbb{N}, A)$ is computably embeddable in $(\mathbb{N}, B)$ if and only if $A \leq B$.

Herein lies a crucial aspect of component-wise reducibility. Whereas m-reducibility is only concerned with computability, a component-wise reduction also implies a structural similarity between the relations. For example, if S and R are two equivalence relations, and $S \leq R$ via f, then $g : \mathbb{N}/S \to \mathbb{N}/R$ given by $[x] \mapsto [f(x)]$ is one to one.

The concept also arises from category theory ([ALM$^+$12]), via the theory of enumerations of Ershov ([Epш69]). An enumeration is a function mapping the natural numbers onto some set. Any such $a : \mathbb{N} \to A$ defines an equivalence relation, $\{(x, y) \mid a(x) = a(y)\}$, and any equivalence relation over $\mathbb{N}^2$ can be defined by some enumeration: namely, $x \mapsto \min[x]$. It thus makes sense to class enumerations by the complexity of the equivalence relation they define. In the terminology of [Epш69], the *positive* enumerations are those which define Σ_1^0 equivalence relations and the *negative* define Π_1^0, and there

Figure 2.1: A component-wise reduction is also a morphism of enumerations

is no difficulty in likewise associating a class of enumerations with every level of the arithmetical hierarchy.

A *morphism of enumerations* is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that for two enumerations a and b, $a \circ f = b$. This gives us another means to view component-wise reducibility: f is a morphism from a to b if and only if f is a reduction between the induced equivalence relations, E_a and E_b, as shown in Figure 2.1. Note that this induces the existence of a unique $g : A \to B$ such that the resulting diagram commutes. In fact g is precisely the injection from $\mathbb{N}/E_a$ to $\mathbb{N}/E_b$ we have seen already.

If we thus fix a category of enumerations of a given level in the arithmetical hierarchy, the terminal object in this category is an enumeration with a morphism from every other enumeration; an equivalence relation to which there exists a reduction from any other equivalence relation: precisely the complete equivalence relation that is the focus of our study.

2.3 Basic results

In this section we will prove a few simple results about the behaviour of computable component-wise reducibility for two reasons. The first is to convince ourselves that this notion of reducibility is, indeed, substantially different from the standard $\leq_m$ or $\leq_1$ reducibilities used in computability theory, and the second is to see that there nevertheless exist some interesting connections between them.

Proposition 2.1. *For binary relations A and B, $A \leq B \Rightarrow A \leq_m B$, but not vice versa.*

Proof. By definition, $A \leq B$ implies the existence of a computable f satisfying:

$$(x, y) \in A \qquad \Longleftrightarrow \qquad (f(x), f(y)) \in B.$$

Define f' by $f'(x, y) = (f(x), f(y))$. As f is computable, clearly so is f'. This gives us:

$$(x, y) \in A \qquad \Longleftrightarrow \qquad f'(x, y) \in B,$$

which is to say, $A \leq_m B$.

The simplest way to see that $A \leq_m B$ does not imply $A \leq B$ is to recall the observations of Section 2.2: $A \leq B$ implies that in the corresponding computable structures

A is embeddable in B, and as such B restricted to the image of f satisfies the same relational properties as A. It follows that, for example, an A that contains a reflexive pair cannot be component-wise reduced to a B that is everywhere irreflexive. Thereby if we use K to denote the halting problem, namely:

$$K = \{(e, w) \mid \exists y, t \; \varphi_{e,t}(w) = y\},$$

and if e is the index of some never-halting machine, $K \times K \not\leq K \times \{(e, w)\}$. However, clearly $K \times K \leq_m K \times \{(e, w)\}$ via $((i, v), (i', v')) \mapsto ((j, \langle v, v' \rangle), (e, w))$, where j is the machine that will first run machine i on v then machine i' on v'. $\qquad \square$

A corresponding statement fails for $\leq_1$. We can construct binary relations where $\leq$ and $\leq_m$ coincide, and as $\leq_m$ does not imply $\leq_1$, it follows that component-wise reducibility as defined here is incomparable with 1-reducibility.

Proposition 2.2. *There exist binary relations such that $A \leq B$ but $A \not\leq_1 B$.*

Proof. Let C and D be any two sets such that $C \leq_m D$ but $C \not\leq_1 D$. Shift the elements of these sets up by one, creating $C^+ = \{x + 1 \mid x \in C\} \cup \{0\}$ and likewise with D. Let $A = C^+ \times \{0\}$, $B = D^+ \times \{0\}$.

Clearly, $A \leq B$: if f is the m-reduction from C to D, let f^+ be given by the maps $x + 1 \mapsto f(x) + 1$ and $0 \mapsto 0$. Observe that $(x, y) \in A$ if and only if $(f^+(x), f^+(y)) \in B$. However, $A \not\leq_1 B$. Suppose to the contrary that there exists a one to one g satisfying:

$$(x, y) \in A \qquad \Longleftrightarrow \qquad g(x, y) \in B.$$

Let g_1 be the first component of g. As the second component is always 0, it must be the case that g_1 is one to one, and it is clearly computable. This gives us:

$$x \in C \qquad \Longleftrightarrow \qquad g_1(x) - 1 \in D,$$

which contradicts the assumption on C and D. $\qquad \square$

As a useful starting point in the investigation of the complexity of sets under component-wise reducibility, we will demonstrate that there exist Σ_n^0 and Π_n^0-complete relations for every n. On the contrary, for $n \geq 2$ there are no Δ_n^0-complete relations. Note that here we are speaking of general binary relations, so this in no way contradicts the result we will later present in Theorem 5.9.

Proposition 2.3. *For every n, there is a Σ_n^0 and a Π_n^0 complete relation. However, there are no Δ_n^0-complete relations for $n \geq 2$.*

Proof. Let us first consider the case with $n = 1$. As Σ_1^0 relations are precisely those enumerated by some machine, we can fix some encoding of pairs and treat W_e as an r.e subset of $\mathbb{N}^2$. It then makes sense to speak of the eth Σ_1^0 relation: precisely that which is enumerated by W_e.

The eth Σ_1^0 relation for any e can then be reduced to $S = \{(\langle x, e \rangle, \langle y, i \rangle) \mid e = i \wedge (x, y) \in W_e)\}$ via the function $x \mapsto \langle x, e \rangle$. As such this relation is Σ_1^0 complete.

As with sets, the complement of a Σ_1^0-complete relation is complete for Π_1^0. This can be seen by considering that for any R in Σ_1^0:

$$(x, y) \in \overline{R} \iff (x, y) \notin R \iff (f(x), f(y)) \notin S \iff (f(x), f(y)) \in \overline{S}.$$

For $n > 1$ the argument is identical, except in this case a Σ_n^0 relation is one enumerated by a machine with access to a Π_{n-1}^0 oracle. Once we fix an enumeration of the sets generated by those, the rest follows.

For the Δ_n^0 case, recall that for $n \geq 2$ there are no Δ_n^0-complete sets ([Ers70]). We can see this by letting A be any Δ_n^0 set, $n \geq 2$, and letting B be as follows:

$$B = \{e \mid \exists y, t \; \varphi_{e,t}(e) = y \to \varphi_e(e) \notin A\}.$$

That is, B consists of the indices of partial recursive functions that either diverge on their own index or whose value on their own index is not in A. Note that B is Δ_n^0: determining whether a partial recursive function diverges on a given input is $\Pi_1^0 \subseteq \Delta_n^0$, and since A is a Δ_n^0 set determining whether an element is not in A is Δ_n^0.

We claim that $B \not\leq_m A$: suppose to the contrary that there exists a computable f such that $x \in B$ if and only if $f(x) \in A$. As $f = \varphi_e$ for some e, we can consider the behaviour of f on its own index. Suppose $e \in B$. By the definition of B, either $f(e)$ diverges, which is impossible as f is computable, or $f(e) \notin A$, which contradicts our assumption on f. It follows that $e \notin B$. However, that is impossible as it would necessitate that $f(e) \notin A$ and so, by the definition of B, $e \in B$.

This shows that for $n \geq 2$ there can therefore be no Δ_n^0-complete set, so by Proposition 2.1 there can be no Δ_n^0-complete relation. $\qquad\qquad\square$

In the sequel, we will not concern ourselves with arbitrary binary relations. Our objects of study are equivalence relations and preorders, and it is of interest that the argument in Proposition 2.3 can be lifted to establish the existence of equivalence relations and preorders complete for Σ_n^0.

Note that for $W_e \in \Sigma_1^0$, if we abuse notation then the transitive closure of W_e can be expressed as:

$$W_e \cup \{(x, y) \mid \exists k \; ((x, z_1) \in W_e \wedge (z_1, z_2) \in W_e \wedge \cdots \wedge (z_k, y) \in W_e)\}. \qquad (2.4)$$

To make the statement more precise, k would need to be a computable encoding of a finite tuple of indices. Regardless, as we are merely adding an additional existential quantifier over a finite conjunction, the resulting relation is still Σ_1^0.

Since every Σ_1^0 equivalence relation arises from the transitive closure of some symmetric and reflexive Σ_1^0 relation, it makes sense to speak of the eth Σ_1^0 equivalence relation: for a given W_e add the reflexive and symmetric pairs and take the transitive closure. We can then modify the proof of Proposition 2.3 to obtain a Σ_n^0-complete equivalence relation. This gives us the same construction of a universal equivalence relation as in Proposition 2.10 of [CHM12].

We can likewise speak of an eth Σ_1^0 preorder: add the reflexive pairs to W_e and take the transitive closure. A complete preorder exists by the same construction.

There are other ways we can derive complete equivalence relations from more general classes. Of use to us will be the simple observation that every equivalence relation arises as the symmetric fragment of some preorder, and the symmetric fragment of a Σ_n^0, Π_n^0 or Δ_n^0 preorder is still Σ_n^0, Π_n^0 or Δ_n^0 respectively.

Proposition 2.4. *The existence of a Σ_n^0, Π_n^0 or Δ_n^0-complete preorder implies the existence of a Σ_n^0, Π_n^0 or Δ_n^0-complete equivalence relation respectively.*

Proof. Let S be a preorder complete for some level of the arithmetical hierarchy. Let S_{sym} be the symmetric fragment of S; that is, $\{(x,y) \mid Sxy \wedge Syx\}$. We claim that S_{sym} is the required equivalence relation.

It is straightforward to verify that S_{sym} is indeed a Σ_n^0 (Π_n^0, Δ_n^0) equivalence relation: it is symmetric, transitive and reflexive, and Σ_n^0 (Π_n^0, Δ_n^0) languages are closed under intersection.

To see that it is complete, note that as any Σ_n^0 (Π_n^0, Δ_n^0) equivalence relation E is also a preorder, there exists a computable f satisfying:

$$Exy \qquad \Longleftrightarrow \qquad Sf(x)f(y).$$

However as E is symmetric, S has to preserve that property under the image of f and we get:

$$Exy \iff Exy \wedge Eyx \iff Sf(x)f(y) \wedge Sf(y)f(x) \iff S_{sym}f(x)f(y).$$

The proposition follows. $\qquad\qquad\square$

As our enquiry was motivated by equivalence relations, we will end this section with an interesting observation between the connection of component-wise and m-reducibility with respect to equivalence classes. This will later be seen in the construction of a Π_1^0-complete equivalence relation in Section 5.

Proposition 2.5. *If E is a Π_1^0 (Σ_1^0) equivalence relation, then $A = \{x \mid x = \min[x]\}$ is Σ_1^0 (Π_1^0).*

Proof. Let $E \in \Pi_1^0$. Observe that x is the minimum element of its equivalence class if and only if for all $y < x$, $(x,y) \notin E$. E is the complement of an r.e set, so $E = \mathbb{N}^2 \setminus W_e$ for some e. As such:
$$A = \{x \mid \exists t\ [0,x]^2 \setminus W_{e,t} = \{(x,x)\}\}.$$

For $E \in \Sigma_1^0$, $E = W_e$ so we can simply pick out the elements $y < x$ such that (x,y) never enters W_e:
$$A = \{x \mid \forall t \forall y < x\ (x,y) \notin W_{e,t}\}.$$

$\qquad\qquad\square$

2.3.1 A Δ_1^0-complete preorder

While the existence of a Δ_1^0-complete equivalence relation has been known for some time, the case of preorders has as yet escaped attention. We will hence here show how a complete preorder can be obtained in the computable case.

As opposed to subsequent sections, we are not here concerned with motivating the result as mathematically natural. Instead we focus purely on the technical aspect of the problem. The main difficulty lies in defining a preorder general enough to allow any computable preorder to be embedded in it. Intuitively, we address this by piecing every computable preorder together. However as there is no effective enumeration of Δ_1^0 sets, let alone the transitive ones, we in fact have to consider approximations of computable preorders in a given number of steps. As such we will define our preorder by letting $(x, e, t) \preceq (y, i, r)$ whenever $x = y, e = i, t = r$, or the following hold:

- $e = i$.

- The eth machine halts on all pairs in $[0, x]$ within t steps, and all pairs in $[0, y]$ within r steps, and the pairs accepted constitute a preorder.

- The pair (x, y) is accepted by the eth machine.

A relation so defined is clearly reflexive and computable. Once we establish transitivity, showing it is complete will be straightforward.

Theorem 2.6. *The preorder $\preceq$ described above is complete for Δ_1^0 preorders.*

Proof. First let us verify that $\preceq$ is transitive. Suppose $(x, e, t) \preceq (y, i, r) \preceq (z, k, s)$ for distinct triples. For contradiction, let $(x, e, t) \not\preceq (z, k, s)$. The possibility that $e \neq k$ is ruled out because $e = i$ and $i = k$. The machine in question also needs to halt on all pairs in $[0, x]$ and $[0, z]$ in order to have $(x, e, t) \preceq (y, i, r)$ and $(y, i, r) \preceq (z, k, s)$, so $(x, e, t) \not\preceq (z, k, s)$ cannot stem from the failure of the machine to define a preorder in the required number of steps. As such the only remaining possibility is that the machine rejects the pair (x, z). But that is clearly impossible as the machine accepts (x, y) and (y, z), and the pairs accepted in $[0, \max(x, y, z)]$ constitute a preorder.

It remains to define the reduction f, and the choice is clear: given a computable preorder P and an element x, f evaluates P on all pairs in $[0, x]$ and maps x to (x, e, t), where e is the index of the machine deciding P and t is the maximum of the time required to decide any of the pairs. $\square$

Corollary 2.7. *There exists a Δ_1^0-complete equivalence relation.*

Proof. By application of Proposition 2.4. We will see a much simpler example in the next section. $\square$

3 Literature Review

We can trace the study of component-wise reductions through three fields, distinguished by the requirements placed on the reduction in each case: by Borel functions in descriptive set theory, polynomial time functions in complexity theory and computable functions in computability theory. This is not to say that these are the only component-wise reducibilities present in the literature; for instance hyperarithmetical reductions are considered by Fokina and Friedman ([FF12]), effectively Borel (Δ_1^1) reductions by Fokina, Friedman and Törnquist ([FFT10]) and infinite-time computable reductions by Coskey and Hamkins [CH11].

We choose to focus on the three selected here due to their affinity in spirit to the sort of problems we will study in Sections 4 and 5.

3.1 Component-wise reductions in Descriptive Set Theory

Component-wise reductions are in fact the standard, rather than a novel, concept in descriptive set theory, and the modern interest in them in complexity and computability theory is likely inspired by these earlier results.

The notion of Borel reducibility is introduced in Friedman and Stanley ([FS89]), Definition 1 and 2. Denoted by $\leq_B$, if A and B are classes of structures then $A \leq_B B$ if and only if there exists a Borel function f such that for all $x, y \in A$ we have $f(x), f(y) \in B$ and:

$$x \cong y \iff f(x) \cong f(y). \tag{3.1}$$

As expected, we say B is Borel-complete if B is Borel and for all Borel A, $A \leq_B B$.

The classes of trees, linear orders and groups are shown to be Borel-complete as well as fields of a prime or zero characteristic. Examples of non-complete classes are finitely branching trees, Abelian torsion groups and fields of a prime or zero characteristic degree with a finite transcendence rank. It is interesting that restricted versions of these natural mathematical structures display similar behaviour in our framework also: we will present a result for subgroups of $\mathbb{Q}$ in Section 4.3 and for polynomial time trees in Section 5.1.

The only equivalence relation studied by [FS89] is thus isomorphism, however the framework was readily extended to a more general setting. Much of these results are covered by Gao ([Gao09]).

Two of the more famous results arising out of the fields are Silver's Dichotomy ([Sil80]) and the Glimm-Effros dichotomy (Harrington, Kechris and Louveau [HKL90]).

Silver's dichotomy states that a Π_1^1 equivalence relation's position in the degree hierarchy is either at most equality on $\mathbb{N}$ or at least equality over $2^{\mathbb{N}}$. Another consequence of [Sil80] is that equality over $2^{\mathbb{N}}$ is the simplest Borel equivalence relation with uncountably many equivalence classes. The Glimm-Effros dichotomy states that there is no Borel equivalence relation between equality over $2^{\mathbb{N}}$ and E_0, where E_0 is the relation of almost equality, i.e. $(A, B) \in E_0$ if the symmetric difference of A and B is finite. Part of the work in the computable case has been motivated by the desire to find similar results.

3.2   Component-wise reductions in Computability Theory

Given the breadth of the field, it is to be expected that the notion of reducibility means different things to different authors. We will briefly examine some alternate definitions before turning to those which correspond to our own.

Calvert, Cummins, Knight and Miller ([CCKM04]) consider the degree structure of classes of finite structures under *enumeration reducibility*, denoted by $\leq_c$. If A and B are two classes of finite structures, then $A \leq_c B$ means there is a partial computable function φ such that if $a, a' \in A$ and $a \cong a'$ then $\varphi(a) \cong \varphi(a')$. In our framework such an enquiry would have been trivial: all finite structures are computable, so the only distinction between relations on such would be the number of equivalence classes. The key difference between the approach of [CCKM04] and the one considered in this work is that in our study we are concerned with relations on the natural numbers, i.e. subsets of $\mathbb{N} \times \mathbb{N}$, whereas in [CCKM04] structures are encoded as subsets of $\mathbb{N}$, so the isomorphism relation studied there is in fact a subset of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$.

The resulting framework imposes a further structural constraint on reducibility. Their encoding ensures that if $a \subseteq b$, then the structure encoded by a is a substructure of that encoded by b. In Proposition 1.1, they show that this induces that if $A \leq_c B$ via f, then $a \subseteq b$ implies $\varphi(a) \subseteq \varphi(b)$. In other words, reducibility must respect substructures as well as isomorphism.

The resulting degree structure is rich. There are uncountably many classes incomparable via $\leq_c$ (Proposition 4.1), as well as countably infinite chains (Proposition 4.5). A maximal element of the ordering is that of finite graphs, and an open problem left by the authors is whether the class of finite graphs is reducible to the class of finite linear orders. This parallels the Complexity Theory results of [BCF+11], where the authors likewise find the class of graphs to be maximal under strong isomorphism reducibility, but are unable to determine whether the class of linear orders with a unary relation lies strictly below graphs or not. The notion of computable reducibility on structures encoded in $2^{\mathbb{N}}$ is further explored by Calvert and Knight ([CK06]). Knight, Miller and Vanden Boom ([KMB07]) consider the same setting but where the reducibility function is only required to be Turing.

Gao and Gerdes ([GG01]) use the same notion of reducibility as we do, labelling it m-reducibility by analogy with classical recursion theory. They demonstrate that the degree structure of equivalence relations has an initial segment of type $\omega + 1$. Namely, if $Id(n)$ is taken to mean equality modulo n and Id equality over the the natural numbers,

we have the following tower:

$$Id(1) \lneq Id(2) \lneq \cdots \lneq Id. \tag{3.2}$$

Every computable equivalence relation with n equivalence classes is bi-reducible with $Id(n)$, and those with infinitely many equivalence classes are bi-reducible with Id (Proposition 3.4). Together this gives a complete characterisation of the computable equivalence relations: precisely those which have a computable invariant function.

The authors further show that it is possible to embed the 1-degrees into the degrees induced by component-wise reducibility. Given an r.e. set A, the authors define $R_A = \{(x, y) \mid x = y \vee x, y \in A\}$ and show that $A \leq_1 B$ implies $R_A \leq R_B$. This result is strengthened by Coskey, Hamkins and Miller ([CHM12]) to show that if A, B are non-computable then $R_A \leq R_B$ implies $A \leq_1 B$. In other words, even though $\leq$ does not automatically imply $\leq_1$, the degree structure of equivalence relations nevertheless contains a copy of the 1-degrees.

It should be noted that while, as we have seen, the number of equivalence classes is crucial to the complexity of a computable equivalence relation, the size of these classes does not seem to matter much. This pattern persists throughout the arithmetical hierarchy, and an equivalence relation does not need to have an overly complex equivalence class structure to rank highly with respect to $\leq$. Fokina and Friedman ([FF12]) show that it is in fact possible to construct a properly Σ_1^1 equivalence relation, vastly more complex than anything we deal with here, which has equivalence classes of size one or two only (Claim 5.1).

Much of [CHM12] focuses on the parallels of Borel reducibility in a computable setting. They show that a version of Silver's Dichotomy fails: equality of of r.e. sets does not appear to have any special status in the computable world. There are relations strictly below and above it, even within Π_2^0. In fact, there exist infinite chains and arbitrarily large finite antichains of equivalence relations lying below equality of r.e. sets (Corollary 4.14). The authors leave the question as to whether a parallel of the Glimm-Effros dichotomy exists open.

Most similar in spirit to our enquiry is that of Fokina, Friedman and Nies ([FFN12]). Their study is of equivalence relations complete for Σ_3^0. Their main result (Theorem 1) demonstrates that 1-equivalence on r.e. sets is complete, which they later use to demonstrate that computable isomorphism of computable predecessor trees, Boolean algebras and metric spaces is Σ_3^0-complete. This is a technique we will use in Sections 4.3 and 5.1: establish the completeness of a relation that may only seem meaningful to a computer scientist, then use that result to show the completeness of more natural relations in mathematics.

3.3   Component-wise reductions in Complexity Theory

Polynomial time component-wise reductions seem to be first introduced by Fortnow and Grochow ([FG11], Definition 4.13). As the kernel of a polynomial time function f is the

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set of (x, y) satisfying $f(x) = f(y)$, [FG11] define a *kernel reduction* from E to S to be a polynomial time function satisfying:

$$Exy \iff Sf(x)f(y). \tag{3.3}$$

In other words, x, y, lie in the kernel of f modulo S.

The interest in this notion of reduction was due to the fact that an equivalence relation kernel-reduces to equality if and only if it has a polynomial time computable invariant. The paper leaves as an open question whether kernel reductions are actually distinct from standard polynomial time reductions, and if so whether P has a maximum element under kernel reductions.

The issue was picked up on by Buss, Chen, Flum, Friedman and Müller ([BCF$^+$11]) in their investigation of *strong isomorphism reductions*, $\leq_{iso}$, and *strong equivalence reductions*, $\leq_{eq}$, depending on whether the underlying relation concerned the isomorphism of finite objects or a general equivalence relation.

More specifically, they define a strong isomorphism reduction to be a polynomial time f between classes of arbitrarily large structures closed under isomorphism, where a structure is a finite tuple of relations over a finite domain. Then we can say for classes C, D that $C \leq_{iso} D$ if and only if for all $\mathcal{A}, \mathcal{B} \in C$:

$$\mathcal{A} \cong \mathcal{B} \iff f(\mathcal{A}) \cong f(\mathcal{B}). \tag{3.4}$$

Strong equivalence reductions are defined more in line with the component-wise reductions we defined in (2.1). As such $\leq_{eq}$ does not range over classes, but rather equivalence relations. As expected, a polynomial time f is a reduction from R to E, denoted $R \leq_{eq} E$, if it satisfies:

$$(x, y) \in R \iff (f(x), f(y)) \in E. \tag{3.5}$$

Within the paper the open question of [FG11] regarding the distinction of $\leq_m^P$ from a component-wise reduction was answered by demonstrating a rich structure of strong isomorphism degrees within P. The authors demonstrate that the countable, atomless Boolean algebra is embeddable into the degree structure induced by $\leq_{iso}$ on polynomial time classes of structures (Theorem 5.1).

There thus exist infinite chains of equivalence relations under strong isomorphism reductions despite being equivalent under $\leq_m^P$; one is therefore able to achieve an infinitely finer gradation of degrees with component-wise reductions in complexity theory.

As we have seen in the previous section, for computable equivalence relations the number of equivalence classes determines the degree structure. [BCF$^+$11] study a parallel notion as *potential reducibility*: if $A(n)$ and $B(n)$ represent the number of isomorphism types of structures of size at most n, the authors observe that if A strong isomorphism reduces to B, it must be the case that for some polynomial p, $A(n) \leq B(p(n))$ for all $n \in \mathbb{N}$. When the latter condition holds A is said to potentially reduce to B.

The authors were unable to determine whether potential reducibility differs from their other notions in the absence of further complexity theoretic assumptions. However

they do demonstrate that if potential reducibility and strong isomorphism reducibility is distinct then P$\neq$ #P. For strong equivalence reducibility, being distinct from potential reducibility would imply P$\neq$NP.

A curious distinction between the case with complexity and computability reductions is that while the existence of complete equivalence relations, at least for Σ_n^0, was never in question, the authors of [BCF$^+$11] were unable to answer the second question of [FG11] definitively. They did, however, manage to derive a necessary and sufficient condition: any of the classes of polynomial time, NP, or CoNP equivalence relations admit a complete element if and only if there exists an effective enumeration of that class.

The "if" direction clearly carries over to the computable case: this is precisely the argument that guarantees the existence of Σ_n^0 complete equivalence relations. The second direction, however, relies on the fact that there exists an effective enumeration of polynomial time functions, and as such given a maximum element the class of polynomial time equivalence relations it would be possible to enumerate all elements of that class via the preimage of the reduction functions. We cannot replicate such a technique in the computable case. However, it does not appear that we need to: in Section 5 we will demonstrate the existence of a Π_1^0-complete equivalence relation, whereas we do not know of any effective enumeration of that class.

4 The case of Σ_n^0

While the existence of Σ_n^0-complete equivalence relations and preorders follow immediately from the transitive closure of a Σ_n^0 relation being Σ_n^0, natural examples of such relations are nevertheless interesting as component-wise reducibility induces a finer granularity on the arithmetical hierarchy than $\leq_m$. In this section we give three examples of preorders and associated equivalence relations from logic, complexity theory and algebra complete for Σ_1^0, Σ_2^0 and Σ_3^0 respectively.

4.1 Σ_1^0 and Logical Equivalence

The reader is reminded that the following problem is Σ_1^0-complete:

$$\{\varphi \mid \exists\Phi : \ \Phi \text{ is a FOL proof of } \varphi\}. \tag{4.1}$$

By FOL we mean first order logic with a full signature; that is, with predicate symbols of any arity[1]. This yields a natural Σ_1^0 equivalence relation, logical equivalence:

$$\{(\varphi, \psi) \mid \exists\Phi : \ \Phi \text{ is a FOL proof of } \varphi \leftrightarrow \psi\}. \tag{4.2}$$

In a sense this result is not new. Montagna and Sorbi ([MS85]) have shown that implication in any consistent, finitely axiomatisable r.e. extension of Peano arithmetic is complete for Σ_1^0-preorders. In an earlier result Pour-El and Kripke ([PEK67]) have shown that finitely axiomatisable theories containing a sufficient portion of Peano Arithmetic are complete with respect to the class of such theories under a deduction-preserving mapping. As the conjunction of any finite set of axioms can be thrown into the antecedent of a first order sentence, both of these frameworks can extend to our result. What we present below, however, is a direct proof, and the only fragment of arithmetic we require is that which is already available in first order logic.

Theorem 4.1. *Logical implication is complete for Σ_1^0 preorders. That is, for every Σ_1^0 preorder P and every x, y there exists a computable f mapping integers to FOL formulae such that:*

$$Pxy \iff \vdash_{FOL} f(x) \to f(y). \tag{4.3}$$

---

[1]Of course, the full strength of first order logic is not required. The papers of Church and Turing ([Chu36],[Tur37]) established this result only needing predicates of arity at most 2. However as having predicates of higher arity is convenient for our construction, we do not place ourselves under any restrictions here.

Proof. Observe that $P \in \Sigma_1^0$ implies that for a given x the set $\{y \mid Pxy\}$ is uniformly r.e., meaning there exists a machine which enumerates it. Our proof relies on representing the computation of such a machine in first order logic. We define a non-deterministic M that on input x begins to enumerate all y above it. Every time a new y is found, the machine makes a non-deterministic choice. One branch will continue with the described computation, whereas the other will "restart" on input y. That is: clear the tape, write y on it, move to the first cell and enter the initial state. Observe that if Pxy, some branch of M on input x will eventually enter the initial configuration of M on input y.

The configurations of M will correspond to formulae in the logic, and our construction will ensure that if φ and ψ are representations of configurations, and M can make a transition from φ to ψ, then $\varphi \to \psi$ will be provable in the logic. Therefore our reduction will be given by an f that maps x to the the initial configuration of M on input x. Readers familiar with such constructions can safely skip the rest of the proof: it contains nothing but technicalities.

Let us be more specific about our machine. We require a non-deterministic M with two tapes: a working tape and a printing tape. For simplicity, M is working on a unary alphabet. The initial configuration of M on input x then consists of the first x cells of the working tape marked, both heads at the leftmost edge of their tapes and the machine in the initial state, q_0. When run on a tape with the first x cells marked the machine will print all y with Pxy. We take printing y to mean that it will mark the first y cells of the print tape, then enter a special state q_{pr}.

After exiting q_{pr} it enters another special state q_{nd} and then makes a non-deterministic branch. One branch continues searching for elements above x in P, the other clears both tapes, marks the first y cells of the working tape, moves the heads to the start of the tapes, enters a special state q_{re}, then enters the initial state q_0. Intuitively, one branch continues with the computation, the other restarts computation on input y. For the sake of convenience, we assume the machine has the ability to keep the head still and not write or read anything during a transition, which it only ever uses in states q_{pr}, q_{nd} and q_{re}. We also require that the *cleaning* states, i.e. those used between q_{nd} and q_{re}, are not used anywhere else in the computation, and the special states q_{pr}, q_{nd} and q_{re} are not used anywhere except in the circumstances outlined above.

The idea behind the construction is to have f map x and y to the initial configurations of M on input x and y respectively, and then represent the computation of M in first order logic. If Pxy, then some computation branch of M on input x will eventually enter the initial configuration of M on input y and as such it will be possible to derive $f(y)$ from $f(x)$.

As a convention, we will treat the first cell on the tape as 1 rather than 0.

We require the following predicate symbols in our logic. For the reader's convenience we note the intended interpretation in parentheses, but of course as far as syntax is concerned these are just arbitrary predicates.

- C_0: 4-ary predicate ($C_0 w_m s_t d_b u_x$ means cell m of the working tape is marked at time t with initial machine input x, and d_b is a counter used to track the branch of the computation).

- C_1: 4-ary predicate (As C_0, but for the output tape).

- Q_i 4-ary predicate for all machine states q_i ($Q_i s_t d_b u_x n_z$ means the machine is in state q_i at time t in branch b with initial input x. The variable n_z is $y + 1$ if the machine is in the process of cleaning the tape to restart on y, otherwise it is 0).

- H_0: 4-ary predicate ($H_0 w_m s_t d_b u_x$ means working head is over cell m at time t with initial machine input x on branch b of the computation).

- H_1: 4-ary predicate (Same as H_0 but for the output head).

- P: 4-ary predicate ($P w_y s_t u_x$ means the machine printed y at time t with initial input x).

- S: Binary predicate (Sxy means $y = x + 1$).

- Z: Unary predicate (Zx means $x = 0$).

- L: Binary predicate (Lxy means $x \leq y$).

- E: Binary predicate (Exy means $x = y$).

A few notes on our use of the variables: the intended interpretation of our domain is the natural numbers, and so we use subscripts above to signal the number a specific variable is meant to represent. As our variables will generally be bounded by quantifiers, we can assume without loss of generality that they are always renamed to conform with the conventions above. E.g. s_{i+1} represents the time step following s_i, w_m is the mth cell and so on. We do however wish to remind the reader that this is merely for notational intuition: these are arbitrary variables, and arithmetic is not a part of our logic.

We require the branch counters on certain predicates because as our machine is non-deterministic, its configuration is not uniquely determined by the clock. The head, for instance, can be in several positions at a given time and if we are to prevent the head in a parallel computation branch from interfering, we need to keep track of where in the computation tree each is located. This is purely a technicality, the only places in which we will touch the branch counter is in state q_{nd}, where the choice happens, and in state q_{re}, where the machine restarts.

The variable n_z is carried by the state predicate purely so that we can "remember" the value we are supposed to restart the machine on. We alter it only after the print state, and when the machine is restarted.

It is necessary to record the initial input via u_x because a single predicate captures not the entire machine configuration, but just a component of it: it is entirely possible for two different configurations to exist at the same time and same branch number of the computation, but with different initial input. We do not wish these to interfere with each other. The only time we change u_x is when we reset the machine.

Next we define some formulae to capture the behaviour of M and the characteristics of the natural numbers. As is usual in logic we cannot guarantee the uniqueness of zero and successor, so we have to settle for uniqueness modulo E.

1. $\forall x\ Lxx$ (reflexivity of L).

2. $\forall xyz\ Lxy \wedge Lyz \rightarrow Lxz$ (transitivity of L).

3. $\forall xy\ Lxy \vee Lyx$ (totality of L).

4. $\forall xy\ Exy$ (reflexivity of E).

5. $\forall xyz\ Exy \wedge Eyz \rightarrow Exz$ (transitivity of E).

6. $\forall xy\ Exy \rightarrow Eyx$ (symmetry of E).

7. $\forall xy\ Zx \rightarrow Lxy$ (zero is least element).

8. $\forall xy\ Sxy \rightarrow Lxy$ (L respects successor).

9. $\forall xy\ Zx \rightarrow \neg Syx$ (zero has no predecessor).

10. $\forall x \exists y\ Sxy$ (every number has a successor).

11. $\forall x \exists yz\ Sxy \wedge Sxz \rightarrow Eyz$ (uniqueness of successor modulo E).

12. $\forall x \exists y\ \neg Zx \rightarrow Syx$ (every non-zero number has a predecessor).

As well as an indiscernibility of identicals clause, stating that Exy implies that the predicates we defined above are true for x if and only if they are true for y. Note that this is a valid first order formulae because we have defined only finitely many predicate symbols. The conjunction of the above we will label NAT.

We next turn to the machine behaviour. Recall that every transition of M can be represented as a sextuple (T, q_i, R, W, M, q_j), where q_i is the machine state before the transition, $R \in \{0, 1\}$ represents whether or not the cell under the head on tape T is marked, $W \in \{0, 1\}$ the action of the machine on the cell under the head on tape T, $M \in \{Le, Ri\}$ the direction the head moves on tape T and q_j the end state.

1. For every transition of the form (T, q_i, R, W, M, q_j), where $q_i \notin \{q_{pr}, q_{nd}, q_{re}\}$:

$$\forall w_m s_t d_b u_x n_z \exists s_{t+1}\ (Ss_t s_{t+1} \wedge (ANT \rightarrow (Q_j s_{t+1} d_b u_x n_z \wedge TAPE \wedge HEAD))).$$

   ANT states the machine read R in state q_i on tape T. Note that whether or not the last clause is present is contingent on the value of R:

$$Q_i s_t d_b u_x n_z \wedge H_T w_m s_t d_b u_x \wedge C_T w_m s_t d_b u_x [\text{if } R = 1].$$

   $TAPE$ states that the cells not under the head on tape T remain unchanged:

$$\forall w_k k(\neg H_T w_k s_t d_b u_x \rightarrow$$

$$((C_T w_k s_t d_b u_x \rightarrow C_T w_k s_{t+1} d_b u_x) \wedge (C_{1-T} w_k s_t d_b u_x \rightarrow C_{1-T} w_k s_{t+1} d_b u_x))).$$

$HEAD$ states that the T-head writes W and moves M, while the other head remains where it is. Note that the formulae will vary depending on the values of W, R and M:

$C_T w_m s_{t+1} d_b u_x$ [if $W = 1$] $\wedge$
$(\forall i\ Zi \to$
$((Si w_m \to H_T w_m s_{t+1} d_b u_x) \wedge (\neg Si w_m \to (\exists w_{m-1}\ S w_{m-1} w_m \wedge H_T w_{m-1} s_t d_b u_x))))$
[if $M = Le$] $\wedge$
$(\exists w_{m+1}\ S w_m w_{m+1} \wedge H_T w_{m+1} s_{t+1} d_b u_x)$ [if $M = Ri$] $\wedge$
$(\forall w_m s_t d_b u_x\ H_{1-T} w_m s_t d_b u_x \to H_{1-T} w_m s_{t+1} d_b u_x)$.

2. For every transition with $q_i = q_{pr}$, we remind the reader that M prints the contents of the output tape and transitions to Q_{nd} without moving the heads or writing anything. So we have:

$$\forall w_m s_t d_b u_x n_z \exists s_{t+1}\ (S s_t s_{t+1} \wedge$$

$$(Q_{pr} s_t d_b u_x n_z \to (Q_{nd} s_{t+1} d_b u_x n_z \wedge TAPE \wedge HEAD \wedge PRINT))).$$

$TAPE$ preserves the tape contents:

$$(C_0 w_m s_t d_b u_x \to C_0 w_m s_{t+1} d_b u_x) \wedge (C_1 w_m s_t d_b u_x \to C_1 w_m s_{t+1} d_b u_x).$$

$HEAD$ preserves the head locations:

$$(H_0 w_m s_t d_b u_x \to H_0 w_m s_{t+1} d_b u_x) \wedge (H_1 w_m s_t d_b u_x \to H_1 w_m s_{t+1} d_b u_x).$$

$PRINT$ prints the contents of the output tape:

$$\forall w_y w_m (C_1 w_y s_t d_b u_x \to (C_1 w_m s_t d_b u_x \to L w_m w_y)) \to P w_y s_{t+1} u_x.$$

3. For every transition with $q_i = q_{nd}$ and q_j a cleaning state, we remind the reader that M makes a transition to q_j without doing anything else. All we need to do is preserve the tape contents, update the branch variable and set n_z to remember the printed value. Note that this will induce that all cleaning steps have an odd branch counter:

$$\forall w_m s_t d_b u_x n_z w_y \exists s_{t+1} d_{b+1}\ (S s_t s_{t+1} \wedge S d_b d_{b+1} \wedge$$

$$(P w_y s_t u_x \wedge Q_{nd} s_t d_b u_x n_z \to (Q_j s_{t+1} d_{b+1} u_x w_y \wedge TAPE \wedge HEAD))).$$

$TAPE$ preserves the tape contents:

$$(C_0 w_m s_t d_b u_x \to C_0 w_m s_{t+1} d_{b+1} u_x) \wedge (C_1 w_m s_t d_b u_x \to C_1 w_m s_{t+1} d_{b+1} u_x).$$

$HEAD$ preserves the head locations:

$$(H_0 w_m s_t d_b u_x \to H_0 w_m s_{t+1} d_{b+1} u_x) \wedge (H_1 w_m s_t d_b u_x \to H_1 w_m s_{t+1} d_{b+1} u_x).$$

4. For every transition with $q_i = q_{nd}$ and q_j a non-cleaning state, we remind the reader that M makes a transition to q_j without doing anything else. All we need to do is preserve the tape contents and update the branch variable. Note that this induces that all the non-cleaning steps have an even branch counter:

$$\forall w_m s_t d_b u_x n_z \exists s_{t+1} d_{b+1} d_{b+2} \ S s_t s_{t+1} \wedge S d_b d_{b+1} \wedge S d_{b+1} d_{b+2} \wedge$$

$$(Q_{nd} s_t d_b u_x n_z \rightarrow (Q_j s_{t+1} d_{b+2} u_x n_z \wedge TAPE \wedge HEAD)).$$

$TAPE$ preserves the tape contents:

$$(C_0 w_m s_t d_b u_x \rightarrow C_0 w_m s_{t+1} d_{b+2} u_x) \wedge (C_1 w_m s_t d_b u_x \rightarrow C_1 w_m s_{t+1} d_{b+2} u_x).$$

$HEAD$ preserves the head locations:

$$(H_0 w_m s_t d_b u_x \rightarrow H_0 w_m s_{t+1} d_{b+2} u_x) \wedge (H_1 w_m s_t d_b u_x \rightarrow H_1 w_m s_{t+1} d_{b+2} u_x).$$

5. For every transition with $q_i = q_{re}$ recall that $q_j = q_0$ and all other parameters are ignored. The heads should already be at the start of the tapes, so all we need to do is preserve the tape contents and reset the counters. We then have:

$$\forall w_m s_t d_b u_x n_z i \exists u_y w_1 \ (Z i \wedge Q_{re} s_t d_b u_x n_z \rightarrow$$

$$(S u_y n_z \wedge S i w_1 \wedge Q_0 i i u_y i \wedge TAPE \wedge H_0 w_1 i i u_y \wedge H_1 w_1 i i u_y)).$$

Since the output tape should be empty, $TAPE$ is:

$$C_0 w_m s_t d_b u_x \rightarrow C_0 w_m i i u_y.$$

The conjunction of the above we will label TRA.

We will use $CONF(x)$ as shorthand for the formula representing the initial configuration of M on input x. That is:

$$\exists w_1 \ldots w_x u_1 \ldots u_x \ Z0 \wedge$$

$$\bigwedge_{i=0}^{x-1} (S w_i w_{i+1} \wedge S u_i u_{i+1}) \wedge Q_0 00 u_x 0 \wedge H_0 w_1 00 u_x \wedge H_1 w_1 00 u_x \wedge \bigwedge_{i=1}^{x} C_0 w_i 00 u_x.$$

We now have all the components necessary to define f. Namely:

$$f(x) = NAT \wedge TRA \wedge CONF(x).$$

It remains to show that Pxy is equivalent to $f(x) \rightarrow f(y)$.

First, suppose that $\neg Pxy$. Recall that the completeness of first order logic implies a formula is provable if and only if it is true under all interpretations. In particular, it must also be true under the interpretation we have been using throughout the construction: that of the simulation of M. As $\neg Pxy$ implies that M on input x will never reach the

initial configuration of M on input y, $f(x) \to f(y)$ is false under some interpretation and thus cannot be provable in the logic.

Now suppose Pxy. Observe that this implies there exists a computation history of M starting with the initial configuration of M on x and leading to the initial configuration of M on y. Let $HIST(x,t)$ be the formula representing the machine configuration at time t of this history. We will show that for $0 \le t \le r-1$, $\bigwedge_{i \le t} HIST(x,i) \to HIST(x,t+1)$. This established, by the principle of strong induction we will have $f(x) \to f(y)$.

Note that there is no generality lost in assuming that $r-1$ is the only step in the history where the state is q_{re}. By the transitvity of P, whatever elements M may reach after restarting on another element, it will eventually reach by taking the non-cleaning branches until the desired number is printed.

Before we proceed, we must be explicit about what we mean by $HIST(x,t)$ representing the machine configuration. Let p be the farthest cell to the right on either tape either marked or with a head above it, b the current branch counter and n either 0 or $y+1$ depending on whether the machine has entered the cleaning stage. Let $t \ne r$.

$$HIST(x,t) = \exists w_1 \ldots w_p s_1 \ldots s_t u_1 \ldots u_x d_1 \ldots d_b n_1 \ldots n_z$$

$$NAT \wedge TRA \wedge Z0 \wedge Succ(x,t) \wedge State(x,t) \wedge Heads(x,t) \wedge Tape(x,t).$$

$Succ(x,t)$ associates all active variables with the numbers they represent:

$$Succ(x,t) = S0w_1 \wedge S0s_1 \wedge S0u_1 \wedge S0d_1 \wedge S0n_1 \wedge$$

$$\bigwedge_{i=1}^{p-1} Sw_i w_{i+1} \wedge \bigwedge_{i=1}^{t-1} Ss_i s_{i+1} \wedge \bigwedge_{i=1}^{x-1} Su_i u_{i+1} \wedge \bigwedge_{i=1}^{z-1} Sn_i n_{i+1}.$$

$State(x,t)$ is the current state, $Heads(x,t)$ is the positions of the heads and $Tape(x,t)$ is the contents of the tape. Note that this is consistent with our requirement that $HIST(x,0) = f(x)$. $HIST(x,r)$ we let equal to $f(y)$ as mentioned above. We will show that the implication holds for every component individually.

NAT, TRA and $Z0$ are constant throughout, so clearly $NAT \wedge TRA \wedge Z0 \to NAT \wedge TRA \wedge Z0$ via the law of identity.

The $Succ(x,t)$ clause is also simple. To show $Succ(x,t)$ we need only show the existence of sufficiently many successor elements, but NAT allows us to always derive the existence of a new variable that is the successor to some existing variable. As such for all t, $NAT \wedge Z0 \to Succ(x,t+1)$ simply by introducing the correct number of successors and renaming them as needed.

If the machine is in state i at step t, then $State(x,t)$ is simply $Q_i tbw_x z$. It is easy to verify that if $t < r-1$ then we can derive $Q_j(t+1)bw_x z$ for the appropriate j via one of 1-4, and hence $State(x,t) \to State(x,t+1)$. When $t = r-1$, however, while 5 says that we can derive $Q_0 00u_y 0$, we must be careful not to be deceived by notation: u_y in this context is a simply another name for n_{z-1} (n_z being such that $Q_{re}s_{r-1}d_b u_x n_z$ holds at time $r-1$), whereas in $State(x,s) = Q_0 00u_y 0$ it is specifically the yth successor of zero.

To prove the implication holds we must therefore show that $En_{z-1}u_y$ holds at time r; that is, is derivable from $\bigwedge_{t<r} HIST(x,t)$.

Let t' be the time in which the machine prints y. Observe that the machine is in state q_{pr}, so 2 applies. From the $PRINT$ statement we can derive $Pw_ys_{t'+1}u_x$. In the context of $PRINT$, w_x is the rightmost cell on the output tape. Since at time t' the output tape has the first y cells marked, w_y is the yth successor of zero. As NAT accounts for the uniqueness of successor, $(HIST(x,t') \wedge NAT) \to Ew_yu_y$. In state $t'+1$, the machine enters q_{nd} where from 3 we get $Q_js_{t'+2}d_{b+1}u_xw_y$. Note that from this point on the last argument of the state predicate is w_y where Ew_yu_y, we can rename variables to obtain $Q_{re}s_{r-1}d_bu_xn_z$ and En_zu_y as required. This gives us $\bigwedge_{i \le t} HIST(x,i) \to State(x,t)$.

For the $Heads(x,t)$ clause we must preserve the position of the heads that do not move, and move the heads that do. In states q_{pr}, q_{nd} and q_{re} both heads are stationary, and this is accounted for by the $HEAD$ clause of 2-5. If a head does move, we are in the case covered by 1 and it is easy to verify that the $HEAD$ clause there does all that is required: move the active head left or right, keeping it still if it tries to move off the edge of the tape, and preserving the position of the other head. For the transition to $Heads(x,r)$ we must also verify that u_y is indeed the yth successor of zero, which we do by the same argument we used for $State(x,t)$, giving us $\bigwedge_{i \le t} HIST(x,i) \to Heads(x,t)$.

For the $Tape(x,t)$ clause we must make sure the marked tape cells are preserved and if the head marks a new cell it is accounted for. Note that if the machine is in state q_{pr}, q_{nd} or q_{re}, the head takes no action and the preservation of the tape follows immediately from the $TAPE$ clause of 2-5. If the machine is in a different state the head must either mark or clear the cell it is over. Note that the $HEAD$ clause in 1 provides for exactly this, and the $TAPE$ clause preserves the rest of the tape. It follows that $Tape(x,t) \to Tape(x,t+1)$ for $t < r-1$. For the last implication we need to assure ourselves that u_y is the yth successor of zero, which we do by the same argument as before, giving us $\bigwedge_{i \le t} HIST(x,i) \to Tape(x,t)$.

From all of the above it follows that for $t < r$, $\bigwedge_{i \le t} HIST(x,i) \to HIST(x,t+1)$. By strong induction it follows that if Pxy then we can derive $f(y)$ from $f(x)$, completing the theorem. $\qquad\square$

Corollary 4.2. *Logical equivalence is complete for Σ_1^0 equivalence relations.*

Proof. Follows from Proposition 2.4, as equivalence is the symmetric fragment of implication. $\qquad\square$

4.2 Σ_2^0 and Polynomial Time Equivalence

With two lead quantifiers, Σ_2^0 admits a wider array of candidates for our study as it is now simple to speak of the behaviour of mappings from one structure to another: we can use the lead existential quantifier to select a function from a suitable class, and the universal to dictate the function's behaviour on all input. We choose to consider a preorder and a resulting equivalence relation from complexity theory: that of polynomial

time reducibility of EXPTIME sets. In fact we do not need the full power of EXPTIME, treating it as $\text{DTIME}(2^n)$ would suffice.

We first prove a lemma which establishes a connection between component-wise and m-reducibility. Since when viewed as a set of pairs a Σ_n^0 preorder is just a Σ_n^0 set, we can bootstrap classical completeness results to provide a useful characterisation of the preorder in question. The technique is quite general, and we make further use of it in Sections 4.3 and 5.2.

Lemma 4.3. *For every Σ_2^0 preorder $\preceq$ there exists a sequence of non-empty r.e. sets $\{V_{xy} \mid x, y \in \mathbb{N}\}$ such that $x \preceq y$ iff V_{xy} is finite, and given x and y we can retrieve the machine enumerating V_{xy} in time linear in $\log x, \log y$. That is, linear in the size of the input.*

Proof. Recall that the following problem is m-complete for Σ_2^0 (see, for instance, [Soa87] Theorem IV:3.2):

$$\{e \mid W_e \text{ is finite}\}.$$

It follows that there exists a computable f such that $(x, y) \in \preceq \iff W_{f(x,y)}$ is finite. Such a $W_{f(x,y)}$ is then precisely the V_{xy} required. Observe that there is no generality lost in assuming V_{xy} is non-empty: we can merely define V_{xy} to be $W_{f(x,y)} \cup \{0\}$, thereby guaranteeing that it has at least one element. It remains to show that we can compute f in linear time.

As $\preceq$ is a Σ_2^0 set, there exists a recursive predicate R such that $x \preceq y \iff \exists v \forall u \ Rvuxy$. Let M be the (always halting) machine computing R and N the machine that on input $\langle x, y \rangle$ runs M on $\langle v = 0, u, x, y \rangle$, for all u. If M ever rejects, N prints v and runs M on $\langle v+1, u, x, y \rangle$ for all u, and so on. Observe that if $x \preceq y$ then N will only print finitely many values, as eventually it will reach such a v such that $Rvuxy$ holds for all u. We will therefore use N_{xy} to denote N with input fixed to $\langle x, y \rangle$, N_{xy} is precisely the machine enumerating V_{xy}.

Let $|N|$ be the size of N. Observe that to obtain N_{xy} from N, it is sufficient to add $2|\langle x, y \rangle|$ states to N: $|\langle x, y \rangle|$ to print $\langle x, y \rangle$ on the tape, $|\langle x, y \rangle|$ to move the head back to the first cell and then enter the initial state of N. The size of N_{xy} is then $|N| + 2|\langle x, y \rangle|$, so if the encoding is linear in $\log x, \log y$, then we can obtain a machine enumerating V_{xy} in time linear in $\log x, \log y$. $\qquad\square$

Theorem 4.4. *Polynomial time reducibility of EXPTIME sets is complete for Σ_2^0 preorders. That is, if A_x is the xth exponential set, then for every Σ_2^0 preorder $\preceq$ there exists a computable function f satisfying:*

$$x \preceq y \iff A_{f(x)} \leq_m^P A_{f(y)}.$$

Proof. First, let us establish that this relation is in fact Σ_2^0. Note that $A_i \leq_m^P A_k$ if and only if there exists a polynomial time t such that $x \in A_i \iff t(x) \in A_k$. Taking t_e to be the eth polynomial time function, the desired relation can be expressed as follows:

$$\{(i, k) \mid \exists e \forall x \ x \in A_i \leftrightarrow t_e(x) \in A_k\}.$$

The matrix is computable: we can query A_i in exponential time, compute t_e in polynomial and verify whether the result is in A_k in exponential. As a result the relation is Σ_2^0.

Let $\preceq$ be a Σ_2^0 preorder, $\{V_{xy} \mid x, y \in \mathbb{N}\}$ be a sequence of non-empty r.e. sets as in Lemma 4.3, $\{M_e \mid e \in \mathbb{N}\}$ be an enumeration of all polynomial time oracle machines, and p_e be the clock of the eth machine. We will construct EXPTIME sets $\{A_x \mid x \in \mathbb{N}\}$ consisting of strings over the alphabet $\{0, 1\}$ satisfying:

1. V_{xy} finite $\rightarrow A_x \leq_m^P A_y$.

2. V_{xy} infinite $\rightarrow A_x \not\leq_T^P A_y$. That is, $A_x \neq M_e(A_y)$ for any e.

Note that as $\leq_m^P$ implies $\leq_T^P$, this will be sufficient to establish the theorem.

The idea behind the construction is to make the sets extremely sparse, so that elements we add to the set later in the construction will be much too large to affect the computation of any polynomial time reduction. To this end we define the function g by $g(0) = 1$, $g(n+1) = 2^{g(n)}$. We also have need of a unary encoding of 4-tuples that encodes each 4-tuple by a string of length $g(n)$ for some n. For convenience, if (x, y, r, e) is encoded by 0^m then we require that $x, y, r, e \leq m$. Clearly for a function that grows as quickly as g this will be satisfied almost everywhere anyway, but insisting on the restriction reduces the cases needed to consider in the proof.

To ensure (1) the construction will make use of a suffix table. This table will associate a pair (x, y) with a 4-tuple (x, y, r, e) and a polynomial q. The sets $\{A_x \mid x \in \mathbb{N}\}$ will be constructed by stages, and the meaning of the suffix table is that if at stage n of the construction (x, y) is associated with (x, y, r, e) and q then for all w with $|w| \geq g(n)$, $w \in A_x$ at stage n iff $w1^{\langle x, y, r, e \rangle} 0^{q(|w|)} \in A_y$. The table thus defines a polynomial reduction of A_x at stage n into A_y: if w is long enough we can apply the suffix, if it is too short we can verify whether it belongs to A_x directly in constant time. If after some stage of the construction the (x, y) entry is never updated, we will have a reduction at all stages, and hence a reduction of the entire set.

To ensure (2) we will use P_{xye} to denote the requirement that $A_x \neq M_e(A_y)$. This will be done by letting $A_x(w) = 1 - M_e(A_y; w)$ for some w. The construction will ensure that if V_{xy} is large enough then P_{xye} will be satisfied.

Regarding terminology, note that we distinguish between P_{xye} being satisfied and being declared satisfied. When we say that P_{xye} has been declared satisfied, we mean that the stage of the construction below where the declaration is made has been reached. When we say that P_{xye} is satisfied we mean that it is true: that is, $A_x \neq M_e(A_y)$ whether we have declared P_{xye} to be satisfied or not.

Once satisfied, P_{xye} is never injured. The idea is then that if V_{xy} is finite, we will have a reduction via the suffix table. If V_{xy} grows indefinitely, it will eventually result in every P_{xye} being satisfied, meaning no machine can be a reduction.

At stage n of the construction let $m = g(n)$, and as $g(n)$ is the encoding of a 4-tuple, $m = \langle x, y, r, e \rangle$. The purpose of the stage is to determine whether $w = 0^m$ is in A_x.

First verify that the eth element enters V_{xy} at step r. If not, do nothing at this stage.

Update the (x, y) entry of the suffix table with (x, y, r, e) and $q = \sum_{i \leq e} p_e$, and effect all the codings required by the suffix table. That is, for every entry (x', y') associated with (x', y', r', e') and q', place $w'1^{\langle x', y', r', e'\rangle}0^{q'(|w'|)}$ in $A_{y'}$ for every $w' \in A_{x'}$.

Next, verify:

- We can simulate $M_e(Ay; w)$ in 2^m time.

- Placing w into A_x and effecting again all the codings required by the suffix table will not place any string shorter than $p_e(m)$ into A_y.

If either fails, do nothing further. Otherwise let $A_x(w) = 1 - M_e(A_y; w)$, declare P_{xye} satisfied and effect all codings required by the suffix table.

This completes the construction. It remains to verify (1), (2) hold, and that A_x is indeed EXPTIME.

Suppose $|V_{xy}| = i$ and that the ith element enters V_{xy} at step s. Let n be the stage where $g(n) = \langle x, y, s, i\rangle$. At stage n the suffix table is updated to associate (x, y) with (x, y, s, i) and $q = \sum_{j \leq i} p_j$, meaning for any w already in A_x, $w1^{\langle x,y,s,i\rangle}0^{q(|w|)}$ is in A_y. Observe that as no element enters V_{xy} again, the (x, y) entry of the table is never updated. This means that if some other w ever enters A_x, effecting the suffix table will require that $w1^{\langle x,y,s,i\rangle}0^{q(|w|)}$ be in A_y. This gives a polynomial time reduction from A_x to A_y:

$$f(w) = \begin{cases} w1^{\langle x,y,s,i\rangle}0^{q(|w|)} & \text{if } |w| \geq g(n) \\ \text{least } v \in A_y & \text{if } |w| < g(n) \text{ and } w \in A_x \\ \text{least } u \notin A_y & \text{if } |w| < g(n) \text{ and } w \notin A_x. \end{cases}$$

Note that the two latter cases are in fact constant time: the amount of time to find v or u will be the same on any input, and since $|w|$ is bounded by $g(n)$ the time to determine whether $w \in A_x$ is also constant. This establishes (1).

Next, suppose V_{xy} is infinite. We will show that the function computed by M_e is not a reduction from A_x to A_y. As e is arbitrary, this will show that no machine is a reduction, and hence a reduction cannot exist. We will do this via the requirements P_{xye}: first by showing that once a requirement is satisfied, it is never again violated and second by showing that with V_{xy} infinite P_{xye} is eventually satisfied for every e.

Suppose that P_{xye} is declared satisfied at stage n. Observe that P_{xye} is not injured at stage n: we only effect the suffix table codings if they do not place any string shorter than $p_e(|w|)$ into A_y, and as p_e is the clock of M_e, the computation on $M_e(w)$ cannot possibly query the oracle for any strings longer than than $p_e(|w|)$. As such whatever strings may have been added to A_y through the codings do not affect the computation of M_e. So if $A_x(w) = 1 - M_e(A_y; w)$ held when P_{xye} was declared satisfied, it still holds after the codings have been effected. Neither can P_{xye} be injured at any later stage: we have verified that $M_e(A_y; w)$ can be simulated in $2^{|w|} = g(n+1)$ time, so at the next stage of the construction whatever strings may be added to A_y will be too large to affect the computation of $M_e(A_y; w)$.

Next, suppose that P_{xye} has never been declared satisfied. We will show that there exists an $e' > e$ with $M_{e'}(w') = M_e(w')$ for all w', such that $P_{xye'}$ has been declared

satisfied. This will ensure that P_{xye} is, in fact, eventually satisfied, even though it is never declared to be so.

We wish to pad M_e into a large enough $M_{e'}$ so that at some stage $\langle x, y, r, e' \rangle$ satisfies all the verifications of the construction and therefore $A_x(w) \neq M_{e'}(A_y; w)$ and $P_{xye'}$ is satisfied. By padding M_e we mean we add inaccessible states so that the machine index changes but the machine behaviour remains the same on any string.

This means we need to show that for a large enough e', at stage $m = \langle x, y, r, e' \rangle$, the following hold:

- We can simulate $M_{e'}(Ay; w)$ in 2^m time.

- Placing w into A_x and effecting all the codings required by the suffix table will not place any string shorter than $p_{e'}(m) = p_e(m)$ into A_y.

The first will hold eventually as $p_{e'}$ grows polynomially with m while 2^m grows exponentially. The second is violated only if there is a sequence of entries in the suffix table $(z_0, z_1), (z_1, z_2), \ldots, (z_k, z_{k+1})$, $x = z_0$ and $y = z_{k+1}$, with corresponding polynomials $q_0, \ldots, q_k$ satisfying $q_0(m) + q_1(m_1) \cdots + q_k(m_k) \leq p_{e'}(m)$ where $m < \cdots < m_k$. That is, placing 0^m in $A_x = A_{z_0}$ will place $0^m 1^{\langle z_0, z_1, s, i \rangle} 0^{q_0(m)}$ in A_{z_1}, which in turn will add another suffix and put the resulting string in A_{z_2}, and so on through the entire chain, with the final string being shorter than $p_e(m)$. Clearly a necessary (but not sufficient) condition for this is that all the polynomials q_i be smaller than p_e. We will show that this will eventually fail to hold. Note that there is no point in considering the codings effected at later stages in the construction, as the strings considered there will be much too large.

Observe that by the transitivity of $\preceq$, $V_{z_i, z_{i+1}}$ must be infinite for some i. Suppose $d > e$ and the dth element enters $V_{z_i, z_{i+1}}$ at step s. At the stage of the construction corresponding to $\langle z_i, z_{i+1}, s, d \rangle$ we will update the (z_i, z_{i+1}) entry of the suffix table with $q_i > p_e$. As $p_e = p_{e'}$, $q > p_{e'}$. This will ensure that $q_0(m) + q_1(m_1) \cdots + q_k(m_k) > p_{e'}(m)$. As such if $e' > d > e$, it will be large enough to satisfy $P_{xye'}$, and therefore P_{xye}. This establishes (2).

Finally, we must show that A_x is EXPTIME. Observe that any string in A_x begins with $0^{g(n)}$ and is followed by zero or more suffixes of the form $1^i 0^j$. If a string is of the wrong form, we can reject it immediately.

First, perform the first n steps of the construction to determine whether $0^{g(n)} \in A_z$, and to construct the suffix table at that stage. We will demonstrate that the nth step can be performed in exponential time, and accordingly so can the first n. With $g(n) = \langle z, y, r, e \rangle$ we begin by constructing $V_{xy,r}$. As $r \leq g(n)$, this takes at most polynomial time. Next we check whether $M_e(A_y; 0^{g(n)})$ can be run in 2^m steps: this can be done simply by calculating $p_e(g(n))$. After that we verify whether placing $0^{g(n)}$ in A_z will place any strings shorter than $p_e(g(n))$ in A_y. Observe that if we treat the suffix table as an adjacency table for a graph, this corresponds to finding a path of the lowest weight. As the graph has at most two nodes for every $g(k)$, $k < m$, it has very much less nodes than $g(n)$ and therefore than the length of the input string. As such finding such a path can be done in well under exponential time. Finally we need to simulate

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$M_e(A_y; 0^{g(n)})$. Note that as we have already performed all the steps prior to n, we know which strings A_y contains so the simulation only takes polynomial time.

At this stage if we have determined that $0^{g(n)} \notin A_z$, we can safely conclude that the queried string is not in A_x. Otherwise we simply need to verify whether every suffix of the string corresponds to a valid entry in the suffix table. Specifically, if the suffix is of the form $1^{\langle x', y', r', e' \rangle} 0^q$ we perform the first n', $g(n') = \langle x', y', r', e' \rangle$, steps of the construction and verify whether the suffix table entry for (x', y') does, in fact, associate (x', y') with (x', y', r', e') and q. By using the same argument as above we can, mutatis mutandis, verify that every such verification takes no more than exponential time. As there are at most linearly many verifications to make, the entire process is within EXPTIME. $\square$

Corollary 4.5. *Polynomial time equivalence of EXPTIME sets is complete for Σ_2^0 equivalence relations.*

Proof. Follows from Proposition 2.4, as polynomial time equivalence is the symmetric fragment of polynomial time reducibility. $\square$

4.3 Σ_3^0 and Isomorphism of Groups

There are two parts to this section. In 4.3.1 we show that almost inclusion (equality) of r.e. sets is complete for Σ_3^0 preorders (equivalence relations). In 4.3.2 we show that this result is in fact equivalent to embeddability (isomorphism) of computable subgroups of $(\mathbb{Q}, +)$, as a consequence of Baer's ([Bae37]) characterisation of the subgroup structure of $(\mathbb{Q}, +)$.

4.3.1 Almost equality of r.e. sets

We will make use of a number of results from the literature of recursion theory. To facilitate that end we will first introduce some notational conventions for this section.

We will use $A \subset_m B$ and $A \subset_{sm} B$ to denote that A is a major and small-major subset of B respectively. See [Soa87] chapter X:4 for these notions. Maass and Stob ([MS83]) have shown that given any $A \subset_m B$ the lattice generated by almost inclusion on the r.e. sets between them is unique up to isomorphism. We will use $[A, B]$ to denote this lattice, and C^* to denote the equivalence class of C under $=^*$. If A is non-recursive, and $A \setminus B$ is r.e., then B is said to be a split of A, which we will denote with $B \sqsubseteq A$.

We are now ready to prove the theorem.

Theorem 4.6. *Almost inclusion of r.e. sets is complete for Σ_3^0 preorders. That is, for any Σ_3^0 preorder P there exists a computable f such that:*

$$Pxy \iff W_{f(x)} \subseteq^* W_{f(y)}.$$

Proof. Fix a non-recursive A. As A is non-recursive there exists a $D \subset_{sm} A$ (see [Soa87], page 194). Then for every $X \sqsubseteq A$ we have the following ([Nie98], Lemma 4.1.2):

$$X \subseteq^* D \iff X \text{ is recursive.}$$

We can then define the Boolean algebra:

$$\mathcal{B}_D(A) = \{(X \cup D)^* \mid X \sqsubseteq A\},$$

with D^* and A^* as the minimum and maximum elements respectively, the complement of $(X \cup D)^*$ being $((A \setminus X) \cup D)^*$, and $(X \cup D)^* \wedge (Y \cup D)^* = ((X \cap Y) \cup D)^*$. Note that we are guaranteed the existence of complements because X ranges over splits.

The Friedberg splitting theorem ([Fri58]) implies that any non-recursive X can be split into non-recursive X_1, X_2 obtained uniformly from an r.e. index for X. By iterating this process we can obtain a uniform sequence of splits $X_n \sqsubseteq A$. If we use p_n to denote $(X_n \cup D)^*$, $\mathcal{F}$ will denote the subalgebra generated by the sequence $\{p_n \mid n \in \mathbb{N}\}$.

Let P be an arbitrary Σ_3^0 preorder, and $\mathcal{I}_0$ the ideal generated by $\{p_n \wedge \neg p_k \mid Pnk\}$. We claim that:

$$Pnk \iff p_n \wedge \neg p_k \in \mathcal{I}_0.$$

Left to right is clear from definition. For the other direction let $\mathcal{B}_P$ be the boolean algebra generated by subsets of $\mathbb{N}$ of the form $\hat{i} = \{r \mid Pri\}$. Consider the Boolean algebra homomorphism $g : \mathcal{F} \to \mathcal{B}_P$ induced by the map $p_i \mapsto \hat{i}$. As $g(p_n) \subseteq g(p_k)$ whenever $p_n \wedge \neg p_k \in \mathcal{I}_0$, it follows that g maps $\mathcal{I}_0$ to $\emptyset$. If $\neg Pnk$ then $\hat{n} \nsubseteq \hat{k}$, $g(p_n \wedge \neg p_k)$ is non-empty and hence $p_n \wedge \neg p_k \notin \mathcal{I}_0$.

Now let $\mathcal{I}$ be the ideal of $\mathcal{B}_D$ generated by $\mathcal{I}_0$. The above argument then implies:

$$Pnk \iff ((X_n \setminus X_k) \cup D)^* \in \mathcal{I}.$$

Note that $\mathcal{I}$ is a Σ_3^0 ideal. By the base case of the ideal definability lemma ([HN98]) there exists a $B \in [D, A]$ satisfying:

$$(X \cup D)^* \in \mathcal{I} \iff X \subseteq^* B,$$

which of course means:

$$Pnk \iff ((X_n \setminus X_k) \cup D)^* \in \mathcal{I} \iff X_n \subseteq^* X_k \cup B.$$

This gives us the desired reduction. $\qquad\square$

4.3.2 Subgroups of $(\mathbb{Q}, +)$

In this section our goal is to use the result above to show that the isomorphism of computable subgroups of $(\mathbb{Q}, +)$ is complete for Σ_3^0 equivalence relations. Before we proceed, we first need to establish some facts about subgroups of $(\mathbb{Q}, +)$. Namely, such subgroups are isomorphic if and only if they agree on all but a finite number of prime powers. Likewise, there exists an embedding from A to B if and only if the set of prime powers in A is almost included in the set of prime powers in B. These results are based on [Bae37]. We give simplified proofs to better tailor them to the computational setting. For notational convenience, we treat 1 as the 0th prime.

Lemma 4.7. *Every subgroup of $(\mathbb{Q}, +)$ is isomorphic to one with a generating set composed of prime powers; specifically, the negative powers. That is, every subgroup is isomorphic to one generated by elements of the form $1/p_i^k$ where p_i is the ith prime and k is some integer.*

Proof. First observe that every subgroup of $(\mathbb{Q}, +)$ is isomorphic to a subgroup that contains 1: if A be a subgroup and $s/t \in A$, then it is easy to verify that $f(x) = t/s \cdot x$ is an isomorphism of subgroups, and as $f(s/t) = 1$ we have the desired subgroup in $f(A)$. A subgroup of $(\mathbb{Q}, +)$ is then either isomorphic to $(\mathbb{Z}, +)$, in which case it is clearly generated by $\{1/p_0\}$, or it contains elements strictly between 0 and 1.

Such a group's generating set, without loss of generality, consists of g for $0 < g \leq 1$: any element $x > 1$ can be obtained from $x - n$ by adding 1 n times, and for the right choice of n, $x - n$ will lie between 0 and 1.

Next we claim that for coprime a and b, $a/b \in A$ if and only if $1/b \in A$, so in fact we can assume the generating set consists of elements with 1 in the numerator. This is because if a and b are coprime then there exist integers x, y such that $ax + by = 1$. As $a/b \in A$, so is xa/b and as $1 \in A$ so is y. We can then obtain $1/b$ as follows:

$$\frac{xa}{b} + y = \frac{ax + by}{b} = \frac{1}{b}.$$

For the other direction, $1/b \in A$ immediately implies $a \cdot 1/b = a/b \in A$.

We now know that any subgroup is isomorphic to one generated by 1 along with fractions of the form $1/q$. The last step is to show that such a subgroup is also generated by fractions of the form $1/p^k$, where p^k appears in the prime factorisation of some such q.

Let $1/q$ be an arbitrary element of the group, and let $q = p^k r$ for some prime p, where r and p are coprime. It immediately follows that $1/p^k$ is in the group, as $1/p^k = r/q$. For the other direction, suppose we have a group containing $1/p^k$ and $1/r$. As p^k and r are coprime, there exist integers x, y such that $xp^k + yr = 1$. Observe that:

$$\frac{x}{r} + \frac{y}{p^k} = \frac{xp^k + yr}{p^k r} = \frac{1}{q}.$$

By induction, it follows that the only elements necessary in the generating set are the reciprocals of the prime powers appearing in the decomposition of q. $\square$

In light of this result, from now on when we speak of "subgroups of $(\mathbb{Q}, +)$" we will mean those subgroups generated by a what we will term a *standard* generating set. Denoted by $S(A)$, the standard generating set of A contains $1/p^k$ for all primes p and integers k such that $1/p^k \in A$. If A and B are two subgroups, we use $S(A) - S(B)$ to denote the set consisting of $1/p^{k-j}$ where k, j are the largest integers, if such exist, such that $1/p^k$ and $1/p^j$ are in $S(A)$ and $S(B)$ respectively. Note that $S(A) - S(B)$ is finite if $S(A) \subset^* S(B)$.

Proposition 4.8. *For subgroups A, B of $(\mathbb{Q}, +)$, $A \cong B$ if and only if $S(A) =^* S(B)$.*

Proof. First, suppose $S(A) =^* S(B)$, let $S(A) - S(B) = \{1/p_1^{k_1}, \ldots, 1/p_n^{k_n}\}$ and $S(B) - S(A) = \{1/q_1^{j_1}, \ldots, 1/q_n^{j_n}\}$. Note that there is no generality lost in assuming both sets are of cardinality n as we can always set the appropriate exponents to 0. We claim that the desired isomorphism is $f : A \to B$ given by:

$$f(x) = \frac{p_1^{k_1} \cdots p_n^{k_n}}{q_1^{j_1} \cdots q_n^{j_n}} x.$$

It is easy to see that f preserves identity and the group operation simply as a consequence of arithmetic on the rational numbers:

$$f(0) = \frac{p_1^{k_1} \cdots p_n^{k_n}}{q_1^{j_1} \cdots q_n^{j_n}} 0 = 0,$$

$$f(x + y) = \frac{p_1^{k_1} \cdots p_n^{k_n}}{q_1^{j_1} \cdots q_n^{j_n}} (x + y) = \frac{p_1^{k_1} \cdots p_n^{k_n}}{q_1^{j_1} \cdots q_n^{j_n}} x + \frac{p_1^{k_1} \cdots p_n^{k_n}}{q_1^{j_1} \cdots q_n^{j_n}} y = f(x) + f(y).$$

It remains to show that f is bijective, and that the image of f is indeed contained in B.

To see that $f(x) \in B$ for all x, observe that $s/t \in B$ if and only if all prime powers q^m that divide t, $1/q^m \in S(B)$. It follows that for any $u/v \in A \setminus B$, there must exist a $p^i \in S(A) \setminus S(B)$ v such that $p^i \mid v$. Without loss of generality, let this be the largest i with this property. Since $S(A) =^* S(B)$, there must exist a largest k such that $1/p^k \in S(A)$ and a largest j such that $1/p^j \in S(B)$. It follows from our definition of f that f multiplies u/v by p^{k-j}, and as such the exponent of p in the denominator of $f(u/v)$ will be p^{i-k+j}, and $1/p^{i-k+j} \in S(B)$. This allows us to conclude that $f(x) \in B$.

To see that f is onto, let s/t be an arbitrary element of B. We need to show that there exists a $u/v \in A$ satisfying:

$$\frac{p_1^{k_1} \cdots p_n^{k_n} u}{q_1^{j_1} \cdots q_n^{j_n} v} = \frac{s}{t}.$$

Let $t = q_1^{z_1} \ldots q_n^{z_n} P$ where $z_i \leq j_i$ and P is such that $1/P \in A$. Observe that:

$$\frac{p_1^{k_1} \cdots p_n^{k_n}}{q_1^{j_1} \cdots q_n^{j_n}} \cdot \frac{q_1^{j_1-z_1} \cdots q_n^{j_n-z_n} s}{p_1^{k_1} \cdots p_n^{k_n} P} = \frac{s}{t}.$$

The right multiplicand is clearly in A as the denominator only contains primes from $S(A)$.

Finally, the fact that f is one to one follows immediately from rational arithmetic.

For the second direction suppose that $S(A) \setminus S(B)$ is infinite. To proceed, we observe that we can define a notion of divisibility in a group in the standard fashion:

$$x \mid y \iff zx = y \text{ for some } z$$

Clearly, any isomorphism from A to B must preserve divisibility. I.e., $x \mid y$ if and only if $f(x) \mid f(y)$, and moreover if $zx = y$ then $zf(x) = f(y)$.

Let f then be some function from A to B. If f is to be an isomorphism, for every $1/p^k \in S(A) \setminus S(B)$ we must have $f(1) = p^k f(1/p^k)$. That is, if $f(1) = s/t$ and $f(1/p^k) = u/v$ we have:
$$\frac{s}{t} = \frac{p^k u}{v}.$$
Now, observe that $u/v \in B$ and $1/p^k \in S(A) \setminus S(B)$ implies that if p^j appears in the prime decomposition of v, then $j < k$ and hence p^{k-j} must appear in the decomposition of s. However, as there are infinitely many elements in $S(A) \setminus S(B)$ this would mean that s would need to be infinitely large, which is impossible. $\square$

A simpler version of the same reasoning can be adapted to show that subgroup embeddability is almost inclusion of standard generating sets.

Proposition 4.9. *For subgroups A, B of $(\mathbb{Q}, +)$, A is embeddable in B if and only if $S(A) \subseteq^* S(B)$.*

Proof. Suppose $S(A) \subseteq^* S(B)$ and let $S(A) - S(B) = \{1/p_1^{k_1}, \ldots, 1/p_n^{k_n}\}$. We claim that $f(x) = p_1^{k_1} \ldots p_n^{k_n} x$ is an embedding of A into B. This is clearly one to one and preserves both addition and the group operation. The image of f lies within B as multiplying by the primes in $S(A) - S(B)$ will rid the denominator of any primes inadmissible in B.

For the other direction suppose $S(A) \setminus S(B)$ is infinite. Again the preservation of divisibility induces that if $zx = y$ then $zf(x) = f(y)$. Since there are infinitely many members of $S(A) \setminus S(B)$, we cannot obtain this. $\square$

Corollary 4.10. *Subgroups of $(\mathbb{Q}, +)$ are isomorphic if and only if they are bi-embeddable.*

Proof. One direction is clear: an isomorphism is an embedding both ways, so if f is an isomorphism between A and B, f and f^{-1} are the required embeddings.

Next, suppose $A, B \subseteq (\mathbb{Q}, +)$ are bi-embeddable via $f : A \to B$ and $g : B \to A$. We claim that this implies that $S(A) \subseteq^* S(B)$ and $S(B) \subseteq^* S(A)$, hence $S(A) =^* S(B)$ which, by Proposition 4.8, implies an isomorphism. $\square$

We now have everything we need to speak about computable groups.

Definition 4.11. A *computable subgroup of* $(\mathbb{Q}, +)$ is a 4-tuple of computable functions, $(e, \oplus, \ominus, I)$. The natural numbers are taken to encode group elements, not necessarily uniquely. The function e selects the identity element(s), meanings $e(x) = 1$ for identity and 0 otherwise. The group operation is encoded by $\oplus : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $\ominus : \mathbb{N} \to \mathbb{N}$ takes an element to its inverse. I is the interpretation function, which maps group elements to (a fixed encoding of) the rationals. We require that the group axioms be respected through I, namely:

- $I(\oplus(\oplus(a, b), c)) = I(\oplus(a, \oplus(b, c)))$.

- $(e(x) = 1) \to (I(\oplus(a, x)) = I(\oplus(x, a)) = I(a))$.

- $(\oplus(a, \ominus(a)) = x) \to (e(x) = 1)$.

As we are dealing with additive subgroups, we also require that I respects addition on the rational numbers:

- $I(\oplus(a, b)) = I(a) + I(b)$.

Of course, there is no way to enumerate arbitrary 4-tuples of computable functions. However we will now show that we can identify with every r.e. set a computable group, and all computable subgroups of $(\mathbb{Q}, +)$ can be obtained in this fashion. This allows us to speak of the ith computable subgroup, denoted G_i, where $S(G_i)$ is encoded by W_i. As such if $(e, \oplus, \ominus, I)$ is G_i, we will let $\langle e, \oplus, \ominus, I \rangle = i$.

Lemma 4.12. *A subgroup of $(\mathbb{Q}, +)$ is computable if and only if we can uniformly effectively obtain its standard generating set from some W_i.*

Proof. We will treat W_i as consisting of pairs (x, y), closed downward in the second component. We can then define G_i to be the group where $S(G_i)$ is the set containing $1/p_x^y$ for all $(x, y) \in W_i$. We can now show that G_i is computable.

Fix some encoding of finite sequences of integers. Let w_s be the element that enters W_i at stage s if such an element exists, or 0 otherwise. If x is the encoding of $z_0 z_1 \ldots z_n$, let $I(x) = z_0 w_0 + z_1 w_1 + \cdots + z_n w_n$. I is clearly computable, so there exists a partial recursive I^{-1} where $I^{-1}(y)$ is the least x such that $I(x) = y$. This allows us to define $\oplus(x, y)$ as $I^{-1}(I(x) + I(y))$, $\ominus(x)$ as $I^{-1}(-I(x))$ and let $e(x) = 1$ if and only if $I(x) = 0$, and 0 otherwise. All of these are clearly computable and satisfy the axioms of Definition 4.11.

For the other direction, suppose we are given $(e, \oplus, \ominus, I)$. We can enumerate W_i by running I on every number and listing (x, y) each time we come across an element of the form $1/p_x^y$. $\square$

This gives us all we need to construct a Σ_3^0-complete preorder.

Theorem 4.13. *Computable embeddability of computable subgroups of $(\mathbb{Q}, +)$ is complete for Σ_3^0 preorders.*

Proof. First, let us verify that the relation is indeed Σ_3^0. We can represent it as:

$$\{((\langle e_1, \oplus_1, \ominus_1, I_1 \rangle, \langle e_2, \oplus_2, \ominus_2, I_2 \rangle)) \mid \exists i \ \varphi_i \text{ is total } \wedge$$

$$\forall xy \ \varphi_i(x) = \varphi_i(y) \to x = y \ \wedge$$

$$e_1(x) = e_2(\varphi_i(x)) \ \wedge$$

$$\oplus_1(x, y) = \oplus_2(\varphi_i(x), \varphi_i(y))\}.$$

Totality of p.r. functions is a Π_2^0 property, so placing an existential quantifier over it is Σ_3^0. By Lemma 4.12, we have an effective encoding of groups so we can obtain the code for $\oplus$, $\ominus$ and e effectively. As φ_i is computable, the rest of the formula is Π_1^0 so with an existential quantifier over it is Σ_2^0, and thus contained within Σ_3^0.

To show embeddability is complete we invoke Theorem 4.6. Almost inclusion is complete for Σ_3^0 preorders, so Proposition 4.9 and Lemma 4.12 already did all the work for us, and we obtain:

$$W_x \subseteq^* W_y \iff G_x \text{ is embeddable in } G_y.$$

$\square$

It immediately follows that computable bi-embeddability of subgroups of $(\mathbb{Q}, +)$ is complete for Σ_3^0 equivalence relations. However, we can strengthen this result somewhat. By Corollary 4.10 bi-embeddability implies isomorphism, and there is no need to speak of a computable isomorphism because that follows immediately.

Proposition 4.14. *Computable subgroups of $(\mathbb{Q}, +)$ are isomorphic if and only if they are computably isomorphic.*

Proof. One direction is clear. For the other direction observe that the isomorphism constructed in Proposition 4.8 involves multiplying by a rational number. This is clearly computable in our framework: if $(e_1, \oplus_1, \ominus_1, I_1)$ and $(e_2, \oplus_2, \ominus_2, I_2)$ are isomorphic via $f(x) = qx$, we can compute this by $I_2^{-1}(qI_1(x))$. $\square$

Corollary 4.15. *Isomorphism of computable subgroups of $(\mathbb{Q}, +)$ is complete for Σ_3^0 equivalence relations.*

Proof. Theorem 4.13 establishes that computable bi-embeddability of computable subgroups is complete for Σ_3^0 equivalence relations. Propositions 4.10 and 4.14 allow us to lift this result to isomorphism of computable subgroups. $\square$

On the other hand, the restriction to computable subgroups is indeed necessary: there are more isotypes of subgroups of $(\mathbb{Q}, +)$ than there are natural numbers, so some subgroups are inherently uncomputable.

Proposition 4.16. *There are uncountably many isotypes of subgroups of $(\mathbb{Q}, +)$. As such, there exist uncomputable subgroups of $(\mathbb{Q}, +)$, even modulo isomorphism.*

Proof. We will prove the proposition in two steps. First, we will show that there exists an injection from $2^{\mathbb{N}}/ =^*$ to $Sub(\mathbb{Q}, +)/ \cong$, where $Sub(\mathbb{Q}, +)$ is the set of subgroups of $(\mathbb{Q}, +)$. That is, there are at least as many isotypes of subgroups of $(\mathbb{Q}, +)$ as there are equivalence classes of $=^*$ on the natural numbers. Next, we will show that there is an injection from $2^{\mathbb{N}}$ into $2^{\mathbb{N}}/ =^*$. I.e., the cardinality of the powerset of the naturals is no larger than the cardinality of the powerset of the naturals modulo almost equality. This will establish that there are uncountably many isotypes of subgroups of $(\mathbb{Q}, +)$, and therefore they cannot all be computable.

First, let $f : 2^{\mathbb{N}} \to Sub(\mathbb{Q}, +)$ be the function which maps the set A to the subgroup generated by $\{1/p_i \mid i \in A\}$. It is easy to see that f is one to one: suppose $n \in A$, $n \notin B$. We claim that $1/p_n \in f(A)$, $1/p_n \notin f(B)$ and hence $A \neq B$ implies $f(A) \neq f(B)$.

Suppose to the contrary, there are integers $z_0, \ldots, z_k$ and indices $i_0, \ldots, i_k$ with $i_j \neq n$ for all j such that:

$$\sum_{j=0}^{k} \frac{z_j}{p_{i_j}} = \frac{1}{p_n},$$

$$\frac{z_0(p_{i_1} \ldots p_{i_k}) + \cdots + z_k(p_{i_0} \ldots p_{i_{k-1}})}{p_{i_0} \ldots p_{i_k}} = \frac{1}{p_n},$$

$$p_n(z_0(p_{i_1} \ldots p_{i_k}) + \cdots + z_k(p_{i_0} \ldots p_{i_{k-1}})) = p_{i_0} \ldots p_{i_k},$$

but that is clearly impossible.

Next, let $g : 2^{\mathbb{N}}/ =^* \to Sub(\mathbb{Q}, +)/ \cong$ be the function which maps $[A]_{=^*}$ to $[f(A)]_{\cong}$. The fact that this is an injection follows immediately from the observation above and Proposition 4.8.

Finally, let $h : 2^{\mathbb{N}} \to 2^{\mathbb{N}}/ =^*$ be the function which maps A to $\{x \mid x = e_n$ for some $n \in A$, or $x = o_{p_n^k}$ for some $k \in \mathbb{N}, n \in A\}$, where e_n is the nth even number (i.e. $2n$) and o_n is the nth odd number $(2n+1)$. For intuition, this could be thought of as $h(A) = \langle A, A' \rangle$ where A' is the set that contains all powers of the nth prime for every n in A, with the first component of $h(A)$ encoded by the even numbers and the second by the odd.

To see that h is one to one suppose $n \in A$, $n \notin B$. We want to show that $h(A) \triangle h(B)$ is infinite, but this is immediate as $o_{p_n^k} \in h(A)$ for all k, while $o_{p_n^k} \notin h(B)$. $\qquad \square$

5 The case of Π_n^0

There are two results in this section. In 5.1 below we demonstrate the existence of a Π_1^0-complete equivalence relation and preorder, and show how a natural example can be obtained from polynomial time trees. The existence of a Π_1^0-complete equivalence relation is based on a characterisation of every Π_1^0 equivalence relations as the kernel of a computable function. This result can actually be derived from a recent work of Cholak, Dzhafarov, Schweber and Shore on partial orders ([CDSS11]). We present both results below as the works are independent and based on different arguments.

In 5.2 we present a quite different, and striking result: the non-existence of equivalence relations complete for Π_n^0, where $n \geq 2$. As it happens, the same construction will also suffice to establish the non-existence of Δ_n^0-complete equivalence relations as well.

5.1 Π_1^0-complete equivalence relations and preorders

While the existence of Σ_n^0-complete equivalence relations was never in question, with Π_1^0 the case is different. In the absence of an effective means of enumerating the class the technique used to obtain a Σ_n^0-complete equivalence relation did not work, and the question of the existence of any complete equivalence relation, no matter how artificial, was an open and interesting one. In light of this fact we will present the results in a different order to that in Section 4. Rather than letting the existence of a complete equivalence relation follow from the existence of a complete preorder, in 5.1.1 below we will construct a complete equivalence relation directly, demonstrating how a Π_1^0 equivalence relation can be seen as a limit of a family of computable approximations. Then in 5.1.2 we will show how the characterisation of Π_1^0 preorders of [CDSS11] can be used to construct a Π_1^0-complete preorder by the same principles.

5.1.1 A complete equivalence relation

We begin with the observation that if we fix a computable function f, a natural example of a Π_1^0 equivalence relation could look something like this:

$$\{(x,y) \mid \forall n \ f(x,n) = f(y,n)\}. \tag{5.1}$$

In fact, the reader may find it difficult to think of a Π_1^0 equivalence relation that cannot be interpreted in this way. As it turns out, that is because no such equivalence relations

exist: every Π_1^0 equivalence relations arises from the column equality of some computable function. This result, which we will shortly present as Theorem 5.1, proves to be the key step in constructing a Π_1^0-complete equivalence relation.

We first introduce a convenient way of thinking about a Π_1^0 equivalence relation. As every such relation is the complement of an r.e. set, for any $E \in \Pi_1^0$ there exists some W_e such that $E = \mathbb{N}^2 \setminus W_e$. This allows us to construct a computable sequence of approximations to E, $\{E_t \mid t \in \mathbb{N}\}$, satisfying:

$$E_{t+1} \subseteq E_t. \tag{5.2}$$

$$E_t \cap [0, t]^2 \text{ is an equivalence relation over } [0, t]^2. \tag{5.3}$$

$$E = \bigcap_{t \in \mathbb{N}} E_t. \tag{5.4}$$

This is done simply by enumerating W_e until $(\mathbb{N}^2 \setminus W_e) \cap [0, t]^2$ is an equivalence relation over $[0, t]^2$, which it must do eventually. We will use E_t' to denote $E_t \cap [0, t]^2$.

With this in mind, we can show that every Π_1^0 equivalence relation is of the form in (5.1).

Theorem 5.1. *Given a Π_1^0 equivalence relation E, we can effectively obtain a computable f such that for all x, y:*

$$Exy \iff \forall n \ f(x, n) = f(y, n). \tag{5.5}$$

Proof. Intuitively, $f(x, n)$ is an approximation of $\min[x]_E$. It will recurse on $f(z, n)$ where z is the smallest element satisfying $E_{\max(x,n)}'xz$. If the smallest such z is x itself, then $f(x, n) = x$. Observe that if $x < n$ then $f(x, n) = \min[x]_{E_n'}$. The idea is that for a large enough n, $f(x, n)$ will in fact be $\min[x]_E$, so Exy thus implies $f(x, n) = f(y, n)$. If n is not large enough, we will show that Exy nevertheless implies $f(x, n) = f(y, n)$, namely $\min[z]_{E_n'}$ for some z.

Explicitly f is:

$$f(x, n) = \begin{cases} f(z, n) & \text{for } z = \min[x]_{E_{\max(x,n)}'}, \text{ if } z < x \\ x & \text{otherwise.} \end{cases}$$

First let us verify that $\neg Exy$ implies that for some n, $f(x, n) \neq f(y, n)$. Observe that if $(x, y) \notin E$, then for a large enough n, $(x, y) \notin E_n'$. We can without loss of generality assume that this n is larger than x or y, so since $f(x, n) = \min[x]_{E_n'}$ and $f(y, n) = \min[y]_{E_n'}$, we have $f(x, n) \neq f(y, n)$ as required.

It remains to consider the case where $(x, y) \in E$. In this case $(x, y) \in E_n$ for all n by (5.4).

We proceed by double induction on the recursive stack depth of $f(x, n)$ and $f(y, n)$, i and k, with the inductive hypothesis that for $x \leq y$, $(x, y) \in E_{\max(y,n)}$ implies $f(x, n) = f(y, n)$ for functions of depth at most i and k respectively.

Base case: $i = k = 0$. In this case $x = \min[x]_{E'_{\max(x,n)}}$ and $y = \min[y]_{E'_{\max(y,n)}}$. Since $(x, y) \in E$, $(x, y) \in E'_{\max(y,n)}$. It must be the case that $x = y$, so clearly $f(x, n) = f(y, n)$.

Inductive case for i: suppose the inductive hypothesis holds for functions of depth i and k. Let $f(x, n)$ be of depth $i+1$, $f(y, n)$ of at most k. Let $f(x, n) = f(z, n)$. Observe that $f(y, n) = f(w, n)$, $z \leq w \leq x$: the upper bound we get from $(x, y) \in E'_{\max(y,n)}$, the lower from (5.2). We then have $(z, w) \in E'_{\max(w,n)}$ as $E_{\max(x,n)} \subseteq E_{\max(w,n)}$, so we can apply the inductive hypothesis on $f(z, n)$ and $f(w, n)$.

Inductive case for k: suppose the inductive hypothesis holds for functions of depth i and k. Let $f(y, n)$ be of depth $k+1$, $f(x, n)$ of at most i. Note that $f(x, n) = f(z, n)$ and $f(y, n) = f(w, n)$ with $z \leq w$. It remains to show that $(z, w) \in E_{\max(w,n)}$ before we can apply the inductive hypothesis. This is clearly the case as $(y, w) \in E_{\max(y,n)} \subseteq E_{\max(x,n)}$. As $E'_{\max(x,n)}$ is an equivalence relation and contains $(x, y), (y, w)$ and (x, z) it must also contain (z, w). As $E_{\max(x,n)} \subseteq E_{\max(w,n)}$, we obtain what is needed. $\qquad \square$

It should be noted that not only do we obtain $f(x, n) = \min[x]_E$ for a large enough n, but we also know that n is large enough if $f(x, n) = x$. Recall that in Proposition 2.5 we have established that the set of least elements of a Π_1^0 equivalence relation is Σ_1^0 and hence r.e.: f gives us the function which enumerates them. Simply apply f to all values of x and n, and print x whenever $f(x, n) = x$.

We can strengthen this result to hold for polynomial time functions, thereby obtaining a Π_1^0 complete equivalence relation as polynomial time functions have an effective enumeration. We thank Moritz Müller at KGRC Vienna for suggesting the simplified proof.

Theorem 5.2. *The equality of polynomial (in fact, quadratic) functions is complete for Π_1^0 equivalence relations. That is, for every Π_1^0 equivalence relation E and for every x, y, we can effectively obtain quadratic time g_x, g_y such that:*

$$Exy \iff g_x = g_y. \tag{5.6}$$

Proof. The idea behind the proof is to pad the input until it is large enough to allow us to put a bound on the running time of the f we constructed in Theorem 5.1. We will thus construct a quadratic time g mapping from strings to strings such that for a computable p mapping integers to strings,

$$g(p(x), p(n)) = f(x, n). \tag{5.7}$$

This of course gives us:

$$Exy \iff \forall n \ g(p(x), p(n)) = g(p(y), p(n)). \tag{5.8}$$

We will then let $g_x(p(n)) = g(p(x), p(n))$, and 0 if the input is not of the form $p(n)$. Provided we can effectively verify if a string is of the form $p(n)$ for some n, this gives us the theorem.

Let E be a Π_1^0 equivalence relation and f be a computable function mapping strings to strings over a binary alphabet, on input $(1^x, 1^n)$ behaving as on (x, n) in 5.1, and on any other input outputting 0. That is:

$$f(1^x, 1^n) = \begin{cases} f(1^z, 1^n) & \text{for } z = \min[x]_{E'_{\max(x,n)}}, \text{ if } z < x \\ 1^x & \text{otherwise.} \end{cases}$$

On any pair of strings not consisting entirely of ones f outputs 0.

Without loss of generality, let $f(1^x, 1^y)$ be computable in time $t(x, y)$ where t be an increasing function in x, y. This can be achieved, for instance, by having $f(1^x, 1^y)$ first compute the function value for all smaller values of x, y before beginning on $1^x, 1^y$.

We want a time constructible h satisfying $t(x, n) \le h(x) + h(n)$. This can be achieved by having $h(k)$ run $f(1^k, 1^k)$ and counting the number of steps. This satisfies $t(x, n) \le h(x) + h(n)$ as for $x \le n$, $t(x, n) \le t(n, n) = h(n)$, and $h(k) = m$ can be calculated in time quadratic in m: place a binary counter at the start of the tape and simulate f to the right of it. In the worst case scenario for every step of f the machine would iterate over $m + \log m$ cells to update the counter, and whenever the counter gains another bit the entire tape would need to be shifted right. This will happen $\log m$ times, and at most m cells would need to be shifted each time. This would take at most $m \log m$, coming to $m \log m + m(m + \log m)$ or $O(m^2)$.

To compute $g(a, b)$ verify whether $a = 1^x 01^{h(x)}$ and $b = 1^n 01^{h(n)}$. If so output $f(1^x, 1^n)$, else output 0. Observe that g is quadratic time: we can verify whether $a = 1^x 01^z$ is of the form $1^x 01^{h(x)}$ by beginning to compute $f(1^x, 1^x)$, stopping whenever the number of steps exceeds z. If the input is of the right form we can compute $f(1^x, 1^n)$ in time $t(x, n) \le h(x) + h(n)$, which is of the same order as $|a|, |b|$ respectively.

Recall that we have defined $g_x(p(n))$ to be $g(p(x), p(n))$ for some computable p. Let $p(x) = 1^x 01^{h(x)}$. We have already seen that we can verify if a string is of this form in time quadratic in the length of the string. This ensures that for every x, n:

$$g(1^x 01^{h(x)}, 1^n 01^{h(n)}) = f(1^x, 1^n). \tag{5.9}$$

As for all other values the function is 0, $g_x = g_y$ iff $\forall n\; f(1^x, 1^n) = f(1^y, 1^n)$, which by Theorem 5.1 guarantees that Exy. $\qquad\square$

As is often the case, having established the existence of one Π_1^0 complete relation, showing the existence of others is easy, as all we have to do is pick a relation rich enough to encode computable function equality. For a natural mathematical example, we will show that the isomorphism problem for polynomial time trees is Π_1^0 complete.

Definition 5.3. A *subtree of* $\{0, \ldots, c\}^*$ is a language closed under prefixes, T. Implicit in the definition is the notion of a predecessor function, $pred : T \to T$, satisfying $pred(\epsilon) = \epsilon$ and for $w \ne \epsilon$, $pred(w) = v$ if and only if $va = w$ for some $a \in \{0, \ldots, c\}$. An isomorphism of subtrees is a bijective function preserving predecessor. A subtree T is polynomial if there is a procedure for verifying $w \in T$ in time polynomial in $|w|$.

Note that we have an effective enumeration of all such trees. The set of strings accepted by any polynomial time machine on the alphabet $\{0, \ldots, c\}$ is, by definition, a polynomial time language over $\{0, \ldots, c\}$. Every such language induces a prefix-closed language: given a machine M and a string s, run M on every prefix of s and accept iff all prefixes are accepted. If M is polynomial time this incurs at most a linear slowdown, and thus is still polynomial.

Given an enumeration of polynomial time machines we will then say the tree induced by e to mean the prefix-closed language induced by the eth machine.

Theorem 5.4. *Isomorphism of polynomial time subtrees of $\{0, \ldots, c\}^*$ is complete for Π_1^0 equivalence relations.*

Proof. First let us verify that the problem is indeed Π_1^0. We claim that the isomorphism can be thus expressed by:

$$\{(e, i) : \forall d \; T_{(e,d)} \cong T_{(i,d)}\}. \tag{5.10}$$

By $T_{(e,d)}$ we mean the tree induced by e on strings of length no greater than d: that is, the subtree of the tree induced by e of depth d. The matrix is clearly computable, as it concerns isomorphism of finite trees. To show that the isomorphism of subtrees of any depth implies a complete isomorphism we need König's lemma.

First observe that as any isomorphism must preserve the predecessor relation, it must therefore map the root to the root. We will define a tree I in which there exists a branch of length d starting from the root for every isomorphism between $T_{(e,d)}$ and $T_{(i,d)}$, and given such a branch the subbranch of length $d-1$ corresponds to the restriction of that isomorphism to $T_{(e,d-1)}$ and $T_{(i,d-1)}$. That is, the root of I corresponds to the unique function mapping the sole node of $T_{(e,0)}$ to the sole node of $T_{(i,0)}$, and for every isomorphism between $T_{(e,d)}$ and $T_{(i,d)}$ there is a corresponding node at depth d. Since the structures are finite every node has finitely many children, as there are only finitely many functions between them. But if (5.10) holds there are infinitely many nodes in the tree. By König's lemma, there must be a branch of infinite length in I, corresponding to a complete isomorphism between the trees induced by e and i.

Next we will show how given indices e and i of polynomial time machines computing functions f and g respectively, we can construct polynomial time subtrees of $\{0, 1\}^*$, T_e and T_i, such that:

$$T_e \cong T_i \Leftrightarrow \forall x \; f(x) = g(x).$$

This will be sufficient to establish Π_1^0 completeness by Theorem 5.2.

To construct T_e, whenever M_e on input x outputs n we require that $1^x 0^n \in T_e$. We then close the resulting language under substrings. This corresponds to an infinite 1-branch, with a 0-branch corresponding to the machine output at each level, as illustrated in Figure 5.1.1.

Let us verify that the resulting construction is indeed polynomial time. Given a string we must first verify whether it is of the form $1^a 0^b$ for some $a, b \geq 0$, which can be done in linear time. Then we check whether M_e on input a outputs b. Since M_e is polynomial time, so are these operations.
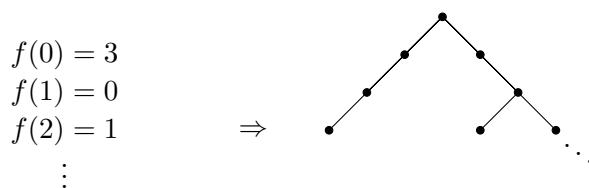
$$f(0) = 3$$
$$f(1) = 0$$
$$f(2) = 1$$
$$\vdots$$
$$\Rightarrow$$

Figure 5.1: The construction of T_e

To see that $T_e \cong T_i \Rightarrow \forall x \ f(x) = g(x)$ we note again that the isomorphism must map the root of T_e to the root of T_i. Having established this, assume for contradiction that $f(c) \neq g(c)$ for some c, but $T_e \cong T_i$. Observe that the isomorphism must map 1^c in T_e to 1^c in T_i as that is the only node in both trees that is at distance c from the root and has an infinite branch below it. However, this is impossible because one has a branch of length $f(c)$ and the other of length $g(c)$.

To see that $T_e \cong T_i \Leftarrow \forall x \ f(x) = g(x)$ one need only observe that with identical functions our construction will produce identical, and a fortiori isomorphic, trees. $\qquad\square$

5.1.2 A complete preorder

Theorem 5.1 can be seen as a corollary of a largely unrelated work. In [CDSS11] the authors in their investigation of partial orders offer a characterisation of Π_1^0 preorders (Proposition 3.1): every such preorder is computably isomorphic to the inclusion relation on a computable family of sets. As an equivalence relation is merely a symmetric preorder, a Π_1^0 equivalence relation is then isomorphic to equality on a computable family of sets. This, of course, is another way to view the equality of computable functions.

As there is no effective enumeration of computable sets the inclusion relation on them is not in itself a Π_1^0 preorder. However, we can use a padding argument as in 5.2 to extend the result of [CDSS11] to polynomial time sets, which does result in a Π_1^0-complete preorder.

Theorem 5.5. *The inclusion relation on polynomial time sets is complete for Π_1^0 preorders. That is, taking X_i to mean the ith polynomial time set, for every Π_1^0 preorder P there exists a computable f satisfying:*

$$Pxy \iff X_{f(x)} \subseteq X_{f(y)}.$$

Proof. By Proposition 3.1 of [CDSS11], for every Π_1^0 preorder P there exists a sequence of computable sets $\{A_i \mid i \in \mathbb{N}\}$ such that Pxy if and only if $A_x \subseteq A_y$. Our goal is to create a sequence of polynomial time sets of binary strings $\{X_i \mid i \in \mathbb{N}\}$ and a computable h satisfying:

$$n \in A_i \iff 1^n 0^{h(n)} \in X_i.$$

Once we establish this the proof will be complete: by [CDSS11] we shall have Pxy if and only if $A_x \subseteq A_y$ which, by our construction, holds if and only if $X_x \subseteq X_y$, and we can obtain X_i from A_i computably via h.

As with equivalence relations, we will require computable approximations to our preorder P. That is, a computable sequence $\{P_t \mid t \in \mathbb{N}\}$, satisfying:

$$P_{t+1} \subseteq P_t. \tag{5.11}$$

$$P_t \cap [0, t]^2 \text{ is a preorder over } [0, t]^2. \tag{5.12}$$

$$P = \bigcap_{t \in \mathbb{N}} P_t. \tag{5.13}$$

The sequence $\{A_i \mid i \in \mathbb{N}\}$ in [CDSS11] is constructed by stages. We give their construction below:

At stage t, for all $i < t$, if $P_t it$ add to A_t the contents of A_i. Consider every $i, k \leq t$. Let n be the smallest number not added to any set thus far. If $P_t ik$, let $n \in A_k$. Else, let $n \notin A_k$.

Let us verify that this construction behaves as claimed. First, the sequence $\{A_i \mid i \in \mathbb{N}\}$ is clearly computable. Next, suppose Pik. If $i < k$, at stage k all elements placed thus far into A_i will be placed into A_k. Thereon, at any stage t an element could only be placed into A_i if there exists a j with $P_t ji$. However, as P_t is transitive at that stage, $P_t jk$ and that element is placed in A_k as well. Next, suppose $k < i$. Clearly for $t > i$, any element placed into i at stage t is also placed into k as P_t is transitive. The only other time an element enters i is at stage i where we add the contents of A_j to A_i, for all j where $P_i ji$. For contradiction, let us assume this process violates our requirement that $A_i \subseteq A_k$. Let j be the smallest number such that $P_i ji$ but $A_j \nsubseteq A_k$. Clearly $j > k$, otherwise we have already seen that all the elements of A_j would be in A_k. It is easy to see that this implies that at stage j there must have been a j' with $P_j j' j$ (by transitivity, $P_j j' k$) with $j' > k$ and $A_{j'} \nsubseteq A_k$. However as there are only finitely many numbers between k and i, eventually this would no longer be possible. As such Pik implies $A_i \subseteq A_k$.

Finally, suppose $\neg Pik$. This means there exists a stage t with $\neg P_t ik$. At that stage we will place some element into A_i (because clearly $P_t ii$), but not into A_k, meaning that $\neg Pik$ implies $A_i \nsubseteq A_k$

We observe that the following is a decision procedure for determining whether $n \in A_i$: perform the first $\max\{\lceil \sqrt{2n} \rceil, i\}$ steps of the construction, and observe whether n has been added to A_i thus far. To see that this is sufficient note that there are only two ways an element n could be added to A_i:

1. n has already been placed into some A_k, and $P_i ki$ held at stage i. We would then add the contents of A_k to A_i at the beginning of that stage.

2. At some stage t, n was the smallest integer not placed in any set so far, and was placed into A_i because $P_t ik$ for some i.

The first case happens at stage i. In the second case, note that at every stage s we have s sets under consideration, and each set induces the introduction of a new integer. So the largest integer placed in any set at stage t is $\sum_{s \leq t} s = t^2/2$, so $t \leq \lceil \sqrt{2n} \rceil$.

While we can thus establish that we need to run no more than $\max\{\lceil\sqrt{2n}\rceil, i\}$ steps of the construction, this tells us nothing about how much time each step will actually take: at stage t we must first construct P_t, and that could take an arbitrarily long time. This is the purpose of the padding function h.

Recall that the sequence $\{P_t \mid t \in \mathbb{N}\}$ is computable. That is, there exists a machine M that on input t returns the code of the machine deciding P_t. As in Theorem 5.2, $h(n)$ is simply the function that counts the number of steps M takes on input n. As before, we can assume that $h(n-1) \leq h(n)$.

We can then define X_i via its decision procedure as follows: to determine whether $1^n 0^k \in A_i$, first verify whether $k = h(n)$. If not, reject the string. Otherwise perform the first $\max\{\lceil\sqrt{2n}\rceil, i\}$ steps of the $\{A_i \mid i \in \mathbb{N}\}$ construction and determine whether $n \in A_i$.

It is clear that this construction guarantees $n \in A_i$ if and only if $1^n 0^{h(n)} \in X_i$. It remains to verify that X_i is polynomial time.

We can ignore the case where $\max\{\lceil\sqrt{2n}\rceil, i\} = i$ as it takes constant time. As such, we will consider the time needed to perform the first $t = \lceil\sqrt{2n}\rceil$ steps of the construction of $\{A_i \mid i \in \mathbb{N}\}$. More precisely, we will consider the amount of time needed to perform the tth step, as performing all preceding step requires at most t times more work, which is sublinear in the length of the input.

As in Theorem 5.2, the first step is to ensure the input string is of the right form. That is, on input $1^n 0^m$ we must first verify whether $h(n) = m$. The argument is as before: begin calculating $h(n)$ by simulating the machine which counts the number of steps M takes on input t, where M is the machine that on input t returns the code of the machine deciding P_t. It should be clear whether the machine takes too little or too many steps in time at most quadratic in m. If the string is of the wrong form, reject it.

Given that the string is of the form $1^n 0^{h(n)}$, it takes at most $h(t)$ time to construct the required approximation of P, whereas the input is of length $n + h(n)$. Following this we copy the elements of A_k into A_t for all k with $P_t kt$: there are at most t such sets, and each set contains at most $n-1$ elements, so the number of operations is polynomial in n.

Finally, we consider every $k \leq t$, of which there are t, and for every j add a new element into A_j if $P_t kj$: at most t^2 operations, or linear in n. $\qquad\square$

We would like to complete the parallel with the equivalence relation case by showing that the embeddability of polynomial time trees is complete for Π_1^0 preorders. Unfortunately, embeddability in the natural sense of an injective function preserving predecessor does not appear to be Π_1^0 but rather Δ_2^0: we could guess the node the root is mapped to and require that for every level below that the embedding is preserved, or we could require that for every depth of the preimage there exists a depth of the image enabling an embedding.

It is also worth noting that with this notion of an embedding bi-embeddability would not, in fact, imply isomorphism as illustrated in Figure 5.2.

Instead we will consider *root-preserving* embeddability: the existence of an injective function that maps the root to the root and respect predecessor. This notion has a
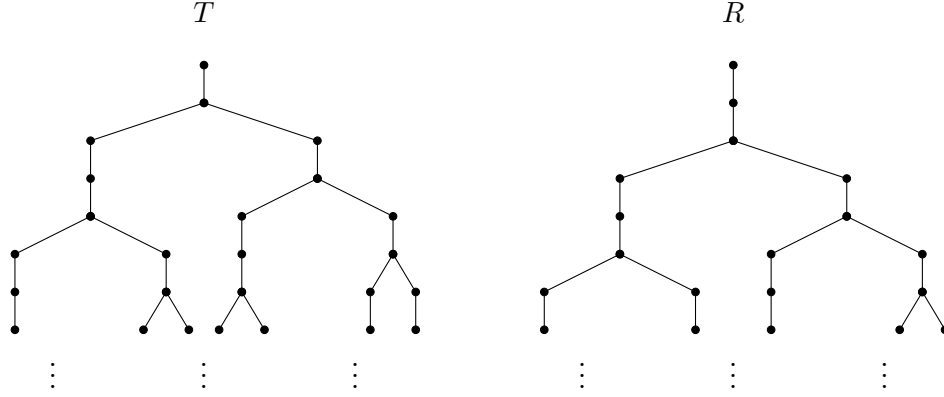
Figure 5.2: Bi-embeddable, but non-isomorphic trees

natural interpretation. T_a is root-preserving embeddable into T_b if and only if T_a is isomorphic to a set of prefixes of T_b. That is, the nodes in T_a can be renamed in such a way so that $T_a \subseteq T_b$. This will also give us an alternative proof of Theorem 5.4.

Theorem 5.6. *Root-preserving embeddability of polynomial time subtrees of $\{0, \ldots, c\}^*$ is complete for Π_1^0 preorders.*

Proof. The proof follows the same lines as Theorem 5.4. The desired relation can be expressed as:

$$\{(e, i) : \forall d\ T_{(e,d)} \text{ is root-preserving embeddable in } T_{(i,d)}\}. \tag{5.14}$$

To obtain the theorem we will show how given indices e, i of polynomial time sets we can construct polynomial time subtrees of $\{0, 1\}^*$, T_e and T_i, satisfying:

$$X_e \subseteq X_i \iff T_e \text{ is root-preserving embeddable in } T_i.$$

Let X_e and X_i be polynomial time sets. Let $1^x 0 \in T_e$ for every $x \in X_e$ and $1^x 0 \in T_i$ for every $x \in X_i$. Let $1^y \in T_e, T_i$ for all y. Suppose $X_e \subseteq X_i$. Note that our construction guarantees that $T_e \subseteq T_i$, and as such the identity function is the required embedding. Next, suppose that f is a root preserving embedding of T_e into T_i. Observe that f must map $1^x 0$ to $1^x 0$ as it is the only string of length $x + 1$ with no suffixes in the language. As such a string is in T_i if and only if $x \in X_i$, this establishes that $X_e \subseteq X_i$. $\qquad\square$

Proposition 5.7. *If T_a and T_b are polynomial time subtrees of $\{0, \ldots, c\}^*$ then T_a and T_b are root-preserving bi-embeddable if and only if $T_a \cong T_b$.*

Proof. We only need to prove the proposition for finite trees, as the rest follows from König's lemma. One direction is obvious. For the other direction suppose T_a and T_b are finite trees with f a root-preserving embedding of T_a into T_b and g a root-preserving

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embedding of T_b into T_a. We claim that f is actually an isomorphism. As f is already an injective, predecessor preserving mapping, it is sufficient to show that f is onto.

As f and g are injective functions from T_a to T_b and T_b to T_a respectively, it follows that $|T_a| \leq |T_b|$ and $|T_b| \leq |T_a|$. As $|T_a| = |T_b|$, and both are finite sets, it follows that any one to one function between them is also onto, thus f is an isomorphism. $\qquad\square$

Corollary 5.8. *Another way of obtaining Theorem 5.4.*

5.2 The end of completeness

In this section we present a modified proof of a result of Russell Miller and Keng Meng Ng from [INMN12], concerning the non-existence of Π_n^0-complete equivalence relations for $n \geq 2$. We then show that it is immediately extendable to the Δ_n^0 case, thereby completing our investigation of equivalence relations complete under computable component-wise reducibility.

Theorem 5.9. *For $n \geq 2$, there are no Π_n^0-complete equivalence relations.*

Proof. We present the proof for the case $n = 2$. For a general n we simply relativise to sets Π_2^0 with an appropriate oracle.

Let E be an arbitrary Π_2^0 equivalence relation. Recall that $\{e \mid W_e \text{ is infinite}\}$ is complete for Π_2^0 sets (see, for instance, [Soa87] Theorem IV:3.2). It follows that there exists an r.e. sequence of sets $\{V[x, y] \mid x, y \in \mathbb{N}\}$ satisfying:

$$V[x, y] \text{ is infinite} \iff (x, y) \in E.$$

We will construct an $F \in \Pi_2^0$ satisfying $F \not\leq E$ by constructing an r.e. sequence $\{U_{xy} \mid x, y \in \mathbb{N}\}$ where:
$$U_{xy} \text{ is infinite} \iff (x, y) \in F.$$

To ensure F is an equivalence relation we will let $U_{xx} = \mathbb{N}$ for all x and we will only explicitly construct sets U_{xy} for $x < y$, taking $U_{yx} = U_{xy}$ as given. We will verify that F is transitive after the construction.

We will give the procedure for enumerating $\{U_{xy} \mid x, y \in \mathbb{N}\}$ by giving a subroutine that we will dovetail over to eventually cover all natural numbers. We will use x_e, y_e and z_e to denote the $3e + 1$, $3e + 2$ and $3e + 3$ respectively. I.e., the eth number with remainder 1,2 and 0 on division by 3. The subroutine is run on x_e, y_e and z_i:

1. Wait for $\varphi_e(x_e), \varphi_e(y_e)$ and $\varphi_e(z_i)$ to converge.

2. If any of $\varphi_e(x_e), \varphi_e(y_e)$ or $\varphi_e(z_i)$ output the same value, end the subroutine.

3. Enumerate $V[\varphi_e(x_e), \varphi_e(z_i)]$ and $V[\varphi_e(x_e), \varphi_e(y_e)]$, adding a new element to $U_{x_e z_i}$ for every step of the computation. If an element enters $V[\varphi_e(x_e), \varphi_e(z_i)]$ go to step 4. If an element enters $V[\varphi_e(x_e), \varphi_e(y_e)]$ go to step 5.

4. Enumerate $V[\varphi_e(y_e), \varphi_e(z_i)]$ and $V[\varphi_e(x_e), \varphi_e(y_e)]$, adding a new element to $U_{y_e z_i}$ for every step of the computation. If an element enters $V[\varphi_e(y_e), \varphi_e(z_i)]$ go to step 3. If an element enters $V[\varphi_e(x_e), \varphi_e(y_e)]$ go to step 5.

5. Restart the subroutine on x_e, y_e, z_j where z_j is the least multiple of 3 not yet used in any subroutine.

As mentioned, to enumerate $\{U_{xy} \mid x, y \in \mathbb{N}\}$ we dovetail the above subroutine on the smallest x_e, y_e and z_i not yet used.

Let us now verify that F is a Π_2^0 equivalence relation as intended. That F is indeed Π_2^0 is immediate from its definition via infinite r.e. sets. We have already dealt with symmetry and reflexivity. We will demonstrate that F has no equivalence class with more than 2 members, thus obtaining transitivity automatically.

Observe that $(x_e, y_d) \notin F$ for any d, e as we never place any element into $U_{x_e y_d}$. We also claim that it is not possible for $(x, z_i), (x, z_j) \in F$ for $i < j$ as we would only run the subroutine on the same x and z_j if we reach step 5, and once we reach step 5 we would no longer place any new elements into $U_{x z_i}$, so it could not be infinite. As such the only candidate for a an equivalence class with more than two members is $\{x_e, y_e, z_i\}$ via $U_{x_e z_i}$ and $U_{y_e z_i}$ being infinite. Observe that this is impossible: if the subroutine ever reaches step 5 then $U_{x_e z_i}$ and $U_{y_e z_i}$ would be finite. If it never reaches step 5, then either after a finite amount of steps it stays at step 3 or 4 for ever, in which case either $U_{y_e z_i}$ or $U_{x_e z_i}$ would be finite, or it must jump between 3 and 4 infinitely many times. As such a jump happens only when an element enters $V[\varphi_e(y_e), \varphi_e(z_i)]$ and $V[\varphi_e(x_e), \varphi_e(z_i)]$ receive a new element respectively, these sets must be infinite. That is not possible because E is an equivalence relation, so $V[\varphi_e(x_e), \varphi_e(y_e)]$ would need to be infinite as well. But if an element ever enters $V[\varphi_e(x_e), \varphi_e(y_e)]$, we would have gone to step 5.

We are now ready to show that $F \not\le E$. Suppose for contradiction that it is: there then exists an $f = \varphi_e$ such that $(x_e, z_i) \in F$ iff $(\varphi_e(x_e), \varphi_e(z_i)) \in E$. As we have argued above $(x_e, z_i) \in F$ only if after a certain stage the subroutine stays at step 3 for ever, but the subroutine will leave step 3 if an element ever enters $V[\varphi_e(x_e), \varphi_e(z_i)]$, so that set must be finite and $(\varphi_e(x_e), \varphi_e(z_i)) \notin E$. The same argument shows that $(y_e, z_i) \in F$ would necessitate $(\varphi_e(y_e), \varphi_e(z_i)) \notin E$. The only remaining possibility is where $(x_e, z_i), (y_e, z_i) \notin F$ for any i. This means an infinite number of subroutines on x_e, y_e must have reached step 5, but since such a subroutine would enter step 5 only if an element enters $V[\varphi_e(x_e), \varphi_e(z_i)]$ or $V[\varphi_e(y_e), \varphi_e(z_i)]$, either $(\varphi_e(x_e), \varphi_e(z_i))$ or $(\varphi_e(y_e), \varphi_e(z_i))$ must be in E, which contradicts the assumption that φ_e is a reduction. $\square$

Corollary 5.10. *There is no Π_n^0-complete preorder for $n \ge 2$.*

Proof. The existence of such a preorder would, by Proposition 2.4, contradict the theorem. $\square$

Proposition 5.11. *For every Π_2^0 equivalence relation E there exists a Δ_2^0 equivalence relation F such that $F \not\le E$.*

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Proof. We will modify the construction in Theorem 5.9 to create a computable sequence $\{U[x, y] \mid x, y \in \mathbb{N}\}$ where:

$$U[x, y] \text{ is cofinite} \iff (x, y) \in F.$$

$$U[x, y] \text{ is finite} \iff (x, y) \notin F.$$

As Fxy is then determined both by $U[x, y]$ being infinite and $\overline{U[x, y]}$ being finite, it is both a Π_2^0 and Σ_2^0 relation.

As before, we let $U[x, x] = \mathbb{N}$ and $U[x, y] = U[y, x]$. Likewise x_e, y_e and z_e denotes $3e + 1$, $3e + 2$ and $3e + 3$ respectively. The subroutine is run on x_e, y_e and z_i:

1. Wait for $\varphi_e(x_e), \varphi_e(y_e)$ and $\varphi_e(z_i)$ to converge. Place a new element into $\overline{U[x_e, y_e]}$, $\overline{U[x_e, z_i]}$ and $\overline{U[y_e, z_i]}$ for each step of the computation.

2. If any of $\varphi_e(x_e), \varphi_e(y_e)$ or $\varphi_e(z_i)$ output the same value, end the subroutine.

3. Enumerate $V[\varphi_e(x_e), \varphi_e(z_i)]$ and $V[\varphi_e(x_e), \varphi_e(y_e)]$, adding a new element to $U[x_e, z_i]$, $\overline{U[x_e, y_e]}$ and $\overline{U[y_e, z_i]}$ for every step of the computation. If an element enters $V[\varphi_e(x_e), \varphi_e(z_i)]$ go to step 4. If an element enters $V[\varphi_e(x_e), \varphi_e(y_e)]$ go to step 5.

4. Enumerate $V[\varphi_e(y_e), \varphi_e(z_i)]$ and $V[\varphi_e(x_e), \varphi_e(y_e)]$, adding a new element to $U[y_e, z_i]$, $\overline{U[x_e, y_e]}$ and $\overline{U[x_e, z_i]}$ for every step of the computation. If an element enters $V[\varphi_e(y_e), \varphi_e(z_i)]$ go to step 3. If an element enters $V[\varphi_e(x_e), \varphi_e(y_e)]$ go to step 5.

5. Place all remaining natural numbers into $\overline{U[x_e, z_i]}$ and $\overline{U[y_e, z_i]}$. Restart the subroutine on x_e, y_e, z_j where z_j is the least multiple of 3 not yet used in any subroutine.

Note that when dovetailed this routine enumerates $\{U[x, y] \mid x, y \in \mathbb{N}\}$ and its complement, meaning it is a computable sequence as required.

It remains to see that if any set in the sequence is infinite it is in fact cofinite. As we have seen in Theorem 5.9 the subroutine cannot go between steps 3 and 4 for ever, if either of $U[x_e, z_i]$ or $U[y_e, z_i]$ is infinite, the subroutine must eventually remain at that step. As such, only a finite number of elements could have been placed in the complement. $\square$

Corollary 5.12. *For $n \geq 2$, there is no Δ_n^0-complete equivalence relation.*

Proof. We relativise the proof to an appropriate oracle. $\square$

6   Conclusion

At the end of our enquiry we emerge with a fair idea of what the world of complete equivalence relations and preorders looks like.

The Σ_n^0 relations are relatively well behaved. There exist complete equivalence relations for every level, and in the lower rungs these correspond to natural mathematical notions. We further note that for the case of Σ_4^0, Turing equivalence of r.e. sets is shown to be complete in [INMN12]. The search for examples in Σ_5^0 or higher is, perhaps, futile, as it would stretch the spirit of the word to refer to relations of quantifier depth five or more as "natural".

In Π_n^0 we find a world starkly different from that in the standard theory of m-reducibility. For Π_1^0 a complete equivalence relation exists, but beyond that any familiarity of degree structure breaks down. Even to find a Π_n^0 equivalence relation that dominates all Δ_n^0 equivalence relations is impossible. Thus the Δ_n^0 case, too, lacks complete members beyond that shown complete for Δ_1^0 in [GG01].

Our departure from the literature with an excursion into the study of preorders, while offering greater generality, did not yield any additional surprises. Wherever we investigated, preorders behaved in the same manner as equivalence relations did. This suggests an open question to researchers in related fields:

Open Question 6.1. What can be said about (complexity, analytic, set-theoretic) classes that admit complete equivalence relations, but not complete preorders, with respect to some notion of component-wise reducibility?

Of course the relevance of this question is in itself contingent on whether a parallel of Proposition 2.4 holds. In a setting where classes are not closed under intersection, the connection between equivalence relations and preorders may be lost altogether.

The degree structure of equivalence relations under component-wise reducibility is rich and complex, and largely beyond the scope of this work. Our contribution concerns merely the maximum degrees - the complete equivalence relations, but we hope that such as it is it may nevertheless shed some light for those who will gaze deeper into the depths than we have managed in the course of this thesis.

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