Category theory

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1 Introduction

The contrast between set theory and categories is instructive for the different motivations they provide. Set theory says encouragingly that it's what's inside that counts. By contrast, category theory proclaims that it's what you do that really matters.

Category theory is the study of mathematical structures by means of the relationships between them. It provides a framework for considering the diverse contents of mathematics from the logical to the topological. Consequently we present numerous examples to explicate and contextualize the abstract notions in category theory.

Category theory was developed by Saunders Mac Lane and Samuel Eilenberg in the 1940s. Initially it was associated with algebraic topology and geometry and proved particularly fertile for the Grothendieck school. In recent years category theory has been associated with areas as diverse as computability, algebra and quantum mechanics.

Sections 2 to 3 outline the basic grammar of category theory. The major reference is the expository essay [Sch01].

Sections 4 to 8 deal with *toposes*, a family of categories which permit a large number of useful constructions. The primary references are the general category theory book [Bar90] and the more specialised book on topoi, [Mac82].

Before defining a topos we expand on the discussion of limits above and describe exponential objects and the subobject classifier. Once the notion of a topos is established we look at some constructions possible in a topos: in particular, the ability to construct an integer and rational object from a natural number object. Lastly we briefly describe the effective topos - a universe where everything is computable. The exposition of the effective topos comes from [Oos08].

In Sections 9 to 11, the focus turns to the *abelian categories*, a broad family of categories which satisfy a set of axioms similar to the category of modules. The aim of these sections is to illustrate the use of categorical methods in algebra.

The example explored is the classical Krull-Schmidt theorem for modules satisfying mild finiteness conditions, which guarantees the existence of a unique decomposition as a sum of indecomposable modules. We gradually develop the tools needed to re-formulate the Krull-Schmidt theorem and its proof in the language of abelian categories, its natural, more general setting.

The exposition of the elementary theory of abelian categories follows [Fre64]. The treatment of Grothendieck categories and the categorical Krull-Schmidt theorem is based on those in [Par70] and [BD68]. Lang's algebra text [Lan80] was used as a reference for the classical Krull-Schmidt theorem.

The discussion of toposes is the work of the first author; that of abelian categories, of the second author. The basic material was studied and written jointly. The authors are grateful to their supervisor Andre Nies for his expertise, his advice and his unfailing encouragement.

2 Fundamental concepts

2.1 Categories

Definition 1. *A* category $\mathfrak{A}$ *comprises*

(i) a class of objects $\mathrm{Ob}(\mathfrak{A})$*;*

(ii) for any pair $A, A' \in \mathrm{Ob}(\mathfrak{A})$*, a set* $\mathbf{Hom}(A, A')$ *of* morphisms*;*

(iii) for any triple $A, A', A'' \in \mathrm{Ob}(\mathfrak{A})$*, a* composition *map*

$$\mathbf{Hom}(A', A'') \times \mathbf{Hom}(A, A') \to \mathbf{Hom}(A, A''),$$

denoted $(f, g) \mapsto f \circ g$*,*

which satisfy the following axioms:

1. *The sets* $\mathbf{Hom}(A, A')$ *are disjoint.*
2. *Composition is associative where defined. That is, for any quadruple* $A, A', A'', A''' \in \mathrm{Ob}(\mathfrak{A})$*, and any triple*

   $$(f, g, h) \in \mathbf{Hom}(A, A') \times \mathbf{Hom}(A', A'') \times \mathbf{Hom}(A'', A'''),$$

   we demand that

   $$(h \circ g) \circ f = h \circ (g \circ f).$$

3. *For each object* $A \in \mathrm{Ob}(\mathfrak{A})$*, there exists an* identity *morphism*

   $$id_A \in \mathbf{Hom}(A, A)$$

   with the obvious composition properties.

We often write $A \in \mathfrak{A}$ for objects of $\mathfrak{A}$ rather than $A \in Ob(\mathfrak{A})$, and $f : A \to A'$ for morphisms rather than $f \in \mathbf{Hom}(A, A')$.

The categories typically studied are standard classes of mathematical structure, together with the appropriate structure-preserving functions.

Example 2. *The following are categories.*

1. *The category* **Set** *has sets as objects and functions as morphisms.*

2. *Define a* pointed set *to be an ordered pair (X, p), where X is a set and $p \in X$ – as objects. Define a* point-preserving function $f : (X, p) \to (Y, q)$ *to be a function $f : X \to Y$ such that $f(p) = q$. The category* **pSet** *has pointed sets as objects and point-preserving functions as morphisms.*

3. *Let G be a group. Define a (left) group action on a set X to be a function $G \times X \to X$ given by $(g, x) \longmapsto g \cdot x$ satisfying the obvious composition and identity properties. A set X equipped with an action of G on X is said to be a G*-set*. Given two G-sets X and Y, call a function $f : X \to Y$ a G*-map*, if $f(g \cdot x) = g \cdot f(x)$. The category* ***G-Set*** *has G-sets as objects and G-maps as morphisms.*

4. *The categories* **Grp**, **Ring**, *and (for any fixed ring unitary R)* ***R-Mod*** *have, respectively, groups, rings, and unitary R-modules as objects, and group homomorphisms, ring homomorphisms, and module homomorphisms as morphisms.*

5. *The category* **Top** *has topological spaces as objects and continuous maps as morphisms. The category* **Met** *has metric spaces as objects and uniformly continuous maps as morphisms.*

Definition 3. *The category $\mathfrak{B}$ is a* subcategory *of the category $\mathfrak{A}$, if the following conditions are satisfied:*

1. $\mathrm{Ob}(\mathfrak{B})$ *is a subclass of* $\mathrm{Ob}(\mathfrak{A})$.

2. *For each pair $B, B' \in \mathrm{Ob}(\mathfrak{B})$, the set $\mathbf{Hom}_{\mathfrak{B}}(B, B')$ of $\mathfrak{B}$-morphisms from B to B' is a subset of the set $\mathbf{Hom}_{\mathfrak{A}}(B, B')$ of $\mathfrak{A}$-morphisms from B to B'.*

3. *The composition operation in $\mathfrak{B}$, where defined, is the restriction of the composition operation in $\mathfrak{A}$.*

A subcategory $\mathfrak{B}$ of $\mathfrak{A}$ is full, *if for each pair B, B' of objects in $\mathfrak{B}$, the set $\mathbf{Hom}_{\mathfrak{B}}(B, B')$ is equal to all of the set $\mathbf{Hom}_{\mathfrak{A}}(B, B')$.*

Example 4. *The category* **Ab** *of abelian groups is a full subcategory of* **Grp**. *The category* **Met** *of metric spaces is a subcategory, not full, of* **Top**.

It must be stressed that a category depends as much on its objects as its morphisms; morphisms usually form the basis of definitions of categorical notions. Quite different categories can therefore be constructed whose underlying classes of objects are the same. For instance, the category **hTop**, with topological spaces as objects and homotopy equivalence classes of continuous maps as morphisms, differs from the category **Top**.

By contrast, families of categories 'extremal' in some way often reduce to well-studied classes of mathematical object.

Example 5. *A category with one object is essentially a monoid. The elements of the monoid are the morphisms from the unique object to itself, and monoid multiplication is composition of morphisms.*

Example 6. *Let* $\mathfrak{J}$ *be a* small category*; that is, a category whose class of objects is a set. Suppose that there is at most one morphism* $f : J \to J'$ *between any pair of objects in* $\mathfrak{J}$*, and that* $\mathbf{Hom}(J, J')$ *and* $\mathbf{Hom}(J', J)$ *are both nonempty only if* $A = A'$*.*

Such a category is essentially a partially ordered set. The elements of the poset are the objects of $\mathfrak{J}$*. The order relation on* $\mathfrak{J}$ *is defined by*

$$J \leq J' \quad \textit{iff} \quad \mathbf{Hom}(J, J') \textit{ is nonempty.}$$

Reflexivity and transitivity are proved respectively by the existence of identity morphisms and of morphism composition.

Example 7. *There is a unique category* 1 *with one object and one morphism.*

Example 8. *A directed graph* G *gives rise to a small category, whose objects are* G*'s vertices and whose morphisms are* G*'s paths.*

Duality pervades category theory and for any concept there is a dual notion. Given a category $\mathfrak{A}$ the dual category $\mathfrak{A}^{op}$ consists of the same objects as $\mathfrak{A}$ but $Hom_{\mathfrak{A}^{op}}(A, A') = Hom_{\mathfrak{A}}(A', A)$. The dual notion is always found by simply reversing all the morphisms.

2.2 Morphisms

Definition 9. *A morphism* $m : A \to A'$ *in* $\mathfrak{A}$ *is monic if for any object* K *and any pair of morphisms* $f, g : K \to A$*, the equality* $m \circ f = m \circ g$ *implies that* $f = g$*.*

$$K \underset{g}{\overset{f}{\rightrightarrows}} A \xrightarrow{m} A'$$

Dual to monic morphisms are the epic morphisms.

Definition 10. *A morphism* $e : A' \to A$ *in* $\mathfrak{A}$ *is epic if for any object* K *and any pair of morphisms* $f, g : A \to K$*, the equality* $f \circ e = g \circ e$ *implies that* $f = g$*.*

$$A' \xrightarrow{e} A \underset{g}{\overset{f}{\rightrightarrows}} K$$

Example 11. *In* **Set** *and in* **Grp***, the monic and epic morphisms are the injective and surjective functions respectively.*

For **Grp** *this is not entirely obvious. For instance, in the epic case, the proof consists of the construction, for each subgroup* H *of a group* G*, of a group* K *and nontrivial homomorphism* $f : G \to K$*, such that* f *is trivial outside* H*.*

Example 12. *In the category* **Haus** *of Hausdorff spaces and continuous maps, a morphism is epic precisely if its image is dense in its range.*

Definition 13. *A morphism* $f : A \to A'$ *is an* isomorphism *if there exists a morphism* $f' : A' \to A$ *such that* $f \circ f' = id_{A'}$ *and* $f' \circ f = id_A$*.*

In many common categories, such as **Set**, **Grp** and R-**Mod**, all morphisms which are monic and epic are isomorphisms. A general explanation for this phenomenon develops in Section 10. On the other hand, the example of the category **Haus** shows that this need not always be the case.

2.3 Functors

In a given category a morphism acts between objects of that category. This concept of a morphism can be expanded beyond the context of one category to talk about an operation between categories.

Definition 14. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be arbitrary categories. A* functor $F : \mathfrak{A} \to \mathfrak{B}$ *is an operation which acts as follows;*

1. *An object $A \in \mathfrak{A}$ is assigned to $FA \in \mathfrak{B}$;*
2. *If $f : A \to A'$ is a morphism in $\mathfrak{A}$ then $F(f) : FA \to FA'$ is a morphism in $\mathfrak{B}$;*
3. *For any $A \in \mathfrak{A}$, $F(id_A) = id_{FA}$;*
4. *If $g \circ f$ is defined in $\mathfrak{A}$ then $F(g \circ f) = F(g) \circ F(f)$ in $\mathfrak{B}$.*

Example 15. *The diagonal functor $\Delta : \mathfrak{A} \to \mathfrak{A} \times \mathfrak{A}$ sends each object $A \in \mathfrak{A}$ to $(A, A) \in \mathfrak{A} \times \mathfrak{A}$.*

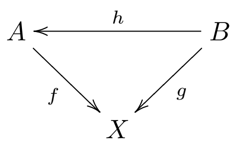
Example 16. *The forgetful functor* **For** : **Grp** $\to$ **Set** *sends each group to the underlying set. By comparison the free functor* **Free** : **Set** $\to$ **Grp** *sends each set to the free group on that set.*

2.4 Subobjects

The useful concept of a subset or subgroup is traditionally defined in terms of element membership. As category theory is based on morphisms rather than elements, the categorical definition of a subobject is instead based on the idea of an inclusion morphism.

For instance, consider the category **Set** and objects X and S where $S \subseteq X$. The image of the inclusion function $\iota : S \to X$ is S. While ι is monic, there are many other monics $m : S' \to X$ whose image is S. All such monics define the subset S in a way equivalent to ι. We can define the subobjects of X by formalising the notion of equivalent monics.

Definition 17. *A monomorphisms $f : A \to X$* dominates *a monomorphism $g : B \to X$, if there exists a morphism $h : B \to A$ such that $g = f \circ h$.*



Two monomorphisms are equivalent*, if each dominates the other.*

Example 18. *The above analysis shows that for an inclusion function $\iota : S \to X$ in* **Set***, the equivalence class $[\iota]$ is precisely the set of all injective functions into the set X whose image is the subset S.*

Definition 19. *A* subobject *of an object A in $\mathfrak{A}$ is an equivalence class of monomorphisms.*

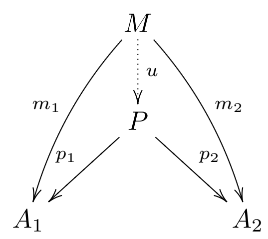
The domination relation bestows a natural partial order on an object's class of subobjects.

Dually, a *quotient object* of an object A in $\mathfrak{A}$ is defined to be an equivalence class of epimorphisms out of A, under the appropriate equivalence relation.

2.5 Products, equalizers, limits

Let A_1 and A_2 be objects in a category $\mathfrak{A}$.

Definition 20. *A* product *of A_1 and A_2 is an object P in $\mathfrak{A}$ along with a pair of morphisms $p_i : P \to A_i$ for $i = 1, 2$ such that for any object M and pair of morphisms $m_i : M \to A_i$ for $i = 1, 2$, there exists a unique morphism $u : M \to A_1 \times A_2$ making $p_i \circ u = m_i$.*



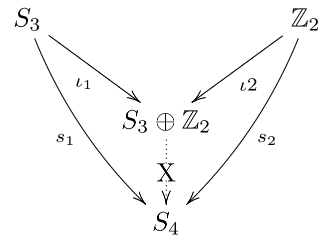
A product of A_1 and A_2, should it exist, is unique up to isomorphism. We denote the product of A_1 and A_2 by $A_1 \times A_2$.

A categorical product captures the notion of cartesian products in **Set** and direct products in **Grp**.

Example 21. *Consider a partially ordered set as a category with the order relation as the morphisms, then the product of two elements is their greatest lower bound.*

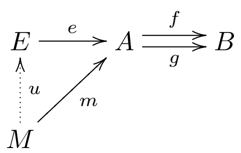
The dual notion is that of *coproduct.* In **Set** the coproduct is the disjoint union with the usual inclusion maps. The coproduct is sometimes referred to as the sum but while a coproduct in **Ab** is a direct sum, a coproduct in **Grp** is not.

Example 22. *Given S_3 and $\mathbb{Z}_2$ in* **Grp***, the direct sum $S_3 \oplus \mathbb{Z}_2$ with inclusion maps ι_1, ι_2 is not a coproduct. This can be seen by looking at the group S_4 and* *maps $s_1 : S_3 \to S_4$ and $s_2 : \mathbb{Z}_2 \to S_4$.*



If we assume the existence of a unique map $u : S_3 \oplus \mathbb{Z}_2 \to S_4$ such that $u \circ \iota_i = s_i$ for i=1,2 then we get the contradiction that the image of u has order 24 while the domain of u has order 12. Hence the direct sum is not a coproduct in **Grp**.

Definition 23. *Given two morphisms $f, g : A \to B$ an* equalizer *is an object E together with a morphism $e : E \to A$ such that $f \circ e = g \circ e$ with the property that for any other morphism $m : M \to A$ where $f \circ m = g \circ m$ there exists a unique morphism $u : M \to E$ such that $e \circ u = m$.*



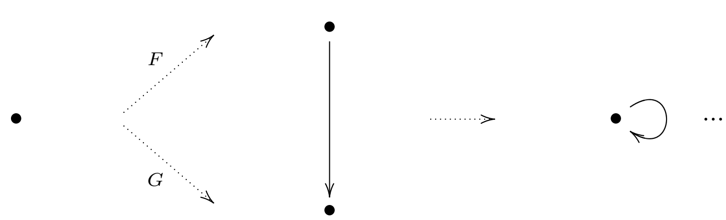
Example 24. *In* **Set** *an equalizer for f, g is the set $E = \{x : f(x) = g(x)\}$ with inclusion map.*

For algebraic categories such as groups, rings and vector spaces the equalizer is constructed in the same way as in **Set**.

The dual notion, *coequalizer*, is not as simple.

Example 25. *In* **Set** *a coequalizer of f, g is a quotient of the set B by the smallest equivalence relation $\sim$ such that $f(a) \sim g(a)$ for all $a \in A$.*

Example 26. *In the category of small categories let* **1** *be the one object category with only the identity morphism and let* **2** *be the two object category with exactly one non identity morphism. The coequalizer of the only two unique functors $F, G :$* **1** *$\to$* **2** *is the monoid of natural numbers under addition, $\mathbb{N}$.*



Lemma 27. *Equalizers are monic. Coequalizers are epic.*

There are obvious analogies in the definitions of products and equalizers. We can generalize to the notion of a *limit* as a suitable object and family of morphisms defined over a collection of objects and morphisms.

To make this precise, we first formalise the notion of a collection of objects and morphisms, secondly we give the criteria for being a suitable object with morphisms over this collection and lastly the conditions that make this suitable object and morphism a limit.

We call a collection of objects and morphisms a diagram which is like an indexed set that also accounts for the arrangement of the morphisms.

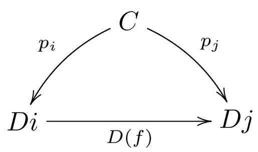
Definition 28. *A* diagram *into the category* $\mathfrak{A}$ *is a functor* $D : \mathfrak{J} \to \mathfrak{A}$*, such that the* index category $\mathfrak{J}$ *is small.*

Example 29. *An ordered pair in* $\mathfrak{A}$ *is a diagram into* $\mathfrak{A}$ *from the index category*

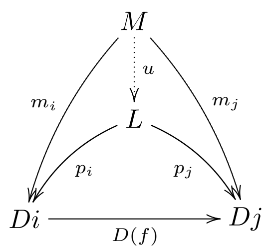
$$\bullet \quad \bullet$$

Let $D : \mathfrak{J} \to \mathfrak{A}$ be a diagram into $\mathfrak{A}$.

Definition 30. *A* cone *over* D *is an object* $C \in \mathfrak{A}$*, together with morphisms* $p_i : C \to Di$ *for each object* i *in the index category* $\mathfrak{J}$*, such that for any morphism* $D(f) : Di \to Dj$ *in the diagram* $D(f) \circ p_i = p_j$*.*



Definition 31. *A* limit *of* D *is a cone* L *over* D *with morphisms* $(p_i)_{i \in \mathfrak{J}}$*, such that for any other cone* M *with morphisms* $(m_i)_{i \in \mathfrak{J}}$*, there exists a unique morphism* $u : M \to L$ *such that for all* $i \in \mathfrak{J}$*, we have* $m_i = p_i \circ u$*.*



A limit of D, should it exist, is unique up to isomorphism.

Example 32. *An equalizer is a limit over a diagram whose index category is*

$$\bullet \rightrightarrows \bullet$$

Example 33. *A product is a limit over a diagram whose index category is*

$$\bullet \quad \bullet$$

Like any category the index category of a diagram must have identiy morphisms. For the purpose of establishing a limit over a diagram the identity morphisms are irrelevant as they commute with every morphism. For simplicity we ignore the identity morphisms in an index category.

A *terminal object* is a limit over the empty diagram. Suppose $\mathfrak{A}$ has a terminal object T. Then any other object $A \in \mathfrak{A}$ is a cone over the empty diagram, and therefore there exists a unique map $u : A \to T$.

Example 34. *In* **Set** *the terminal objects are the singleton sets. There is only one function from any set A into a singleton set $\{x\}$.*

(On the other hand, there may be many functions $f : \{x\} \to A$; these have the useful property of picking out elements of A.)

3 Universal properties

3.1 Natural transformations

Just as a functor is like a morphism between two categories, a natural transformation is like a morphism between two functors. Let $\mathfrak{A}, \mathfrak{B}$ be two categories and $F, G : \mathfrak{A} \to \mathfrak{B}$ be two functors between the categories.

If $h : X \to Y$ is a morphism in $\mathfrak{A}$ then $F(h)$ and $G(h)$ are morphisms in $\mathfrak{B}$.

$$\begin{array}{ccc} FX & & GX \\ {\scriptstyle F(h)}\downarrow & & \downarrow{\scriptstyle G(h)} \\ FY & & GY \end{array}$$

A natural transformation between the functors F, G guarantees a way to connect the two $\mathfrak{B}$ morphisms $F(h)$ and $G(h)$.

Definition 35. *A natural transformation η between the functors F and G is a family of $\mathfrak{B}$ morphism $\eta_X : FX \to GX$, one for each X in $\mathfrak{A}$ called the component at X. The family of morphisms η_X satisfy the condition that for any $h : X \to Y$ in $\mathfrak{A}$ the following square commutes,*

$$\begin{array}{ccc} FX & \xrightarrow{\eta_X} & GX \\ {\scriptstyle F(h)}\downarrow & & \downarrow{\scriptstyle G(h)} \\ FY & \xrightarrow[\eta_Y]{} & GY \end{array}$$

Let V be a vector space over the field $\mathbb{K}$, and V^{**} its double dual. Both V and V^{**} are objects in the category $\mathfrak{V}_{\mathbb{K}}$ of vector spaces over $\mathbb{K}$.

Example 36. *There is a natural transformation η from the identity functor $I : \mathfrak{V}_{\mathbb{K}} \to \mathfrak{V}_{\mathbb{K}}$ to the double dual functor $-^{**} : \mathfrak{V}_{\mathbb{K}} \to \mathfrak{V}_{\mathbb{K}}$. The components of the natural transformation are defined as $\eta_U : U \to U^{**}$ for each $U \in \mathfrak{V}_{\mathbb{K}}$.*

Let $\eta_U(x) = \sigma_x$ for any $x \in U$. As σ_x is in U^{**} it is a linear function from U^* to $\mathbb{K}$ which we define as $\sigma_x(\varphi) = \varphi(x)$ for any $\varphi \in U^*$.

To show that η is a natural transformation from I to $-^{**}$ we will show that for vector spaces U, V and a linear function $f : U \to V$ the following square commutes,

$$\begin{array}{ccc} U & \xrightarrow{\eta_U} & U^{**} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{**}} \\ V & \xrightarrow[\eta_V]{} & V^{**} \end{array}$$

Given $f : U \to V$ define $f^* : V^* \to U^*$ then $f^{**} : U^{**} \to V^{**}$ as follows,

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ & {\scriptstyle \varphi\circ f}\searrow & \downarrow{\scriptstyle \varphi} \\ & & \mathbb{K} \end{array} \qquad \begin{array}{l} \text{for } \varphi \in V^* \\ f^*(\varphi) = \varphi \circ f \end{array} \qquad \begin{array}{ccc} U^* & \xleftarrow{f^*} & V^* \\ {\scriptstyle \phi}\downarrow & \swarrow{\scriptstyle \phi\circ f^*} & \\ \mathbb{K} & & \end{array} \qquad \begin{array}{l} \text{for } \phi \in U^{**} \\ f^{**}(\phi) = \phi \circ f^* \end{array}$$

To check that the natural transformation square commutes for all $x \in U$ we need $f^{**} \circ \eta_U(x) = \eta_V \circ f(x)$.

$$\begin{array}{rcll} LHS & = & f^{**} \circ \eta_U(x) & \\ & = & f^{**} \circ \sigma_x & \text{by definition of } \eta \\ & = & \sigma_x \circ f^* & \text{by definition of } f^{**} \\ RHS & = & \eta_V \circ f(x) & \\ & = & \sigma_{f(x)} & \text{by definition of } \eta \end{array}$$

To see that $\sigma_x \circ f^*$ is the same as $\sigma_{f(x)}$ notice that both are linear functions from V^* to $\mathbb{K}$. Hence we need to check that they act the same on any $\varphi \in V^*$.

$$\begin{array}{rcll} \sigma_x \circ f^*(\varphi) & = & \sigma_x \circ \varphi \circ f & \text{by definition of } f^* \\ & = & \varphi \circ f(x) & \text{by definition of } \sigma_x \\ & = & \sigma_{f(x)}(\varphi) & \text{by definition of } \sigma_{f(x)} \end{array}$$

Let $\mathfrak{A}, \mathfrak{B}$ be two categories with functors $F, G : \mathfrak{A} \to \mathfrak{B}$ and a natural transformation η from F to G.

Definition 37. *If for all $X \in \mathfrak{A}$ the component of that natural transformation η_X is an isomorphism then η is a natural isomorphism from F to G.*

In the category $\mathbf{Set}^{\mathfrak{A}}$, where $\mathfrak{A}$ is some small category, the objects are functors $F : \mathfrak{A} \to \mathbf{Set}$ and the morphisms are natural transformations. A natural isomorphism in $\mathbf{Set}^{\mathfrak{A}}$ is an isomorphism in that category.

3.2 Adjoint functors

Let $\mathfrak{A}$ and $\mathfrak{B}$ be categories with functors $F : \mathfrak{A} \to \mathfrak{B}$ and $G : \mathfrak{B} \to \mathfrak{A}$. We define the functor $\mathbf{Hom}_{\mathfrak{A}}G : \mathfrak{A}^{op} \times \mathfrak{B} \to \mathbf{Set}$ as follows:

1. Let (A, B) be an object in $\mathfrak{A}^{op} \times \mathfrak{B}$. Then

$$\mathbf{Hom}_{\mathfrak{A}}G\ (A, B) = \mathbf{Hom}_{\mathfrak{A}}(A, GB).$$

2. Let $(f, g) : (A, B) \to (A', B')$ be a morphism in $\mathfrak{A}^{op} \times \mathfrak{B}$, where $f : A' \to A$ and $g : B \to B'$. Then $\mathbf{Hom}_{\mathfrak{A}}G$ sends (f, g) to the morphism

$$\mathbf{Hom}_{\mathfrak{A}}G\ (f, g) \quad : \quad \mathbf{Hom}_{\mathfrak{A}}(A, GB) \to \mathbf{Hom}_{\mathfrak{A}}(A', GB')$$

in **Set** defined by, for each $a \in \mathbf{Hom}_{\mathfrak{A}}(A, GB)$,

$$[\mathbf{Hom}_{\mathfrak{A}}G\ (f, g)]\ (a) = Gg \circ a \circ f.$$

Then define a second functor $\mathbf{Hom}_{\mathfrak{B}}F : \mathfrak{A}^{op} \times \mathfrak{B} \to \mathbf{Set}$ similarly, so that for any object (A, B) in $\mathfrak{A}^{op} \times \mathfrak{B}$,

$$\mathbf{Hom}_{\mathfrak{B}}F\ (A, B) = \mathbf{Hom}_{\mathfrak{B}}(FA, B).$$

Definition 38. *The functor G is* adjoint *to F, and the functor F is* coadjoint *to G, if the functors $\mathbf{Hom}_{\mathfrak{A}}G$ and $\mathbf{Hom}_{\mathfrak{B}}F$ are naturally isomorphic.*

Lemma 39. *The adjoint of a functor, if it exists, is unique up to natural isomorphism.*

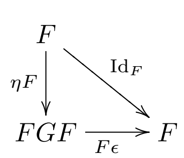
The same is true of the coadjoint.

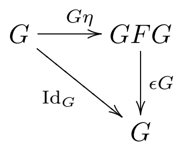
Lemma 40. *The functor F is adjoint to G precisely if there are natural transformations*

$$\eta : \mathrm{Id}_{\mathfrak{B}} \to FG$$

$$\epsilon : GF \to \mathrm{Id}_{\mathfrak{A}}$$

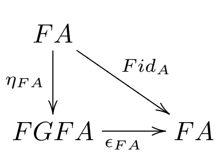
which satisfy the two triangle conditions $F\epsilon \circ \eta F = \mathrm{Id}_F$ *and* $\epsilon G \circ G\eta = \mathrm{Id}_G$*,*

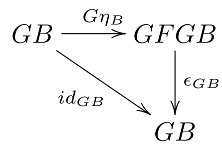




We call η the *unit*, and ϵ the *co-unit*, of the adjunction.

The triangle conditions, as stated in natural transformations, are shorthand for two more concrete families of identities in morphisms between objects. For any objects $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$, we must have $F\epsilon_A \circ \eta_{FA} = id_{FA}$ and $\epsilon_{GB} \circ G\eta_B = id_{GB}$:





Example 41. *Let* Free : **Set** → **Grp** *be the functor mapping a set to the free group on that set. Then* Free *is coadjoint to the forgetful functor* For : **Grp** → **Set**.

That is, there is a natural isomorphism between the functors

$$\mathbf{Hom}_{\mathbf{Set}}(\mathrm{For}) \quad \textit{and} \quad \mathbf{Hom}_{\mathbf{Grp}}(\mathrm{Free}).$$

The component morphisms of this natural isomorphism are, for each set X and each group G, the canonical bijection

$$\mathbf{Hom}(X, \mathrm{For}(G)) \to \mathbf{Hom}(\mathrm{Free}(X), G),$$

which arises since a homomorphism into G from $\mathrm{Free}(X)$ *is uniquely determined by a function into G from* $\mathrm{Free}(X)$*'s generating set X.*

Example 42. *An isomorphism is adjoint to its inverse.*

Example 43. *Let* Poly : **Ring** → **pRing** *be the functor mapping a commutative ring R with unity to the pointed commutative ring with unity $(R[X], X)$ of polynomials over R. Then* Poly *is coadjoint to the forgetful functor* For : **Ring** → **pRing**.

That is, there is a natural isomorphism between the functors

$$\mathbf{Hom}_{\mathbf{Ring}}(\mathrm{For}) \quad \textit{and} \quad \mathbf{Hom}_{\mathbf{pRing}}(\mathrm{Poly}).$$

The component morphisms of this natural isomorphism are, for each ring R and each pointed ring (S, s), the canonical bijection

$$\mathbf{Hom}(R, \mathrm{For}(S, s)) \to \mathbf{Hom}(\mathrm{Poly}(R), (S, s)),$$

which arises since a point-preserving ring homomorphism from $(R[X], X)$ into (S, s) is uniquely determined by a ring homomorphism from R into S.

Example 44. *Let* **CMet** *be the category of complete metric spaces. (It is a subcategory of the category* **Met** *of metric spaces.) Let* Fill : **Met** → **CMet** *be the functor mapping a metric space X to its completion $\overline{X}$. Then* Fill *is coadjoint to the inclusion functor ι :* **CMet** → **Met**.

The component morphisms of this natural isomorphism between the functors

$$\mathbf{Hom}_{\mathbf{Met}}(\iota) \quad \textit{and} \quad \mathbf{Hom}_{\mathbf{CMet}}(\mathrm{Fill}).$$

are, for each metric space X and each complete metric space Y, the canonical bijection

$$\mathbf{Hom}(X, Y) \to \mathbf{Hom}(\overline{X}, Y),$$

which arises since a uniformly continuous function from X into Y is uniquely determined by its (also uniformly continuous) restriction to X.

Example 45. *Let $\mathfrak{A}$ be a category. Let $\Delta : \mathfrak{A} \to \mathfrak{A} \times \mathfrak{A}$ be the* diagonal functor *mapping an object A to its self-pairing (A, A). Then Δ is coadjoint precisely if $\mathfrak{A}$ has finite products; if it does, the product functor* $\mathrm{Prod} : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ *is Δ's adjoint.*

The component morphisms of this natural isomorphism between the functors

$$\mathbf{Hom}_{\mathfrak{A}}(\mathrm{Prod}) \quad \textit{and} \quad \mathbf{Hom}_{\mathfrak{A}\times\mathfrak{A}}(\Delta).$$

are, for each triple A, A_1, A_2 of objects in $\mathfrak{A}$, the canonical bijection

$$\mathbf{Hom}(A, A_1 \times A_2) \to \mathbf{Hom}((A, A), (A_1, A_2)),$$

which arises since a morphism from A into $A_1 \times A_2$ is uniquely determined by its two component morphisms from A into A_1 and A_2 respectively.

Example 46. *Let $\mathfrak{A}$ be a category, and let $\mathbf{1}$ be the category with one object and one morphism. Consider the unique functor $! : \mathfrak{A} \to \mathbf{1}$. The functor **!** is coadjoint precisely if $\mathfrak{A}$ has a terminal object; if it does, then the adjoint of $!$ is the functor $T : \mathbf{1} \to \mathfrak{A}$ which maps $\mathbf{1}$'s one object onto $\mathfrak{A}$'s terminal object.*

Example 47. *The preceding two examples can be generalized to a statement for limits generally. Let $\mathfrak{A}$ be a category, $\mathfrak{J}$ be a small category, and $\mathbf{Diag}_{\mathfrak{J}}\mathfrak{A}$ the category of diagrams from $\mathfrak{J}$ into $\mathfrak{A}$. Consider the* diagonal functor

$$\Delta : \mathfrak{A} \to \mathbf{Diag}_{\mathfrak{J}}\mathfrak{A},$$

where Δ sends an object $A \in \mathfrak{A}$ to the diagram $\Delta A : \mathfrak{J} \to \mathfrak{A}$, which sends each object in $\mathfrak{J}$ to A and each morphism in $\mathfrak{J}$ to id_A.

Then Δ is coadjoint precisely if $\mathfrak{A}$ has finite $\mathfrak{J}$-limits. If it does, the $\mathfrak{J}$-limit functor $\mathfrak{J} - \mathrm{Lim} : \mathbf{Diag}_{\mathfrak{J}}\mathfrak{A} \to \mathfrak{A}$ is Δ's adjoint.

Lemma 48. *Adjoint functors preserve limits. Coadjoint functors preserve colimits.*

Example 49. *From Example 41, the free functor $F : \mathbf{Set} \to \mathbf{Grp}$ is coadjoint. It therefore preserves colimits. For this reason the coproduct of two free groups* $\mathrm{Free}(X), \mathrm{Free}(Y) \in \mathbf{Grp}$ *is their* free product*; that is, the free group generated by the disjoint union (coproduct in* **Set***) of their generating sets X and Y.*

3.3 Reflections

The adjointness of the completion functor from Example 44 can also be considerably generalized.

Definition 50. *A subcategory $\mathfrak{A}$ of a category $\mathfrak{B}$ is* reflective*, if the inclusion functor $\iota : \mathfrak{A} \to \mathfrak{B}$ is adjoint. It is* coreflective*, if the inclusion functor is coadjoint.*

Let $\mathfrak{A}$ be a reflective subcategory of $\mathfrak{B}$. We call the functor $R : \mathfrak{B} \to \mathfrak{A}$ coadjoint to ι the *reflector*, and the image of an object $B \in \mathfrak{B}$ under R its *reflection* in $\mathfrak{A}$.

Example 51. *The category* **Ab** *of abelian groups is a reflective subcategory of the category* **Grp** *of groups. The reflector is the functor $Q : \mathbf{Grp} \to \mathbf{Ab}$ which sends a group to its quotient by its commutator subgroup.*

To see this, observe that there is a canonical bijection, for each group G and abelian group A, between the sets of group homomorphisms

$$\mathbf{Hom}(G, A) \quad \textit{and} \quad \mathbf{Hom}(G/[G, G], A),$$

given since a homomorphism of G into the abelian group A must send each of G's commutators to 1_A.

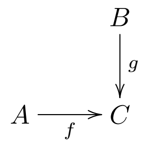
Example 52. *The fields (with morphisms the field embeddings) are a reflective subcategory of the category of integral domains. The reflector sends each integral domain to its field of fractions.*

4 Limits

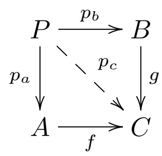
For the next few sections, the focus is on the theory of toposes. We start with a closer look at limits, which were first introduced in 2.5.

In general a limit is a cone over an arbitrary diagram such that all other cones have a unique morphism into the limit. So far we have only considered the limit over the empty diagram, a pair of objects and a parallel pair of morphisms which are the terminal object, the product and the equalizer respectively.

Another important example is the pullback which categorically expresses notions such as the inverse image and an equivalence relation. The pullback is a limit over the following diagram.

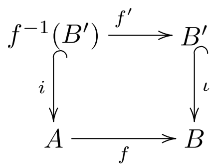


A pullback, P with morphisms p_x makes a commuting square as $f \circ p_a = p_c = g \circ p_b$. We typically leave out the diagonal and state that $f \circ p_a = g \circ p_b$.



Example 53. *In* **Set** *a standard example of a pullback for the diagram above is $P = \{(a, b) : f(a) = g(b)\}$ with corresponding projections.*

Example 54. *Let $f : A \to B$ be a function in* **Set**. *The inverse image of $B' \subseteq B$ is found by the pullback of the two functions f and the inclusion $\iota : B' \to B$,*



where $f^{-1}(B') = \{x \in A : f(x) \in B'\}$ and f' is the restriction of f to the domain $f^{-1}(B')$ and $i : f^{-1}(B') \to A$ is an inclusion.

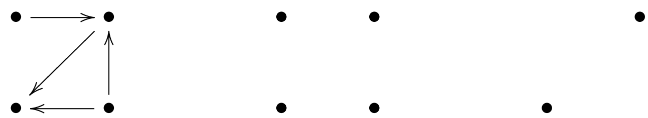
We do not need to establish the existence of each sort of limit individually. Once we know that a category has limits over any finite diagram, then we know that the category has a multitude of tools for expressing the interaction of its objects.

Theorem 55. *If the category $\mathfrak{A}$ has equalizers and all finite products then $\mathfrak{A}$ has limits over all finite diagrams.*

Proof. Let $\mathfrak{J}$ be a finite index category. A limit over the diagram $D : \mathfrak{J} \to \mathfrak{A}$ can be constructed as the equalizer over a suitable pair of morphisms between two products; the product of the objects in D and the product of all the objects which are codomains in D.

As $\mathfrak{J}$ is a finite category let $Ob\mathfrak{J}$ be the finite set of objects in $\mathfrak{J}$.

Let $Ob\mathfrak{J}$ be the index category consisting only of objects from $\mathfrak{J}$ with no morphisms. Construct a second index category $Co\mathfrak{J}$ that contains a subset of the objects of $Ob\mathfrak{J}$. In particular $Co\mathfrak{J} = \{u : \exists n \neq u \in Ob\mathfrak{J}, \mathbf{Hom}_{\mathfrak{J}}(n, u) \neq \emptyset\}$. $Co\mathfrak{J}$ is the index category with all of the codomain objects of non-identity morphisms from $\mathfrak{J}$ as illustrated by the following diagram.



$\mathfrak{J}$ $\qquad$ $Ob\mathfrak{J}$ $\qquad$ $Co\mathfrak{J}$

Objects and morphisms in the diagram D will be denoted Dn and $D(f)$ respectively where n is an object and f a morphism in $\mathfrak{J}$. Throughout this proof n is used for an object in $Ob\mathfrak{J}$ while u denotes an object in $Co\mathfrak{J}$.

Define two products; the first over all the objects of D which are indexed by $Ob\mathfrak{J}$ and the second over all the codomain objects of D indexed by $Co\mathfrak{J}$.

$$\prod Ob\mathfrak{J} = \prod_{n \in Ob\mathfrak{J}} Dn \quad \text{with morphisms } (O_n : \prod Ob\mathfrak{J} \to Dn)_{n \in Ob\mathfrak{J}}$$

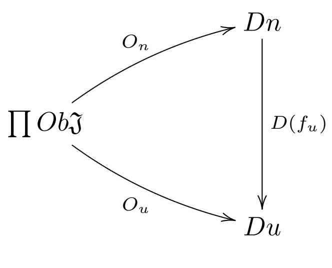
$$\prod Co\mathfrak{J} = \prod_{u \in Co\mathfrak{J}} Du \quad \text{with morphisms } (C_u : \prod Co\mathfrak{J} \to Du)_{u \in Co\mathfrak{J}}$$

Construct two morphisms, $r, s : \prod Ob\mathfrak{J} \to \prod Co\mathfrak{J}$ as follows.

By definition, for each $u \in Co\mathfrak{J}$ there exists some morphism $f_u : n \to u$ in $\mathfrak{J}$ where $n \neq u \in Ob\mathfrak{J}$. For each $u \in Co\mathfrak{J}$ let $r_u, s_u : \prod Ob\mathfrak{J} \to Du$ be defined as

(1) $\quad r_u = D(f_u) \circ O_n$

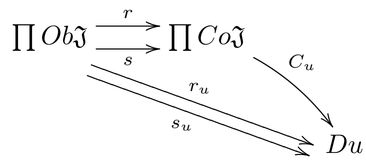
(2) $\quad s_u = O_u$



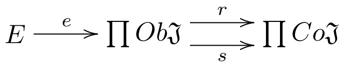
For each $u \in Co\mathfrak{J}$ there exists $r_u, s_u : \prod Ob\mathfrak{J} \to Du$ hence there exists $r =< .., r_u, .. >$, $s =< .., s_u, .. >: \prod Ob\mathfrak{J} \to \prod Co\mathfrak{J}$ which commute as follows.

(3) $C_u \circ r = r_u$

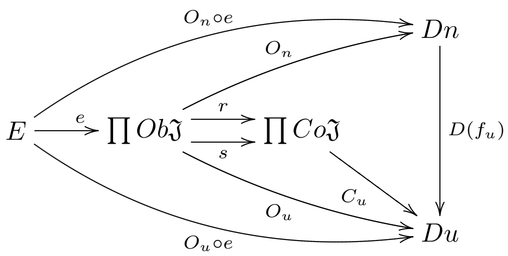
(4) $C_u \circ s = s_u$



Given the parallel pair of morphisms r, s there exists an equalizer, E with morphism $e : E \to \prod Ob\mathfrak{J}$.



What remains to be shown in that E is the limit over the diagram D with the family of morphisms $(O_n \circ e : E \to Dn)_{n \in Ob\mathfrak{J}}$.

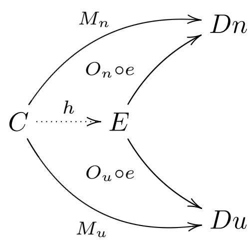


First we require that E is a cone over the diagram. This means the morphisms associated with E must commute with morphisms in the diagram.

Given $D(f_u) : Dn \to Du$ we require $D(f_u) \circ O_n \circ e = O_u \circ e$.

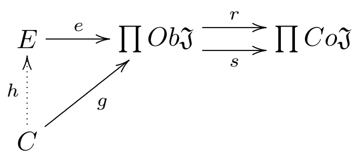
$$\begin{aligned} D(f_u) \circ O_n \circ e &= C_u \circ r \circ e && \text{by (1) and (3)} \\ &= C_u \circ s \circ e && \text{by definition of an equalizer} \\ &= O_u \circ e && \text{by (2) and (4)} \end{aligned}$$

Secondly, if E is a limit cone then any other cone over the diagram D must factor through E. Let C be an arbitrary cone over D with morphisms $(M_n : C \to Dn)_{n \in Ob\mathfrak{J}}$. We need to show that there exists a unique morphism $h : C \to E$ such that $O_n \circ e \circ h = M_n$ for all $n \in Ob\mathfrak{J}$.



Using the properties of E as an equalizer of r and s we can show that the unique h must exist.

If there is a morphism $g : C \to \prod Ob\mathfrak{I}$ such that $s \circ g = r \circ g$ then the unique $h : C \to E$ exists by the property of the equalizer E. It remains to be shown that such a g exists.



As C is a cone over the diagram indexed by $\mathfrak{I}$ it is also a cone over the diagram indexed by $Ob\mathfrak{I}$. By the property of the limit $\prod Ob\mathfrak{I}$ there exists a unique morphism $g : C \to \prod Ob\mathfrak{I}$ such that

$$(5) \quad O_n \circ g = M_n.$$

To verify that $s \circ g = r \circ g$ it is sufficient to check that $C_u \circ s \circ g = C_u \circ r \circ g$ for each $u \in Co\mathfrak{I}$.

$$\begin{aligned} C_u \circ s \circ g &= O_u \circ g && \text{by (4) and (2)} \\ &= M_u && \text{by (5)} \\ &= D(f_u) \circ M_n && \text{by property of the cone } C \\ &= D(f_u) \circ O_n \circ g && \text{by (5)} \\ &= C_u \circ r \circ g && \text{by (3) and (1)} \end{aligned}$$

Therefore the unique morphism $h : C \to E$ exists by the property of the equalizer such that $g = e \circ h$. This guarantees that $O_n \circ e \circ h = M_n$ so the cone C factors through E, and E is the limit over D. □

This is a useful theorem to guarantee the existence of finite limits. Equivalent results are possible in terms of different sorts of limits.

Lemma 56. *If a category $\mathfrak{A}$ has finite products and pullbacks, then $\mathfrak{A}$ has finite limits.*

This result follows from above because an equalizer of $f, g : A \to B$ is also a pullback of f and g. We can just rephrase the previous result in terms of pullbacks instead of equalizers.

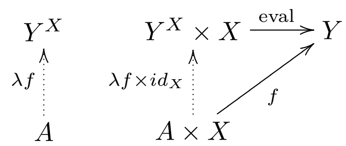
Once we have guaranteed the existence of finite limits in a category we can do many diagram chasing proofs like the one given above.

5 Some Set-like categorical properties

5.1 Exponential objects and evaluation morphisms

The exponential object and evaluation morphism allow us to curry a function. Given some $f : A \times X \to Y$ the curried function will be $\lambda f : A \to (X \to Y)$. This allows us to change from a function of two variables to a single variable.

Definition 57. *Let X and Y be objects in the category $\mathfrak{A}$. An* exponential *of Y by X is an object Y^X along with a morphism* $\text{eval} : Y^X \times X \to Y$ *such that for any object A and morphism $f : A \times X \to Y$ there exists a unique morphism $\lambda f : A \to Y^X$ such that the following triangle commutes,*



An exponential is unique up to isomorphism.

Example 58. *Given X and Y in* **Set** *the functions from X to Y form a set which is the object Y^X. The morphism* eval *is defined by* $\text{eval}(f, x) = f(x)$ *where $f : X \to Y$ and $x \in X$.*

The same construction is valid in **FinSet**. There are only finitly many functions between two finite sets and the eval function is still defined.

Example 59. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two categories in the category of all small categories. The functor category $\mathfrak{B}^{\mathfrak{A}}$ is the exponential object with standard* eval *morphism.*

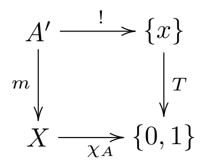
5.2 Subobject classifiers

In set theory a subset is equivalent to a characteristic function. The analogous tool for categories is to characterise subobjects using a specific pullback.

Recall that a subobject of X is an equivalence class of monomorphisms into X. Two monomorphisms are equivalent if each dominates the other.

Let $S \subseteq X$ then the characteristic function $\chi_S : X \to \{0, 1\}$ is defined as $\chi(x) = 1$ if $x \in S$ and 0 otherwise.

Conversely given a characteristic function $\chi_A : X \to \{0, 1\}$ we can determine the subset $A \subseteq X$ using a pullback. Define $T : \{x\} \to \{0, 1\}$ such that $T(x) = 1$.



As $\{x\}$ is a terminal object in **Set** then ! is the unique function from A' into $\{x\}$. Furthermore as T is monic, by a property of pullbacks m must also be monic.

The pullback means that for any $a \in A'$,

$$T \circ !(a) = T(x) = 1 = \chi_A \circ m(a)$$

The elements in the image of m form the subset of X for which χ_A takes the value 1. As other monics $m' : A'' \to X$ satisfy the pullback above $[m]$ is the subobject of X.

The equivalence class of monics, $[m]$ is not the same as the subset A but from a categorical perspective they interact with χ_A in the same way.

Definition 60. *Let* $\mathfrak{A}$ *be a category with terminal object* $\mathbf{1}$*. A* subobject classifier *is an object* Ω *along with the monic* $T : \mathbf{1} \to \Omega$ *such that for any monic* $m : A \to X$ *there exists a unique morphism* $\chi_m : X \to \Omega$ *such that the following is a pullback.*

$$\begin{array}{ccc} A & \xrightarrow{!} & \mathbf{1} \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle T} \\ X & \xrightarrow[\chi_m]{} & \Omega \end{array}$$

The equivalence class of monics $[m]$ *is the subobject of* X *corresponding to* χ_m*.*

In **Set**, **FinSet** and **Grp** the subobject classifier is simply {0,1} with the standard characteristic functions.

Lawvere refers to Ω as the truth value object and the morphism T as a way to single out the value true from Ω.

6 Toposes

We are now equipped to present the axioms for when a category is a topos.

The origin of topoi in category theory came from the Grothendieck school of algebraic geometry. In the 1950s Grothendieck had introduced the idea of an abelian category to unify the categories of sheaves of abelian groups in his work on homological algebra. Turning to the categories of sheaves of sets in the 1960s, these were captured by the concept of a Grothendieck topos.

The topoi we are looking at are not all Grothendieck topoi but came out of Lawvere's work. Lawvere focused on the existence of a truth value object in each Grothendieck topos and their connection to a Grothendieck topology. Subsequently he developed a concept of a topos in the 1960s in terms of the existence of a subobject classifier which is like a truth value object. This was originally referred to as an elementary topos but is now just called a topos.

A motivating factor in Lawvere's work with topoi was the desire to give a categorical version of Cohen's proof of the independence of the continuum hypothesis from the axioms of set theory. This seemed plausible due to connections between Cohen's forcing technique and sheaf theory which is tied up with Grothendieck topoi.

We will give some common examples and counterexamples, as well as methods for constructing new topoi from old ones.

Definition 61. *A category* $\mathfrak{A}$ *is a topos if it has;*

1. *Limits over all finite diagrams;*
2. *For all objects* A, B *in* $\mathfrak{A}$ *an exponential object* A^B *with evaluation morphisms* $\mathrm{eval} : A^B \times B \to A$*;*

3. *A subobject classifier with the object* Ω *and morphism* $T : \mathbf{1} \to \Omega$.

Example 62. *The categories* **Set** *and* **FinSet** *are topoi. The category of countable sets and functions between them,* **CountSet**, *is not a topos because it lacks exponential objects.*

Given the sets $\{0,1\}$ and ω in **CountSet**, assume the exponential object $\{0,1\}^\omega$ exists. Then $Hom(\mathbf{1}, \{0,1\}^\omega)$ would be a countable set. That is a contradiction because $Hom(\mathbf{1}, \{0,1\}^\omega)$ is isomorphic to the uncountable set $Hom(\omega, \{0,1\})$.

Example 63. *The category* **G-Set** *is a topos. Recall that the objects are sets,* X *under the group action of* G *such that* $h \cdot (g \cdot x) = (h \cdot g) \cdot x$ *where* $h, g \in G$ *and* $x \in X$.

Products and equalizers are the same in **G-Set** as in **Set** because they preserve G-actions. Let $m, n : X \to Y$ be two G-maps and consider the standard equalizer from **Set**, $E = \{x \in X : m(x) = n(x)\}$. If E is an object in **G-Set** then it must be closed under G-actions.

Let $x \in E$, then by definition of the G-maps and the equalizer $m(g \cdot x) = g \cdot m(x) = g \cdot n(x) = n(g \cdot x)$. Therefore $g \cdot x \in E$ and E is closed under G-actions. Similarly products are closed under G-actions so **G-Set** has finite limits.

Given the G-Sets X and Y, the exponential object Y^X is the set of all G-maps $f : X \to Y$. If Y^X is an object in **G-Set** then a G-action has to be defined on the maps. Let $f \in Y^X$ and $g \in G$ then define gf to be the map which acts on $x \in X$ by

$$x \longmapsto gf(g^{-1}x).$$

This is a G-action because for $g, h \in G$,

$$\begin{aligned} h(gf)(x) &= h(gf(h^{-1}x)) \\ &= hg(f(g^{-1}h^{-1}x)) \\ &= ((hg)f)(x). \end{aligned}$$

Lastly the subobject classifier is $\Omega = \{0,1\}$ with the trivial group action defined on it and the morphism $T : \mathbf{1} \to \{0,1\}$ which picks out the value 1.

The category **G-Set** is only one example of how to generate new topoi from old ones.

Lemma 64. *If* $\mathfrak{T}_1$ *and* $\mathfrak{T}_2$ *are topoi then the cartesian product* $\mathfrak{T}_1 \times \mathfrak{T}_2$ *is a topos.*

Proof. Finite limits, exponentials and subobject classifer are defined component wise. In particular $(Y_1, Y_2)^{X_1, X_2} = (Y_1^{X_1}, Y_2^{X_2})$ and the subobject classifier is (Ω_1, Ω_2). □

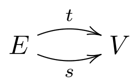
Theorem 65. *For a small category* $\mathfrak{A}$*, the functor category* $\mathbf{Set}^{\mathfrak{A}}$ *is a topos.*

We will give two examples to illustrate this result. Recall that in a functor category the objects are functors $F : \mathfrak{A} \to \mathbf{Set}$ and the morphisms are natural transformations between functors.

The category **G-Set** can be expressed as a functor category $\mathbf{Set}^{\mathbf{G\text{-}m}}$ where **G-m** is a monoid whose morphisms correspond to elements of the group G. A functor from **G-m** to **Set** is like a group action on a set.

The second illustration of the functor category construction is given by the category of directed multi-graphs.

Let **Gr** be the category with two objects, V, E and two non-identity morphisms.



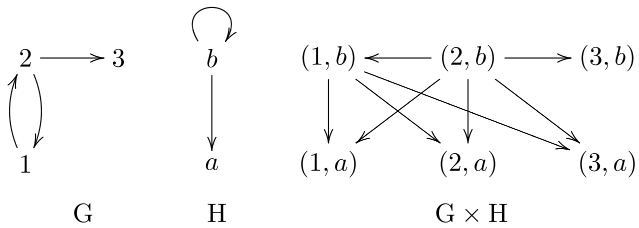
The category **Gr** is like a graph with vertice and edge sets and a way of assigning a source and target vertex to each edge.

The category of all directed multi-graphs is equivalent to the functor category $\mathbf{Set}^{\mathbf{Gr}}$. As **Gr** is a small category, $\mathbf{Set}^{\mathbf{Gr}}$ is a topos.

Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be graphs in $\mathbf{Set}^{\mathbf{Gr}}$.

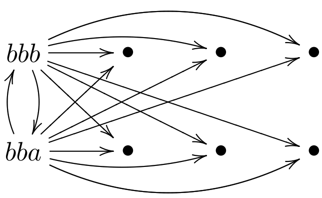
The product $G \times H = (V_G \times V_H, E_G \times E_H)$ such that for (x, y) and (x', y') in $V_G \times V_H$ we have (x, y) is adjacent to (x', y') if $(x, x') \in E_G$ and $(y, y') \in E_H$.

Consider the following example of two graphs G and H and their product.



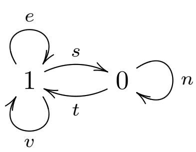
The exponential graph $H^G = (Hom(V_G, V_H), E)$ such that f and g in $Hom(V_G, V_H)$ are adjacent when $(x, x') \in E_G$ implies $(f(x), g(x')) \in E_H$.

Following on from the example above, let $f \in Hom(V_G, V_H)$ be represented as $f(1)f(2)f(3)$, then this is the graph of H^G.



The terminal object is the graph with one vertex and only the identity morphism.

The subobject classifier is the following graph with morphism $T : \mathbf{1} \to \Omega$ that picks out e.



The subobject classifier gives the different possibilities for the relationships between vertices and edges. For the subgraph $m : S \to G$ the classifing map $\chi_m : G \to \Omega$ acts as follows;

- If a vertex is in S it is mapped to 1, otherwise mapped to 0;
- An edge in S is mapped to e;
- An edge that is not in S can be mapped to four possibilities;
  - (v) If the source and target vertices are in the subgraph;
  - (s) If the source vertex is in the subgraph;
  - (t) If the target vertex is in the subgraph;
  - (n) If neither the edge nor vertices are in the subgraph.

Example 66. *A partially ordered set can be treated as a category. If it has no greatest element then there is no terminal object. In this case finite limits are not defined and it is not a topos.*

Example 67. *It is not always possible to put a topology on the set of continuous functions between two topologies. This means* **Top** *lacks exponential objects and is not a topos.*

Example 68. *The category* **Ab** *is not a topos because there does not exist a subobject classifier.*

The terminal object **1** in **Ab** is the zero group. The group homomorphism $T : \mathbf{1} \to \Omega$ must send the zero group to $0 \in \Omega$. Given any $\phi : A \to \Omega$ the pullback must give $Ker(\phi) = \phi^{-1}(0)$ as the subgroup of A.

$$\begin{array}{ccc} Ker(\phi) & \overset{!}{\dashrightarrow} & \mathbf{1} \\ \downarrow & & \downarrow T \\ A & \underset{\phi}{\longrightarrow} & \Omega \end{array}$$

Hence Ω must be an abelian group with a copy of every quotient $A/Ker(\phi)$ for every abelian group A which is impossible.

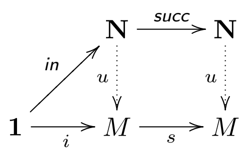
The categories which are topoi have the ability to express a lot of mathematical concepts and operations. For this reason topoi are sometimes presented as an alternative foundation for mathematics. We will now look at the possibility of expressing concepts about the natural numbers in topoi.

7 Natural numbers, integers and rationals

In the category **Set**, $\mathbb{N}$ is an object but the natural number 3 is not. Categorically the singleton $\{3\}$ is indistinguishable from $\{\pi\}$. It is not possible to use the elements to give a categorical definition of $\mathbb{N}$. Instead we should characterise $\mathbb{N}$ in terms of morphisms.

In set theory the natural numbers can be generated by defining an initial element and a successor relation. A similar idea holds in category theory; a morphism from a terminal object is like selecting an initial element and a successor relation is just a particular non-identity morphism from an object to itself.

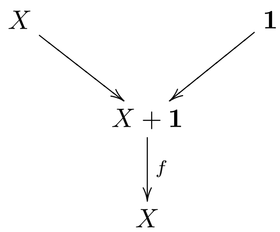
Definition 69. *A* natural number object *in* $\mathfrak{T}$ *is an object* $\mathbf{N}$ *along with two morphisms* $\mathsf{in} : \mathbf{1} \to \mathbf{N}$ *and* $\mathsf{succ} : \mathbf{N} \to \mathbf{N}$ *such that for any other object* M *with morphisms* $i : \mathbf{1} \to M$ *and* $s : M \to M$ *there exists a unique morphism* $u : \mathbf{N} \to M$ *making the following diagram commute.*



Not surprisingly the set of the natural numbers is the natural number object in the topos **Set**. In any category a natural number object is unique up to isomorphism.

To define the conditions for having a natural number object we need to specify which morphisms will accompany it. Thinking back to set theory, the property that the natural numbers form an infinite set can be captured by the existence of an isomorphism from $\mathbb{N} \cup \{x\}$ to $\mathbb{N}$. Freyd used the idea of an appropriate isomorphism to give the condition for a natural number object in a topos.

Theorem 70. *If there exists an object* X *in* $\mathfrak{T}$ *such that there is an isomorphism* f *from the coproduct of* X *and the terminal object* $\mathbf{1}$ *to* X *then* X *is the natural number object in* $\mathfrak{T}$.



The coproduct of X and $\mathbf{1}$ is like taking their disjoint union which suggests we have added something to X. Yet the isomorphism f attests to the fact that $X + \mathbf{1}$ is still no different to X. This expresses the notion of an infinite object categorically.

A category which is not a topos may have a natural number object. What is significant about having a natural number object in a topos is that we have the tools of finite limits at our disposal. This makes it possible to construct an integer object and a rational number object using similar ideas to the set theoretic constructions.

Example 71. *The set theoretic construction of the integers is given by*

$$\mathbb{Z} = \{(n, m) : n, m \in \mathbb{N}\}/ \sim,$$

where $(n, m) \sim (n', m')$ *if and only if* $n + m' = n' + m$. *The equivalence class* $[(n, m)]$ *represents the integer* $n - m$.

In category theory a pullback is used to define an equivalence relation and a coequalizer is used to quotient.

Let $\mathbf{N}$ be the natural number object in a topos $\mathfrak{T}$. Let $+ : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ be the additive morphism that is defined recursively using the property of the natural number object.

Let E be the equivalence relation given by the following pullback.

$$\begin{array}{ccc} E & \xrightarrow{a} & \mathbf{N} \times \mathbf{N} \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle +} \\ \mathbf{N} \times \mathbf{N} & \xrightarrow[+]{} & \mathbf{N} \end{array}$$

In **Set** E would be $\{(n, m, n', m') : +(n, m') = +(n', m)\}$.

Define two morphisms $p, p' : E \to \mathbf{N} \times \mathbf{N}$. In **Set** p and p' would act on E by,

$$p : (n, m, n', m') \longmapsto (n, m),$$

$$p' : (n, m, n', m') \longmapsto (n', m').$$

We define the integer object $\mathbf{Z}$ as the coequalizer of the morphisms p and p'. This quotients by the relation $\sim$.

$$E \underset{p'}{\overset{p}{\rightrightarrows}} \mathbf{N} \times \mathbf{N} \longrightarrow \mathbf{Z}$$

In a general category we do not know explicitly what p and p' are. However we can give their construction in terms of the morphisms from the pullback and product already defined. Then in any category p and p' behave in the same way, and their coequalizer is the integer object of that category.

First from the pullback we have the morphisms a and b which in **Set** were projections of a 4-tuple onto a pair. Secondly, as $\mathbf{N} \times \mathbf{N}$ is a product in the topos it comes with projections $\pi_1, \pi_2 : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$. In **Set** these give the first and second elements of an ordered pair.

Categorically p and p' can be constructed such that $p =< \pi_1 \circ a, \pi_2 \circ b >$ and $p' =< \pi_1 \circ b, \pi_2 \circ a >$.

Using pullbacks and coequalizers in a similar way a rational number object $\mathbf{Q}$ can be constructed from $\mathbf{N}$ and $\mathbf{Z}$.

Example 72. *In the set theoretic construction the rationals are given by*

$$\{(z, n) : z \in \mathbb{Z}, n \in \mathbb{N}\}/\sim,$$

where $(z, n) \sim (z', n')$ *if and only if* $z(n' + 1) = z'(n + 1)$. *The equivalence class* $[(z, n)]$ *represents the rational* $\frac{z}{n+1}$.

Again in the categorical situation a coequalizer will be used to quotient an equivalence relation constructed by a pullback.

In particular given that $\mathbf{N}$ and $\mathbf{Z}$ are objects in $\mathfrak{T}$ the product $\mathbf{Z} \times \mathbf{N}$ exists with morphisms $\pi_1 : \mathbf{Z} \times \mathbf{N} \to \mathbf{Z}$ and $\pi_2 : \mathbf{Z} \times \mathbf{N} \to \mathbf{N}$.

Let $m : \mathbf{Z} \times \mathbf{N} \to \mathbf{Z}$ be a morphism which acts like the multiplication of an integer with a natural number. Then $m \circ (Id_{\mathbf{Z}} \times \mathsf{succ})$ acts like multiplication of an integer with the successor of a natural number.

Define E as the object given by the following pullback,

$$\begin{array}{ccc} E & \xrightarrow{\quad a \quad} & \mathbf{Z} \times \mathbf{N} \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle m \circ (Id_{\mathbf{Z}} \times \mathsf{succ})} \\ \mathbf{Z} \times \mathbf{N} & \xrightarrow[m \circ (Id_{\mathbf{Z}} \times \mathsf{succ})]{} & \mathbf{Z} \end{array}$$

In **Set** E is $\{(z, n', z', n) : z(n'+1) = z'(n+1)\}$.

Define the rational number object $\mathbf{Q}$ as the coequalizer of the following diagram,

$$E \underset{<\pi_1 \circ b, \pi_2 \circ a>}{\overset{<\pi_1 \circ a, \pi_2 \circ b>}{\rightrightarrows}} \mathbf{Z} \times \mathbf{N} \longrightarrow \mathbf{Q}$$

In **Set** the construction of the integer and rational object is the categorical translation of the set theoretic construction. However the advantage of expressing it categorically is that the same result will hold in any other topos with a natural number object.

8 The effective topos

8.1 Definitions

The effective topos, **Eff** was first described by J. M. E. Hyland in 1982. It is based on the idea of Kleene's recursive realizability.

We will give a brief definition of **Eff** before turning to the subject of analysis. Hyland claims that analysis in **Eff** is essentially constructive real analysis similar to the Markov school. We will give one theorem which holds in **Eff** but not in classical analysis.

Objects in **Eff** are pairs $(X, \sim)$ consisting of a set X with a map $\sim : X \times X \to P(\mathbb{N})$.

For $(x, y) \in X$ the image under the map $\sim$ is denoted $(x \sim y)$. The natural numbers in $(x \sim y)$ index partial functions. These partial functions realize something about the similarity of x and y.

Let ϕ_e denote the partial function from $\mathbb{N}$ to $\mathbb{N}$ computed by the e^{th} Turing machine. Define $\phi_e(n) \downarrow$ to mean that ϕ_e halts on input n.

Let $\langle -, - \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a pairing function.

The map $\sim$ must satisfy conditions for symmetry and transitivity. Namely there exists s and t in $\mathbb{N}$ such that;

1. $n \in (x \sim y) \Rightarrow \phi_s(n) \downarrow \wedge \phi_s(n) \in (y \sim x)$
2. $n \in (x \sim y) \wedge m \in (y \sim z) \Rightarrow \phi_t(\langle n, m\rangle) \downarrow \wedge \phi_t(\langle n, m\rangle) \in (x \sim z)$.

Example 73. *The natural number object is* $\mathbf{N} = (\mathbb{N}, \sim)$*, where* $(x \sim y) = \{x\} \cap \{y\}$.

A morphism from $(X, \sim)$ to $(Y, \sim)$ is an equivalence class of strict, single valued, relational and total maps $M : X \times Y \to P(\mathbb{N})$. A full definition can be found in [Oos08].

The subobject classifier is larger than previous examples to account for the different possible subsets of $\mathbb{N}$ which contain the indexes for the significant partial functions.

In particular we have $\Omega = (P(\mathbb{N}), \sim)$ where

$$(X \sim Y) = \{\langle e_0, e_1\rangle : \forall x \in X \phi_{e_0}(x) \in Y \wedge \forall y \in Y \phi_{e_1}(y) \in X\}.$$

8.2 Analysis in the effective topos

There are two set theoretic constructions of the real numbers using Cauchy sequences or Dedekind cuts. Classically the Dedekind reals are isomorphic to the Cauchy reals. A Dedekind real number object and a Cauchy real number object can be constructed in any topos with a natural number object. In a topos the Dedekind reals do not necessarily coincide with the Cauchy reals.

Lemma 74. *In* **Eff** *the object of the Dedekind reals is isomorphic to the Cauchy reals object.*

There is only one real number object up to isomorphism in **Eff**. Let **R** denote this real number object.

The following result in **Eff** contradicts the classical Bolzano-Weierstrass theorem.

Theorem 75. *There exists a bounded monotone sequence of rationals in* **Eff** *which has no limit in* **R**.

Proof. Let $f : \mathbb{N} \to \mathbb{N}$ be an injective function which enumerates the halting set $\{(e, x) : \phi_e(x) \downarrow\}$ without repetition. Using f we will construct a bounded monotone sequence $(r_n)_{n \in N}$ of rationals defined below.

$$r_n = \sum_{i=0}^{n} 2^{-f(i)}$$

Assume for a contradiction that $(r_n)_{n \in N}$ has a limit L.

$$\forall k, \exists N_k, \forall n > N_k \quad |r_n - L| < 2^{-k}$$

There exists a computable function $g(k) = M_k$ where $M_k > N_k$ by countable choice such that,

$$\forall k, \forall n > g(k) \quad |r_n - L| < 2^{-k}$$

Given $k \in \mathbb{N}$, if there exists an i such that $f(i) = k$ then for any $n < i$ we have $r_n + 2^{-k} \leq r_i < L$ hence $|r_n - L| > 2^{-k}$.

For all $k \in \mathbb{N}$ if $f(i) = k$ then $i \leq g(k)$. This contradicts the fact that the image of f is undecidable. □

Classical mathematics will not always be found in a topos as **Eff** demonstrates. This concludes the discussion of topoi.

9 Zero objects

Definition 76. *A* zero object *of a category* $\mathfrak{C}$ *is an object* $O \in \mathfrak{C}$ *such that, for each object* $A \in \mathfrak{C}$*, there is precisely one morphism in* $\mathbf{Hom}(A, 0)$ *and precisely one morphism in* $\mathbf{Hom}(0, A)$*.*

Equivalently, a zero object of $\mathfrak{C}$ is an object that is both terminal and initial. A zero object of $\mathfrak{C}$ is unique up to isomorphism.

Suppose that $\mathfrak{C}$ has a zero object O. For any pair of objects A and B in $\mathfrak{C}$, composing the unique morphisms $f : A \to O$ and $g : O \to B$ yields a distinguished morphism $g \circ f : A \to B$. We call this the *zero morphism* from A to B, and denote it 0_{AB}. It is clear that the zero morphism is well-defined, independent of the choice of zero object.

For the rest of this section, we work in a category $\mathfrak{C}$ with zero object O (and hence with zero morphisms).

Lemma 77 (Composition with zero gives zero)**.** *Let* A *and* B *be objects in* $\mathfrak{C}$*. Then for any morphism* $f : B \to D$*,*

$$f \circ 0_{AB} = 0_{AD}.$$

Likewise, for any morphism $f : D \to A$*,*

$$0_{AB} \circ g = 0_{BD}.$$

Definition 78. *Let* $f : A \to B$ *be a morphism in* $\mathfrak{C}$*. A* kernel *(respectively,* cokernel*) of* f *is an equalizer (respectively, coequalizer) of* f *with* 0_{AB}*.*

From our earlier work on equalizers, a morphism's kernel, if it exists, is unique up to equivalence of subobjects. Likewise a cokernel is unique up to equivalence of quotient objects.

Lemma 79. *A morphism* $f : A \to B$ *is monic, precisely if it has kernel the unique morphism* $0 : O \to A$*. It is epic, precisely if it has cokernel the unique morphism* $0 : B \to O$*.*

Lemma 80. *Let* A *be an object in* $\mathfrak{C}$*. Then* $id_A : A \to A$ *is a kernel of the unique morphism* $0 : A \to O$*, and a cokernel of the unique morphism* $0 : O \to A$*.*

Lemma 81. *Let* $f : A \to B$ *be a morphism in* $\mathfrak{C}$*. Suppose that* $k : K \to A$ *is a kernel of* f*,* $l : A \to L$ *is a cokernel of* k*, and* $m : M \to A$ *is a kernel of* l*. Then* m *is a kernel of* f*.*

Proof. Repeated applications of the definition of kernel and cokernel. □

10 Abelian categories

10.1 Axioms

Definition 82. *An* abelian category *is a category $\mathfrak{C}$ satisfying the following conditions.*

1. *$\mathfrak{C}$ has a zero object.*
2. (i) *Each pair of objects in $\mathfrak{C}$ has a product.*
   (ii) *Each pair of objects in $\mathfrak{C}$ has a coproduct.*
3. (i) *Each morphism in $\mathfrak{C}$ has a kernel.*
   (ii) *Each morphism in $\mathfrak{C}$ has a cokernel.*
4. (i) *Each monomorphism in $\mathfrak{C}$ is some morphism's kernel.*
   (ii) *Each epimorphism in $\mathfrak{C}$ is some morphism's cokernel.*

Example 83. *The category* **Ab** *of abelian groups and group homomorphisms is abelian. More generally, for any unitary ring R, the category R-***Mod** *of unitary R-modules and module homomorphisms is abelian. The zero object is the module* (0).

Example 84. *The category* **Grp** *of groups and group homomorphisms is not abelian. It satisfies all axioms except for 4(i): Not all its monomorphisms are kernels, since not all subgroups are normal.*

Example 85. *The category* **pCompHaus** *of pointed compact Hausdorff spaces and point-preserving continuous maps is not abelian. It satisfies all axioms except for 4(ii).*

Example 86. *Consider the category whose objects are the smooth vector bundles on a manifold M, and whose morphisms are smoothly varying families of linear maps between fibres. This category is not abelian; it satisfies all axioms except for 4(i).*

Indeed, the monomorphisms of this category are the morphisms which restrict to injective linear maps on a dense subset of M. The kernels are a strict subclass of this: they are the morphisms which restrict everywhere to injective linear maps.

Example 87. *The category* **CompHausAb** *of compact Hausdorff abelian topological groups and continuous group homomorphisms is an abelian category. Objects in this category include, for instance,*

- *The finite abelian groups, equipped with the discrete topology.*
- *The torus groups. That is, the circle $S = \mathbb{R}/\mathbb{Z}$, and more generally the products of arbitrarily many copies of S.*
- *For each prime p, the p-adic integers.*

Example 88. *The category of sheaves of abelian groups on a topological space X is an abelian category.*

10.2 Elementary properties

In this section we explore the properties of a fixed abelian category $\mathfrak{G}$.

Theorem 89. *A monic, epic morphism is an isomorphism.*

Proof. Let $f : A \to B$ be both monic and epic. Since f is a monomorphism, Axiom 4(i) implies that it is the kernel of some morphism $g : B \to C$. Since f is an epimorphism, it has cokernel the unique morphism $0 : B \to O$, and, as observed in the section on zero objects, the morphism $0 : B \to O$ has kernel $id_B : B \to B$.

Since id_B is a kernel of 0 is a cokernel of f is a kernel of g, it follows (by Lemma 81) that id_B is a kernel of g.

The uniqueness of kernels therefore implies that id_B and f are equivalent subobjects of B. That is, there exists a pair of mutually inverse isomorphisms $x : A \to B$ and $y : B \to A$, such that $id_B \circ x = f$ and $f \circ y = id_B$. From the former we conclude that $x = f$. Hence f is an isomorphism. □

This proof uses only Axioms 1 and 4(i). (A dual proof using only Axioms 1 and 4(ii) works equally well.) Theorem 89 is therefore also valid in, for instance, the categories **Grp** and **pCompHaus**.

Let us single out one idea from the above proof for future use.

Lemma 90 (Ker-Coker duality)**.** *$g : B \to C$ is a cokernel of a monomorphism $f : A \to B$, then f is a kernel of g. If $f : A \to B$ is a kernel of an epimorphism $g : B \to C$, then g is a cokernel of f.*

Proof. Suppose $g : B \to C$ is a cokernel of a monomorphism $f : A \to B$. By Axiom 4(i), there is some morphism $r : B \to R$ of which f is a kernel. By Axiom 3(i), the morphism g has some kernel $k : K \to B$.

We therefore have that k is a kernel of g is a cokernel of f is a kernel of r. Hence (Lemma 81) k is a kernel of r. So k and f are equivalent as subobjects of B, and so f is a kernel of g.

The proof of the dual statement is similar. □

Rather suggestively, we call a pair (f, g) of morphisms satisfying either of the two (equivalent) conditions in Lemma 90 an *exact sequence.* (However, the concept will not be pursued any further in this report.)

Definition 91. *An* intersection *(respectively,* union*) of two subobjects of an object A, is a greatest lower bound (respectively,* least upper bound*) on them in the canonical partial order on the subobjects of A.*

An intersection *(respectively,* union*) of two quotient objects of an object A, is a greatest lower bound (respectively,* least upper bound*) on them in the canonical partial order on the quotient objects of A.*

Intersections and unions of subobjects (respectively, quotient objects) are unique up to equivalence of subobjects (respectively, quotient objects).

Lemma 92. *Let A be an object in $\mathfrak{G}$. Then each pair of subobjects of A, and each pair of quotient objects of A, has an intersection.*

Proof. Let $b_1 : B_1 \to A$ and $b_2 : B_2 \to A$ be monomorphisms. By Axiom 4(i), the monomorphism b_1 is the kernel of some morphism $f : A \to F$. By Axiom 3(i), the morphism $f \circ b_1$ has some kernel $k : K \to B_2$. Applying repeatedly the definition of a kernel, together with the fact that b_2 is monic, we find that the subobject $b_2 \circ k : K \to A$ of A is an intersection of B_1 and B_2.

A dual proof establishes the statement about quotient objects. □

Corollary 93. *Let A be an object in $\mathfrak{G}$. Then each pair of subobjects of A, and each pair of quotient objects of A, has a union.*

Proof. Let $b_1 : B_1 \to A$ and $b_2 : B_2 \to A$ be monomorphisms. Axiom 3(ii) ensures the existence of cokernels of B_1 and B_2. By the lemma just proved, these cokernels – treated as a pair of quotient objects of A – have an intersection. The kernel of this intersection (which exists by Axiom 3(i)) is a union of the kernels of the cokernels of B_1 and B_2. But by Lemma 90 on Ker-Coker duality, the kernels of the cokernels of B_1 and B_2 are just B_1 and B_2 themselves.

The proof of the dual statement is similar. □

10.3 Direct sums

Let $\mathfrak{G}$ be a category with zero objects and with finite products and coproducts. That is, $\mathfrak{G}$ satisfies the first two of the axioms for an abelian category. As well as the abelian categories, the categories **Grp** and **pCompHaus** are for instance of this kind.

We explore a number of constructions we can make for this family of categories. Let A_1 and A_2 be objects of $\mathfrak{G}$. For each pair $f_1 : F \to A_1$, $f_2 : F \to A_2$ we denote by $(f_1\ f_2)$ the canonical morphism induced from F to $A_1 \times A_2$. Likewise, for each pair $g_1 : A_1 \to G$, $g_2 : A_2 \to G$, we denote by $\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ the canonical morphism induced from $A_1 + A_2$ to G.

In particular, along with the canonical injections and projections

$$\iota_1 : A_1 \to A_1 + A_2, \quad \iota_2 : A_2 \to A_1 + A_2,$$
$$\pi_1 : A_1 \times A_2 \to A_1, \quad \pi_2 : A_1 \times A_2 \to A_2,$$

we have distinguished morphisms

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} : A_1 + A_2 \to A_1, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} : A_1 + A_2 \to A_2,$$
$$(1\ 0) : A_1 \to A_1 \times A_2, \quad (0\ 1) : A_2 \to A_1 \times A_2,$$

as well as a canonical morphism

$$\begin{pmatrix} (1\ 0) \\ (0\ 1) \end{pmatrix} = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) : A_1 + A_2 \to A_1 \times A_2,$$

which we henceforth denote by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Lemma 94. *The morphism* $\binom{0}{1}$ *is a cokernel of the canonical injection* ι_1*.*

Proof. By construction $\binom{0}{1} \circ \iota_1 = 0$. On the other hand, suppose that $g : A_1 + A_2 \to G$ is such that $g \circ \iota_1 = 0$. Then for some $g_2 : A_2 \to G$, we have that $g = \binom{0}{g_2} = g_2 \circ \binom{0}{1}$. □

For the rest of this section, we further suppose that $\mathfrak{G}$ is abelian.

Corollary 95. *The canonical injection* ι_1 *is a kernel of the morphism* $\binom{0}{1}$*.*

Proof. Certainly ι_1 is a monomorphism, and by the preceding lemma, the morphism $\binom{0}{1}$ is ι_1's cokernel. The result follows by Lemma 90 on Ker-Coker duality. □

Corollary 95 (as well as the results to be established in the rest of this section) can certainly fail in categories that satisfy only the first two abelian category axioms. It is instructive, before going further, to consider a counterexample.

Example 96. *In the category* **Grp** *of groups, the coproduct* $A_1 + A_2$ *of two groups* A_1 *and* A_2 *is their free product. Here the kernel of the morphism*

$$\binom{0}{1} : A_1 + A_2 \to A_2$$

is much larger than $A_1 + A_2$*'s subgroup* A_1*. Indeed, it is the set of all reduced words on the elements of* A_1 *and* A_2*, for which the overall product of all the elements on* A_1 *involved is* A_1*'s identity.*

Lemma 97. *The subobjects* $\iota_1 : A_1 \to A_1 + A_2$ *and* $\iota_2 : A_2 \to A_1 + A_2$ *of* $A_1 + A_2$ *have intersection* $0 : O \to A_1 + A_2$*.*

Proof. By the previous corollary, we know that ι_1 is a kernel of $\binom{0}{1}$. Observe that the unique morphism $0 : O \to A_2$ is the kernel of

$$\binom{0}{1} \circ \iota_2 = id_{A_2} : A_2 \to A_2.$$

From the construction in the proof of existence of intersections in Lemma 92, we deduce that the intersection of ι_1 and ι_2 is $A_1 + A_2$'s subobject

$$0 = \iota_2 \circ 0 : O \to A_1 + A_2.$$

□

Theorem 98. *Suppose that* $\mathfrak{G}$ *is abelian. Then the morphism*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : A_1 + A_2 \to A_1 \times A_2$$

is an isomorphism.

Proof. We will prove that it is a monomorphism. It will follow by a dual argument that it is an epimorphism, and therefore by Theorem 89 that it is an isomorphism.

Let $k : K \to A_1 \to A_2$ be a kernel of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$0_{KA_2} = \pi_2 \circ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \circ k = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \circ k.$$

So in the canonical partial order on the subobjects of $A_1 + A_2$, the subobject k is at most the kernel of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, which by Lemma 95 is ι_1.

Likewise k is at most ι_2. So k is at most the intersection of ι_1 and ι_2, which by Lemma 97 is the unique morphism $0 : O \to A_1 + A_2$.

The morphism $0 : O \to A_1 + A_2$ is therefore a kernel of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and so $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a monomorphism. □

In an abelian category, we can therefore define the *direct sum* (unique up to isomorphism) of two objects A_1 and A_2 to be the object $A_1 + A_2 \simeq A_1 \times A_2$. We denote the direct sum of A_1 and A_2 by $A_1 \oplus A_2$.

We are therefore justified in using our previously-established notation as follows. With the direct sum of A_1 and A_2 are associated canonical injections and projections

$$\begin{aligned} \iota_1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} : A_1 \to A_1 \oplus A_2, & \iota_2 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} : A_2 \to A_1 \oplus A_2, \\ \pi_1 &= (1\ 0) : A_1 \oplus A_2 \to A_1, & \pi_2 &= (0\ 1) : A_1 \oplus A_2 \to A_2. \end{aligned}$$

For each pair $f_1 : F \to A_1$, $f_2 : F \to A_2$ we have a canonical morphism

$$(f_1\ f_2) : F \to A_1 \oplus A_2.$$

For each pair $g_1 : A_1 \to G$, $g_2 : A_2 \to G$, we have a canonical morphism

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} : A_1 \oplus A_2 \to G.$$

10.4 The abelian group structure on homomorphisms

Let $\mathfrak{G}$ be an abelian category, and A and B two objects of $\mathfrak{G}$.

We define a binary operation $+$ on $\mathbf{Hom}(A, B)$ as follows: if $x, y \in \mathbf{Hom}(A, B)$, then

$$x + y = (x\ y) \circ \begin{pmatrix} 1 \\ 1 \end{pmatrix} : A \to B.$$

The following striking and fundamental theorem is then straightforward to deduce.

Theorem 99. *The binary operation $+$ defines an abelian group structure on $\mathbf{Hom}(A, B)$, with the morphism 0_{AB} as identity.*

We will make considerable use of this abelian group structure on each **Hom**-set. The following two consequences will be particularly useful.

Corollary 100. *Let A be an object of $\mathfrak{G}$. The binary operations $+$ and $\circ$ define a (not necessarily commutative) ring structure on* $\mathbf{Hom}(A, A)$*, known as the* endomorphism ring *of A and denoted* $\mathbf{End}(A)$*. This ring has additive identity 0_{AA} and multiplicative identity id_A.*

Theorem 101 (Direct sum systems)**.** *Let A_1 and A_2 be objects of $\mathfrak{G}$. The canonical injections and projections of $S = A_1 \oplus A_2$ satisfy*

$$\begin{array}{ll} \pi_1 \circ \iota_1 = id_{A_1}, & \pi_2 \circ \iota_1 = 0_{A_1 A_2} \\ \pi_1 \circ \iota_2 = 0_{A_2 A_1} & \pi_2 \circ \iota_2 = id_{A_2} \\ \iota_1 \pi_1 + \iota_2 \pi_2 = 1_S. & \end{array}$$

Conversely, if for three objects A_1, A_2, S and four morphisms ι_1, ι_2, π_1, π_2 the five conditions above are satisfied, then S is a direct sum of A_1 and A_2, with those four morphisms as its canonical injections and projections.

All the remarks on direct sums thus far made extend directly to products and coproducts of any finite number of objects.

11 Krull-Schmidt decomposition theorem

The aim of this section is to prove an analogue, in a general fixed abelian category $\mathfrak{G}$, of the classical Krull-Scmidt decomposition theorem for modules.

Definition 102. *An object $A \in \mathfrak{G}$ is* indecomposable*, if, for any direct sum*

$$A = B \oplus C,$$

either B or C is the subobject $id_A : A \to A$.

Definition 103. *An object $A \in \mathfrak{G}$ has* finite length*, if for each set of subobjects of A there is a maximal and a minimal subobject.*

Example 104. *Finite-dimensional vector spaces over a field k are finite-length objects in the abelian category k-**Mod** of vector spaces over k.*

Example 105. *Finite abelian groups are finite-length objects in the category* **Ab** *of abelian groups. The additive group $\mathbb{Z}$ of the integers, as an object of* **Ab***, does not have finite length: the set*

$$n \mapsto n, n \mapsto 2n, n \mapsto 4n, n \mapsto 8n, \ldots$$

of subobjects of $\mathbb{Z}$ each contains the last, and hence has no minimum.

Our work will concern the existence and uniqueness of decompositions of finite-length objects $A \in \mathfrak{G}$ as a direct sum of indecomposable modules. Our proof will be to use the additive structure on the endomorphism ring of each object involved. Why this is particularly promising as a strategy is shown by the following lemma.

Lemma 106. *Let $A \in \mathfrak{G}$. If* $\mathbf{End}(A)$ *is local, then A is indecomposable. Conversely, if $A \in \mathfrak{G}$ is indecomposable and has finite length, then* $\mathbf{End}(A)$ *is local.*

Proof. We prove here only the former direction. Suppose $A = B \oplus B'$. Let $\iota : B \to A$ and $\pi : A \to B$ be the natural injection and projection respectively. Consider the morphism $f = \iota \circ \pi : A \to A$. Then $f^2 = f$, so, since $\mathbf{End}(A)$ is local, either $f = 1_A$ or $f = 0_A$. If the former, then $B = A$ and $B' = 0$. If the latter, then $B = 0$ and $B' = A$. □

Theorem 107 (Krull-Schmidt)**.** *Suppose that $A \in \mathfrak{G}$ has finite length. Then it has a decomposition*

$$A = \bigoplus_{i \in I} A_i,$$

with I also finite, and such that each A_i is indecomposable. Moreover, if

$$A = \bigoplus_{j \in J} B_j,$$

is any other such decomposition, then $|I| = |J|$, and there exists a bijection $\varphi : I \to J$, such that for each $i \in I$, we have the isomorphism $A_i \simeq B_{\varphi(i)}$.

Proof. Existence.

Take a maximal decomposition. Then each part is indecomposable.

Uniqueness

Since A has finite length, so does each of the A_i. Using, as guaranteed by Lemma 106, that each ring $\mathbf{End}(A_i)$ is local, we find by induction on $|I|$ that:

Lemma 108. *Let $f \in \mathbf{End}(A)$. Then there exist subobjects $(\iota_i' : B_i \to A)_{i \in I}$, and isomorphisms $(h_i : A_i \to B_i)_{i \in I}$, such that for each $i \in I$ either $\iota_i' \circ h_i = f \circ \iota_i$ or $\iota_i' \circ h_i = (id_A - f) \circ \iota_i$. Moreover,*

$$A = \bigoplus_{i \in I} B_i.$$

□

With some further finiteness assumptions (and some messy set theory), the uniqueness part of theorem extends from direct products more generally to coproducts, holding even when I and J are infinite.

Definition 109. *An abelian category $\mathfrak{G}$ is a* Grothendieck category*, if it has colimits, and satisfies the following conditions:*

1. *The category $\mathfrak{G}$ is* locally small*; that is, for each object $A \in \mathfrak{G}$, the class of subobjects of A is a set.*

2. *Let $A \in \mathfrak{G}$. Let $(I, \leq)$ be an ordered set; let $\{\iota_i : A_i \to A : i \in I\}$ be a set of subobjects of A, such that for each pair $i, j \in I$, we have $i \leq j$ precisely if there is some morphism $f : A_j \to A_i$ such that $\iota_i \circ f = \iota_j$. Let $\iota : B \to A$ be any subobject of A. Then*

$$\bigcup_{i \in I} (A_i \cap B) = \left(\bigcup_{i \in I} A_i \right) \cap B.$$

Theorem 110. *Let $\mathfrak{G}$ be a Grothendieck category, and let $A \in \mathfrak{G}$. Suppose that A decomposes into two coproducts*

$$A = \coprod_{i \in I} A_i = \coprod_{j \in J} B_j,$$

where all B_i are indecomposable, and the endomorphism rings of all A_i are local. Then there exists a bijection $\varphi : I \to J$, such that for all $i \in I$ we have $A_i = B_{\varphi(i)}$.

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