Hyperfinite Equivalence Relations

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Abstract

A Borel equivalence relation on a Polish space X is hyperfinite if it can be written as an increasing countable union of Borel equivalence relations on X with finite classes. A canonical example is the equivalence relation E_0 of eventual equality on infinite binary sequences; this is the most complex hyperfinite equivalence relation under the usual Borel reducibility.

In the first part of this thesis we give three proofs that E_0 is not smooth, using a variety of methods. In the second part we examine the Borel complexity of isometry on two subclasses of Polish ultrametric spaces. For instance, we give an elementary proof of the result of Gao and Kechris that for the class of Heine-Borel spaces this complexity is Borel equivalent to E_0.

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1 Introduction

1.1 Background

Much of pure mathematics is devoted to the classification of objects in a given class of structure by some notion of equivalence, isomorphism for example. A *classification problem* for a given class of mathematical objects asks:

> Given two objects, how can one tell if they are equivalent?

Ideally one would like a *complete set of invariants* for the objects, namely, a uniformly obtainable set of attributes so that the problem of equivalence of objects can be reduced to verifying equality of the attributes. As an example, consider as objects the vector spaces over $\mathbb{Q}$, where the equivalence is isomorphism, and the invariant is the cardinality of a basis. A complete set of invariants needs to:

(1) entail a non-trivial reduction in complexity,

(2) be obtainable by a uniform procedure which one may reasonably hope to carry out in mathematical practice.

It is not immediately clear how one should interpret these intuitive requirements. For instance, at one extreme we have the pathology given by a choice function on the set of equivalence classes of some relation. As most would tend to agree this doesn't provide any satisfying reduction in complexity, nor is there any guarantee that this corresponds to a process which could be carried out in practice. On the other hand, an overly restrictive interpretation of (1) and (2) may prohibit the describing of invariants in many cases which mathematicians are interested in. As is usually the case when an attempt is made to formalize the intuitive, drawing a satisfying line between the pathological and the degenerate is difficult.

Invariant descriptive set theory (IDST) walks this fine line by providing a formal framework which encompasses the intuitive properties (1) and (2). It is the logician's approach to classification problems, where the intuitive condition (2) is interpreted by a formal constraint on definability. This leads to a comparison theory for classification problems, and also allows meaningful statements about when a classification is not possible.

The central concept of IDST is the notion of Borel reduction between equivalence relations on standard Borel spaces (see Definition 2.3 below). There is a strong empirical case for why this provides a meaningful framework for interpreting the intuitive notion of classification: Firstly, canonical benchmark relations (see [6, Ch. 15]) along with dichotomy theorems (see [6, Ch. 6]) lend non-redundant meaning to Borel reductions between mathematically interesting equivalence relations. Secondly, the structure of the pre-ordering $\leq_B$ is rich enough to capture the mathematical nuances of different relations; it is not the case that all relations of mathematical interest are mapped onto one level and the rest is artificial.

There is a strong metamathematical case for why the notion of a Borel reduction is an appropriate mechanism for imposing constraints on definability. Borel sets are set-theoretically 'primordial' in the sense that their codes are hereditarily countable, and each constructible Borel code appears in L below L_{ω_1} . By Shoenfield absoluteness, definability of reductions and equivalence relations is stable across changes in any reasonably chosen set theoretic universe. There is evidence that Borelness is not too strong a constraint on definability Hjorth [11, p.301]: Non-existence of a Borel reduction will mostly persist to non-existence of reductions in higher classes, such as projective or $L(\mathbb{R})$.

1.2 Hyperfiniteness and E_0

A Borel equivalence relation on a Polish space X is hyperfinite if it can be written as an increasing countable union of Borel equivalence relations on X with finite classes. A canonical hyperfinite benchmark is the equivalence relation E_0 of eventual equality on infinite binary sequence; every hyperfinite equivalence relation is Borel reducible E_0. A well- known dichotomy theorem [10] implies that every equivalence relation below E_0 is either Borel equivalent to E_0, or smooth, that is, reducible to the identity on $\mathbb{R}$.

In the first part of the thesis we give three proofs that E_0 is not smooth, illustrating three methods employed in IDST. The standard proof is measure-theoretic via the Kolmogorov 0-1 law. We follow this with a very different proof using Cohen forcing, beneath which we expose some topological phenomena taking place in the background. This leads to a third proof of the same result by means of a Baire category argument.

In the second part we follow Gao and Kechris [7] to apply Borel reducibility to a mathematical notion of equivalence: the isometry of metric spaces. Recall that a metric space is called *ultrametric* if every triangle is isosceles. We examine the Borel complexity of isometry on two subclasses of Polish ultrametric spaces, compact and Heine-Borel. For compact ultrametric spaces, isometry is smooth; we show this by assigning finitely branching rooted trees with real valued labels as invariants. We proceed to an alternate proof of the result of Gao and Kechris that for the class of Heine-Borel spaces, the Borel complexity of isometry is equivalent to E_0, by representing such spaces as locally finite trees.

2 Preliminaries

2.1 Notation

We fix some notation which is used throughout. For sets A and B we denote by B^A the set of functions from A to B. In particular, the notation 2^A represents the set of characteristic functions $A \to \{0, 1\}$. The notation $A^{<\omega}$ denotes the set of all finite sequences of elements in A and for $t, s \in A^{<\omega}$, $t^\frown s$ denotes the element of $A^{<\omega}$ obtained by appending s to t. For the sets 2^ω and ω^ω, we adopt the standard descriptive set theoretic convention of referring to their elements as *reals*.

For an equivalence relation E on a set X, we denote by $[x]_E$ the E-equivalence class of $x \in X$.

When φ is a formula in the language of set theory, we denote its relativization to a structure M by φ^M. Similarly, if A is the class defined by φ, we denote by A^M the relativized class defined by φ^M. We use the lightface class notation for the arithmetical and analytical hierarchies of formulas in the language of second order arithmetic: for $n \in \omega$, the classes Σ^0_n and Π^0_n are the respective n-th arithmetical levels, while Σ^1_n and Π^1_n denote the respective n-th analytical levels.

2.2 Descriptive set theory

The wider field of descriptive set theory is essentially the study of definable subsets of topological spaces which bare some resemblance to the real the line. Recall that a *Polish space* is a separable, completely metrizable topological space. Indeed, this definition is motivated by $\mathbb{R}$, more convenient to work with in a set-theoretic context however are the spaces 2^ω and ω^ω, respectively termed the *Cantor space* and *Baire space*. Each is a Polish space when treated as a countable product of discrete spaces; $\{0,1\}$ and ω respectively. More generally, the discrete topology on any countable set A defines a Polish topology, as does the product topology on a countable cartesian products of Polish spaces.

Let X be a Polish space. Recall that the *Borel* σ-algebra on X is the σ-algebra generated by the open subsets of X. We may stratify the Borel σ-algebra on a Polish space X into a transfinite hierarchy indexed by the countable ordinals $\alpha < \omega_1$. Set $\mathbf{\Sigma}^0_1$ to be the collection of all open subsets of X. At successor ordinal stages we define

$$A \in \mathbf{\Pi}^0_\alpha \iff X \setminus A \in \mathbf{\Sigma}^0_\alpha,$$

$$A \in \mathbf{\Sigma}^0_{\alpha+1} \iff A = \bigcup_{n\in\omega} A_n \text{ such that each } A_n \in \mathbf{\Pi}^0_\alpha.$$

For stages indexed by $\alpha < \omega_1$ a limit ordinal, we have

$$\mathbf{\Pi}^0_\alpha = \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta \quad \text{and} \quad \mathbf{\Sigma}^0_\alpha = \bigcup_{\beta<\alpha} \mathbf{\Sigma}^0_\beta.$$

This series of classes is called the *Borel hierarchy*. As is convention, we refer to sets in the class $\mathbf{\Sigma}^0_2$ as F_σ sets and sets in the class $\mathbf{\Pi}^0_2$ as G_δ sets.

A map $f : X \to Y$ between Polish spaces is called *Borel* if for every Borel subset $B \subseteq Y$, $f^{-1}(B) \subseteq X$ is Borel. If $f : X \to Y$ is a bijection such that both f and f^{-1} are Borel maps then we call f a *Borel isomorphism* and say that X and Y are *Borel isomorphic*. We have the following fundamental result, a proof of which can be found in Kechris [14, Thm. 15.6].

Theorem 2.1 (The Borel isomorphism theorem)**.** *If X and Y are Polish spaces of the same cardinality, then X and Y are Borel isomorphic.*

As a result, the Borel structure of a given space does not depend on the choice of underlying Polish topology. We therefore call a set X along with a σ-algebra $\mathcal{B}$ of subsets of X a *standard Borel space* if there exists a Polish topology on X whose Borel σ-algebra is exactly $\mathcal{B}$. We make frequent use of the following closure property for standard Borel spaces. A proof can again be found in Kechris [14, Cor. 13.4].

Theorem 2.2. *Let X be a standard Borel space. If $A \subseteq X$ is Borel, then A is itself a standard Borel space in the inherited structure.*

2.2.1 Borel equivalence relations

We now introduce some basic notions from the theory of equivalence relations on standard Borel spaces. Concretely, an equivalence relation E on a standard Borel space X is said to be *Borel* if it is a Borel subset of the product $X \times X$.

Definition 2.3. Let X and Y be standard Borel spaces with E and F equivalence relations on X and Y respectively. We say that E is *Borel reducible* to F, written $E \leq_B F$, if there exists a Borel map $f : X \to Y$ such that

$$x \; E \; z \iff f(x) \; F \; f(z).$$

Such a map f is called a *Borel reduction*, which one may interpret intuitively as some sufficiently definable procedure by which questions about E-equivalence of points in X can be transferred to questions about F-equivalence of points in Y. In this sense, the equivalence relation E can be considered at most as complex as the equivalence relation F. We say that E and F are *Borel bireducible*, writing $E \sim_B F$, if both $E \leq_B F$ and $F \leq_B E$ hold. Similarly, we write $E <_B F$ if $E \leq_B F$ and $F \not\leq_B E$.

The familiar order notation is justified: indeed, the relation $\leq_B$ is a preorder (reflexive and transitive) on the class of all equivalence relations defined on standard Borel spaces. Reflexivity is witnessed by the identity map, while transitivity follows from composing Borel reductions.

We note that the definition of a Borel reduction imposes no constraint on the equivalence relations, other than the requirement they be defined on standard Borel spaces. By restricting to various subclasses of the collection of all equivalence relations defined on Standard Borel spaces, a non-trivial structural theory of $\leq_B$ emerges. The simplest examples of equivalence relations with respect to Borel reducibility are the identity relations, denoted, $\mathrm{id}(X)$ for a given standard Borel space X. As a consequence of Theorem 2.1, any two standard Borel spaces of the same cardinality have Borel bireducible identity relations.

On the other hand, the identity relation on a countable standard Borel space is strictly reducible to the identity relation on an uncountable standard Borel space, for example, $\mathrm{id}(\omega) <_B \mathrm{id}(2^\omega)$. The forward reduction is witnessed for example by the map taking a natural n to the characteristic function of its singleton $\{n\}$, while the reverse reduction fails, as it would be necessarily be non-injective.

In general, we call an equivalence relation E on a standard Borel space X *smooth* if $E \leq_B \mathrm{id}(2^\omega)$, that is, the problem of determining E-equivalence is no more complex than checking equality of two real number invariants. We introduce two related notions. A Borel subset $T \subseteq X$ is called a *Borel transversal* for E if T contains exactly one point from each E-equivalence class. Equivalently, a Borel map $s : X \to X$ is called a *Borel selector* for E if for each $x \in X$, $x \mathrel{E} s(x)$ and s is constant on $[x]_E$. The existence of a Borel transversal (or equivalently a Borel selector) is a strictly stronger condition than smoothness, essentially guaranteeing that the quotient X/E is realized within X as a Borel subset. There are many sufficient conditions for the existence of a transversal, see for example [14, Ex. 18.20].

2.2.2 The structure of the $\leq_B$ preorder

A significant amount of effort is directed toward establishing $\leq_B$ dichotomy theorems by restricting focus to certain subclasses of equivalence relations which satisfy some type of definability constraint. The first such dichotomy theorem is due to Silver [18], for which we state a standard variant. A proof can be found in Gao [6, Thm. 5.3.5].

Theorem 2.4 (Silver dichotomy)**.** *Let X be a standard Borel space and E a smooth equivalence relation on X. Then either $E \leq_B \mathrm{id}(\omega)$ or $E \sim_B \mathrm{id}(2^\omega)$.*

Recall from the introduction that E_0 is a canonical example of a non-smooth Borel equivalence relation.

Definition 2.5. Define E_0 to be the equivalence relation on 2^ω given by

$$x \mathrel{E_0} y \iff \exists m\, \forall n \geq m\, \big(x(n) = y(n)\big) \quad \forall x, y \in 2^\omega. \tag{2.1}$$

That is, two binary sequences are E_0-equivalent if they differ only at finitely many coordinates. For this reason E_0 is referred to as the relation of *eventual agreement*.

The equivalence relation E_0 is indeed Borel. This is seen by writing $E_0 = \bigcup_{k\in\omega} F_k$ where each F_k is the closed (hence Borel) equivalence relation given by

$$F_k = \{(x, y) \in 2^\omega \times 2^\omega : \forall n \geq k\, (x(n) = y(n))\}. \tag{2.2}$$

It is useful to express E_0 as the orbit equivalence relation induced by the action of the countable group

$$G = \bigoplus_{n\in\omega} \mathbb{Z}/2\mathbb{Z}$$

on 2^ω by pointwise addition modulo 2:

$$(g \cdot x)(n) = x(n) + g(n) \pmod 2. \tag{2.3}$$

For each fixed $g \in G$, the map $x \mapsto g \cdot x$ is a homeomorphism of 2^ω with respect to the product topology, and the orbits of this action are precisely the E_0-equivalence classes. Since G is countable, we conclude that each E_0-class is countable.

It is easy to see that the identity relation on reals is a lower bound for the $\leq_B$-complexity of E_0.

Proposition 2.6. $\mathrm{id}(2^\omega) \leq_B E_0$.

Proof. Fix a Borel pairing function $\langle \cdot, \cdot \rangle : \omega \times \omega \to \omega$ and define $f : 2^\omega \to 2^\omega$ by

$$f(x)(\langle n, m \rangle) = x(n) \qquad \forall n, m \in \omega.$$

If $x = y$, then $f(x) = f(y)$ and hence $f(x)\ E_0\ f(y)$ trivially. If $x \neq y$, let $n_0 \in \omega$ be least such that $x(n_0) \neq y(n_0)$. Then for all $m \in \omega$ we have

$$f(x)(\langle n_0, m \rangle) \neq f(y)(\langle n_0, m \rangle),$$

so $f(x)$ and $f(y)$ differ on infinitely many coordinates, which implies $f(x)\ \not\!\!E_0\ f(y)$. $\square$

The converse to Proposition 2.6 does not hold; the standard proof of this fact is a measure-theoretic argument. We present a version using Kolmogorov's 0–1 law (see Kechris [14, Ex. 17.1]), which can be generalized to give a sufficient measure-theoretic condition for non-smoothness (see [6, Prop. 6.1.6]).

Proposition 2.7. $E_0 \not\leq_B \mathrm{id}(2^\omega)$.

Proof. Assume towards a contradiction that there is a Borel reduction $f : 2^\omega \to [0, 1]$ of E_0 to $\mathrm{id}([0, 1])$, and let μ be the usual product probability measure on 2^ω. For each Borel $A \subseteq [0, 1]$, the preimage $f^{-1}(A)$ is a union of E_0-classes and so is invariant under changes to finitely many coordinates. In the language of probability theory, the preimage $f^{-1}(A)$ is a measurable tail set to which Kolmogorov's 0–1 law applies.

Define

$$r = \inf\{s \in [0, 1] : \mu(f^{-1}([0, s])) = 1\}.$$

By the 0–1 law, each set $f^{-1}([0, s])$ or $f^{-1}((s, 1])$ has measure 0 or 1, and by the definition of r, it follows that

$$\mu\big(f^{-1}((r, 1])\big) = 0 \quad \text{and} \quad \mu\big(f^{-1}([0, r))\big) = 0.$$

Taking complements, we get that $\mu(f^{-1}(\{r\})) = 1$. But $f^{-1}(\{r\})$ is a single E_0-equivalence class, hence is countable, and therefore has measure zero under μ, a contradiction. $\square$

In fact a much stronger result holds: E_0 is the unique $\leq_B$-least non-smooth Borel equivalence relation (up to Borel bireducibility). This is the Harrington–Kechris–Louveau dichotomy [10].

Theorem 2.8 (Harrington–Kechris–Louveau dichotomy)**.** *Let X be a Polish space and E a Borel equivalence relation on X. If E is not smooth, then $E_0 \leq_B E$.*

We call a Borel equivalence relation *countable* if each of its equivalence classes is at most countable in size. It is a result of Dougherty, Jackson, and Kechris [3] that the class of countable Borel equivalence relations admits a $\leq_B$-greatest element, denoted E_∞, which is strictly above E_0 in terms of $\leq_B$ complexity.

Definition 2.9. Let X be a standard Borel space and let E be a Borel equivalence relation on X.

(i) E is called *hyperfinite* if there exists a countable increasing sequence of Borel equivalence relations $(F_n)_{n\in\omega}$ on X, such that each F_n has finite sized equivalence classes, and

$$E = \bigcup_{n\in\omega} F_n.$$

(ii) E is called *hypersmooth* if there exists an increasing sequence of smooth equivalence relations $(S_n)_{n\in\omega}$ on X such that

$$E = \bigcup_{n\in\omega} S_n.$$

As we have seen in Equation (2.2), E_0 can be written in this way and hence is hyperfinite. In fact, Dougherty, Jackson, and Kechris proved that E_0 is the unique up to Borel bireducibility $\leq_B$-maximum for the class of hyperfinite equivalence relations, along with two other characterizations of hyperfiniteness in [3, Thm. 5.1, 7.1].

Theorem 2.10. *Let X be a standard Borel space and let E be a Borel equivalence relation on X. The following are equivalent.*

1. *E is hyperfinite.*
2. *$E \leq_B E_0$.*
3. *E is the orbit-equivalence relation induced by some Borel action of $(\mathbb{Z}, +)$ on X.*
4. *E is a hypersmooth equivalence relation and $E \leq_B E_\infty$.*

The situation for the unrestricted class of all Borel equivalence relations differs from that of the hyperfinite, countable or hypersmooth ones in that there is no greatest element for $\leq_B$. This was proved by Friedman and Stanley [5, Thm. 1.2.2] with the introduction of their jump operator.

Definition 2.11. For a Borel equivalence relation E on a standard Borel space X, the *Friedman–Stanley jump* E^+ is the equivalence relation on X^ω defined by

$$(x_n)\ E^+(y_n) \iff \{[x_n]_E : n \in \omega\} = \{[y_n]_E : n \in \omega\}. \tag{2.4}$$

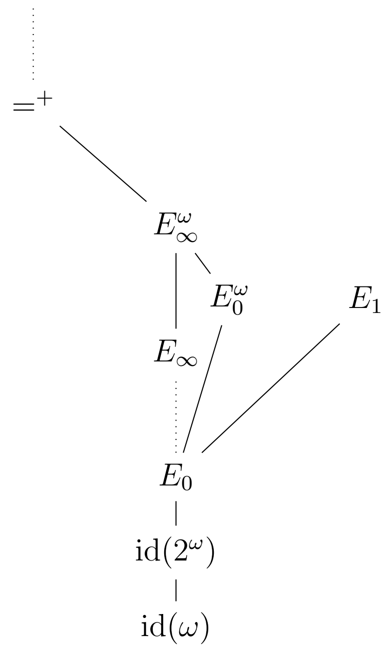

Figure 1: A fragment of the $\leq_B$ preorder on the class of Borel equivalence relations.

2.3 Trees on ω and countable graphs

We introduce two standard Borel spaces of countable combinatorial structures that will be used in later applications: the space of set-theoretic trees on ω and the space of countable graphs on ω.

Recall that a *tree* on ω is a subset $T \subseteq \omega^{<\omega}$ that is closed under taking initial segments. We define the *height* (or *length*) of a tree T to be

$$\mathrm{lh}(T) = \sup\{|s| : s \in T\}.$$

We say that two trees $T, S \subseteq \omega^{<\omega}$ are *isomorphic* if there is a bijection $\varphi : T \to S$ such that

$$u \subset v \iff \varphi(u) \subset \varphi(v).$$

By viewing a tree T by its characteristic function $x_T : \omega^{<\omega} \to \{0,1\}$, we can identify the set of all trees on ω with a closed subset Tr of $2^{\omega^{<\omega}}$ defined by

$$\begin{aligned} \mathrm{Tr} &= \{x_T \in 2^{\omega^{<\omega}} : \forall t \in \omega^{<\omega} \left(x_T(t) = 1 \to \forall s \subseteq t \,(x_T(s) = 1)\right)\} \\ &= \bigcap_{t \in \omega^{<\omega}} \bigcap_{s \subseteq t} \{x_T \in 2^{\omega^{<\omega}} : x_T(t) = 0 \vee x_T(s) = 1\}. \end{aligned}$$

It follows that Tr is itself a Polish space.

A tree T on ω is called *finitely-branching* if every node in T has only finitely many immediate extensions. We denote by $\mathrm{Tr}_{\mathrm{FB}}$ the subset of Tr consisting of characteristic functions of finitely-branching trees on ω.

Proposition 2.12. $\mathrm{Tr}_{\mathrm{FB}}$ *is an isomorphism-invariant standard Borel subspace of* Tr.

Proof. Invariance is clear since a tree isomorphism preserves the number of successors of each node. To see that $\mathrm{Tr}_{\mathrm{FB}}$ is standard Borel, it suffices to exhibit it as a Borel subset of Tr and invoke Theorem 2.2. We have

$$\begin{aligned}\mathrm{Tr}_{\mathrm{FB}} &= \big\{T \in \mathrm{Tr} : \forall t \in \omega^{<\omega}\big(x_T(t) = 1 \to \exists n \in \omega\ \forall m \geq n\ x_T(t^\frown\langle m\rangle) = 0\big)\big\} \\ &= \bigcap_{t\in\omega^{<\omega}} \bigcup_{n\in\omega} \bigcap_{m\geq n} \{x_T \in \mathrm{Tr} : x_T(t) = 0 \vee x_T(t^\frown\langle m\rangle) = 0\},\end{aligned}$$

which is a $\mathbf{\Pi}^0_3$ subset of Tr. □

We have the following result for isomorphism on the class $\mathrm{Tr}_{\mathrm{FB}}$. A proof can be found in Gao [6, Thm. 13.2.3].

Theorem 2.13. *Tree isomorphism on* $\mathrm{Tr}_{\mathrm{FB}}$ *is smooth.*

We note that under an appropriate encoding of the space $\mathrm{Tr}_{\mathrm{FB}}$, the isomorphism relation is equivalent to an orbit equivalence relation of a Borel action of the Polish group S_∞ (see [6, Ch. 13]). By a result of Burgess (see [6, Cor. 5.4.12]), the isomorphism relation on $\mathrm{Tr}_{\mathrm{FB}}$ has a Borel selector.

We turn now to the more general structure of countable simple graphs. Notice that up to choice of enumeration of vertices, we can code the adjacency matrix of a countable graph as an element of $2^{\omega\times\omega}$. We define

$$\mathrm{Gr} = \big\{x \in 2^{\omega\times\omega} : \forall n\,(x(n,n) = 0)\ \wedge\ \forall m,n\,(x(n,m) = x(m,n))\big\},$$

to be the collection of all adjacency matrices of graphs on ω. With respect to the product topology, we see that Gr is a closed subset of $2^{\omega\times\omega}$, and hence is a Polish space.

Recall that a graph is called a *tree* if it is both connected and acyclic. Further, we say a graph is *locally finite* if each vertex is incident to only finitely many edges. We denote by $\mathrm{Gr}_{\mathrm{LF}}$ the subclass of Gr consisting of adjacency matrices coding locally finite trees.

Proposition 2.14. *The collection* $\mathrm{Gr}_{\mathrm{LF}}$ *is a standard Borel space.*

Proof. We show each property defines a Borel subset of Gr. Denote by LocFin, Conn and Acyc the subcollections of Gr consisting of locally finite, connected and acyclic graphs respectively. Each is seen to be Borel:

$$\begin{aligned}
\mathrm{LocFin} &= \{x \in \mathrm{Gr} : \forall n \exists k \forall m > k (x(n,m) = 0)\} \\
&= \bigcap_{n\in\omega} \bigcup_{k\in\omega} \bigcap_{m>k} \{x \in \mathrm{Gr} : x(n,m) = 0\}. \\
\mathrm{Conn} &= \{x \in \mathrm{Gr} : \forall m, n \exists k_1, ..., k_r (x(n,k_1) = x(k_1,k_2) = \cdots = x(k_{r-1},k_r) = x(k_r,m) = 1)\} \\
&= \bigcap_{m,n\in\omega} \bigcup_{s\in[\omega]^{<\omega}} \bigcap_{k\in s} \{x \in \mathrm{Gr} : x(n,k) = 1\}. \\
\mathrm{Acyc} &= \{x \in \mathrm{Gr} : \neg\exists k_1, ..., k_r (x(k_1,k_2) = \cdots = x(k_r,k_1) = 1)\} \\
&= \mathrm{Gr} \setminus \bigcup_{r\geq 3} \bigcup_{\substack{k_1,\ldots,k_r\in\omega,\\ k_i\neq k_j}} \{x \in \mathrm{Gr} : x(k_1,k_r) = 1 \wedge \forall i < r (x(k_i,k_{i+1}) = 1)\}.
\end{aligned}$$

The result then follows by taking $\mathrm{Gr}_{\mathrm{LF}} = \mathrm{Conn} \cap \mathrm{Acyc} \cap \mathrm{LocFin}$. □

We have the following result on the complexity of isomorphism on the class Gr LF which we will later make use of. The result is originally due to Jackson, Kechris, and Louveau [12], but a self-contained proof can be found in Gao [6, Thm. 13.2.4].

Theorem 2.15. *Isomorphism of locally finite graph-theoretic trees is Borel reducible to E_∞.*

Remark 2.16. Let T be a locally finite tree and let $\lambda : T \to \mathbb{R}$ be a labelling function which assigns a real number to each vertex of T . Without loss of generality we may treat T along with the label on each vertex as a locally finite tree via the following. Fix a Borel reduction $f : \mathbb{R} \to \mathrm{Tr}_{\mathrm{FB}}$ of $\mathrm{id}(\mathbb{R})$ to the isomorphism of finitely-branching trees on ω. We can encode T along with the labels on each vertex as a locally finite tree T' obtained by attaching to each vertex $v \in T$ the rooted tree $f(\lambda(v))$.

3 Proving non-reducibility with forcing

In this section we show how forcing can be used to prove the non-existence of a Borel reduction in some instances.

Background on the forcing method can found in Jech [13, Sec. 14] or Kunen [15, Sec. IV]. Recall that a model M of set theory is called *transitive* if $x \in M$ implies $x \subseteq M$. We say that a formula φ is *absolute* between the models M and N if

$$M \models \varphi \iff N \models \varphi.$$

In particular, we make essential use of a strong absoluteness principle proved by Shoenfield [17]. We state a variant which gives a lower bound on the complexity of statements which are able to be proved independent of the axioms of ZFC by the method of forcing. A modern proof can be found in Jech [13, Thm. 25.20].

Theorem 3.1 (Shoenfield Absoluteness)**.** *Let M and N be transitive models of* ZF+DC *such that $M \cap \mathrm{Ord} = N \cap \mathrm{Ord}$ and suppose that $x \in \omega^\omega \cap M \cap N$. If φ is Σ^1_2 or Π^1_2 in the parameter x, then φ is absolute between M and N.*

In particular, φ is absolute between M and any extension of M by forcing.

We apply this to show the absoluteness of several descriptive set-theoretic notions.

Lemma 3.2. *Let E and F be Borel equivalence relations on standard Borel spaces X and Y. The following are absolute between transitive models of ZFC:*

(i) "x is a Borel code".

(ii) "$B \subseteq X$ is a Borel set".

(iii) For $z, w \in X$, "zEw".

(iv) "$f : X \to Y$ is Borel measurable".

(v) "$f : X \to Y$ is a Borel reduction of E to F".

Proof. We show each statement is of at most Σ^1_2 complexity in respective real parameters: (i) x being a Borel code reduces to checking that a certain arithmetical relation R on ω^ω (see [13, Lemma 25.44]) is well-founded below x. The statement of this: $\forall \langle z_0, z_1, \ldots\rangle \in \omega^\omega \exists n \in \omega(z_0 = x \wedge \neg(z_{n+1} R z_n))$ is Π^1_1 in the parameter x. (ii) This is equivalent to the Σ^1_2 statement: $\exists x \forall y (y \in A_x \leftrightarrow y \in B)$. (iii) E is a Borel subset of X^2, so there is a code $c \in \omega^\omega$ with $A_c = E$. Then zEw if and only if $(z, w) \in A_c$ is arithmetical in the parameter c. (iv) Similarly, f is Borel measurable if and only if f is a Borel subset of $X \times Y$ if and only if $\exists c \forall \langle x, y\rangle (f(x) = y \leftrightarrow (x, y) \in A_c)$. This is Σ^1_2. (v) This statement is a conjunction of Σ^1_2 statements in parameters the codes for E and F. $\square$

3.1 A forcing proof that E_0 is not smooth

As discussed in the introduction, here we present two additional proofs of the fact $E_0 \not\leq_B \mathrm{id}(2^\omega)$ (Proposition 2.7). The first is via Cohen forcing. The argument we present appears to be well-known but lacks formal presentation in the literature. One of the few places it seems to be recorded is a sketch by Hamkins [9] on MathOverflow, which we base the following detailed recreation on.

Second proof of Proposition 2.7. Suppose for contradiction that E_0 is smooth. Let M be a countable transitive model of ZFC with $\mathbb{P} = \mathrm{Fn}(\omega, 2) \cap M$ and $f : (2^\omega)^M \to (2^\omega)^M$ a Borel reduction of E_0 to $\mathrm{id}(2^\omega)$ in M. Let $G \subseteq \mathbb{P}$ be generic over M with $c = \bigcup G$.

Let $\dot{c}$ be the $\mathbb{P}$-name for generic element and we note that since f and E_0 are coded by reals in M, each has a canonical $\mathbb{P}$-name.

Given any finite $F \subseteq \omega$, define $\pi_F : \mathbb{P} \to \mathbb{P}$ by:

$$(\pi_F p)(n) = \begin{cases} p(n) & \text{if } n \notin F \\ 1 - p(n) & \text{if } n \in F \end{cases} \quad \forall p \in \mathbb{P}\ \forall n \in \operatorname{dom}(p).$$

If $p \leq q$, then $p \supseteq q$, so for each $n \in \operatorname{dom}(q)$, $(\pi_F p)(n) = (\pi_F q)(n)$. Certainly π_F preserves the top condition $\mathbf{1} = \emptyset$, and since π_F is bijective, it follows that π_F is an automorphism of $\mathbb{P}$.

Let $n \in \omega$ and suppose $p, q \in \mathbb{P}$ are conditions which decide the value of $f(c)(n)$. Then there are $b_p, b_q \in \{0, 1\}$ such that:

$$p \Vdash \check{f}(\dot{c})(\check{n}) = \check{b_p} \qquad \text{and} \qquad q \Vdash \check{f}(\dot{c})(\check{n}) = \check{b_q}.$$

Choose $F = \{k \in \operatorname{dom}(p) \cap \operatorname{dom}(q) : p(k) \neq q(k)\}$, then $\pi_F p \parallel q$. Since $F \subset \omega$ is finite, we have $\mathbf{1} \Vdash \dot{c}\check{E}_0 \pi_F \dot{c}$ which implies $\mathbf{1} \Vdash \check{f}(\dot{c}) = \check{f}(\pi_F \dot{c})$. By the Symmetry Lemma (see [13, Lemma 14.37]), we have:

$$\pi_F p \Vdash \check{f}(\dot{c})(\check{n}) = \check{f}(\pi_F \dot{c})(\check{n}) = \check{b_p}.$$

It follows that $b_p = b_q$, since if this were not the case, we may find an $r \leq \pi_F p, q$, which forces $f(c)(n)$ to take two different values. Since the conditions p and q and coordinate $n \in \omega$ were arbitrary, $\mathbf{1}$ decides $f(c)$. Then there is ground model real $w \in 2^\omega \cap M$ such that $M[G] \models f(c) = w$, for any generic filter $G \subseteq \mathbb{P}$ over M.

The statement $\exists x(f(x) = w)$ holds in $M[G]$ and is Σ^1_1 in w, therefore it relativizes down to M. Thus choose witness $r \in 2^\omega \cap M$ such that $M[G] \models f(r) = w$. Since f is a reduction, in $M[G]$ this implies $rE_0 c$, but if this were the case c would be definable in M as a finite adjustment to r, a contradiction. □

Often seen in the context of forcing is the parenthetical phrase: "*Cohen forcing is basically a Baire category argument*" apparently (see [1, p. 16]) due to Ryll-Nardzewski, although now long canonized into the forcing folklore.

Let M be a countable transitive model of ZFC and let $\mathbb{P} = \operatorname{Fn}(\omega, 2)$. To each $p \in \mathbb{P}$ we may associate the basic open subset of the Cantor space:

$$[p] = \{x \in 2^\omega : p \subset x\},$$

under which a dense (in the sense of forcing) subset $D \subseteq \mathbb{P}$ corresponds to the open dense subset of 2^ω given by

$$[D] = \bigcup_{p \in D} [p].$$

Then since M is countable, it follows by the Baire category theorem (see [14, Thm. 8.4]) that the set

$$S = \bigcap_{D \in M} \{[D] : M \models D \subseteq \mathbb{P} \text{ is dense}\}$$

is a dense G_δ subset of 2^ω.

Claim 3.3. *S is the set of Cohen generic reals over M.*

Proof. Let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over M and $c = \bigcup G$. Then for each $D \subseteq \mathbb{P}$, there is a $p \in G \cap D$, for which it follows that $p \subset c$ and so $c \in S$. On the other hand, each $x \in S$ generates a $\mathbb{P}$-generic filter $G_x = \{p \in \mathbb{P} : p \subset x\}$ such that $x = \bigcup G_x$. □

We need the following fact, adapted from a result in Kechris [14, Thm. 8.38].

Lemma 3.4. *Let X, Y be Polish spaces. If $f : X \to Y$ is Borel measurable, then there is a comeager $C \subseteq X$ such that $f \restriction C$ is continuous.*

Proof. Fix a countable basis $\{A_n\}_n$ of Y. Then each $f^{-1}(A_n)$ is Borel and in particular has the property of Baire. Thus let $\{V_n\}_n$ be open sets in X such that for each $n \in \omega$, $M_n = f^{-1}(A_n) \triangle V_n$ is meager. Then $C = X \setminus \bigcup_{n\in\omega} M_n$ is comeager, and

$$(f \restriction C)^{-1}(A_n) = f^{-1}(A_n) \cap C = V_n \cap C$$

is open relative to C. □

The third proof of Proposition. 2.7 we present follows Hjorth in [11, Ex. 1.6(2)]. We include some additional details.

Third proof of Proposition 2.7. Suppose $f : 2^\omega \to 2^\omega$ is a Borel reduction of E_0 to $\mathrm{id}(2^\omega)$. By Lemma 3.4, let $C \subseteq 2^\omega$ be comeager on which f is continuous. As in Eqn. (2.3), realize E_0 as the continuous action of $G = \bigoplus_\omega \mathbb{Z}/2\mathbb{Z}$ on 2^ω. Then each translate $g \cdot C$ by $g \in G$ is comeager as a homeomorphic image of the comeager C. Since G is countable, the intersection

$$D = \bigcap_{g\in G} g \cdot C$$

remains comeager and is invariant under E_0.

Since f is continuous on D and constant on the dense E_0-equivalence classes, f is constant on all of D. It follows that D is a single E_0-equivalence class and therefore is meager, contradiction. □

3.2 A forcing proof that $=^+$ is not smooth

We present a similar argument to the one given in the second proof of Proposition 2.7 to show that the properness of the first Friedman-Stanley jump of the equality, that is, the equivalence relation on $\mathbb{R}^\omega$ defined by

$$(x_n) =^+ (y_n) \iff \{x_n : n \in \omega\} = \{y_n : n \in \omega\}.$$

is not smooth. The result itself is in fact the well-known *Borel diagonalization theorem* proved by Friedman [4] eight years prior to the introduction of the jump operator.

The proof we present is again established in the folklore but seems to lack any formal presentation in the literature apart from another remark by Hamkins [9]. Recall the *cardinal collapse* notion of forcing $\mathrm{Coll}(\omega, 2^\omega)$, consisting of all finite partial functions $p : \omega \to 2^\omega$ ordered by extension. The generic object is a surjection of ω to the set of ground model reals.

Proposition 3.5. $=^+ \not\leq_B =$.

Proof. Let M be a countable model of ZFC and $\mathbb{P} = \mathrm{Coll}(\omega, 2^\omega) \cap M$ with $G \subseteq \mathbb{P}$ a $\mathbb{P}$-generic filter over M. Suppose $f : (2^\omega)^\omega \to 2^\omega$ is a Borel reduction of $=^+$ to $=$ in M, which by Lemma 3.2 is absolute up to $M[G]$.

Let $F = \bigcup G$, then $F : \omega \to 2^\omega \cap M$ enumerates the ground model set of reals. In particular, since the set of ground model reals is fixed, the image of the generic object F does not depend on the generic filter G which generates it, and so the image of the generic element $f(F)$ is invariant under choice of G.

It follows that $\mathbf{1}$ decides $f(F)$, so there is a ground model real $w \in 2^\omega \cap M$ such that $M[G] \models f(F) = w$. By the same up-and-down absoluteness argument as the one performed in the third proof of Proposition 2.7, there is a ground model ω-sequence of reals $r \in ((2^\omega)^\omega)^M$ such that $M[G] \models f(r) = f(F)$. Then $M[G] \models r =^+ F$, so $M[G] \models \forall x \in (2^\omega)^M \exists n \in \omega (r(n) = x)$. This statement is Δ_0 in the Lévy hierarchy and hence relativizes down to M, giving the contradiction to Cantor's theorem: $M \models |2^\omega| = \aleph_0$. $\square$

We note that the Friedman-Stanley jump in depth through the method of forcing by Shani , in particular giving a forcing proof of its properness at each finite stage in [16, Cor 4.6].

4 Isometry of Polish ultrametric spaces

In this section, we examine Polish ultrametric spaces under various degrees of compactness assumptions. We obtain graph-theoretic representations in the case of compact and Heine-Borel ultrametric spaces, from which we give elementary proofs of several results on the complexity of the isometry relation, first obtained by Gao and Kechris [7].

4.1 The hyperspace of Polish ultrametric spaces

Recall, an *ultrametric space* is a metric space (X, d) whose metric satisfies the strong triangle inequality:

$$d(x, y) \leq \max\{d(x, z), d(y, z)\} \quad \forall x, y, z \in X.$$

We make frequent use of the following easy properties.

Proposition 4.1. *Let X be an ultrametric space and $B_r(x), B_q(y) \subseteq X$ be open balls.*

(i) *If $z \in B_r(x)$, then $B_r(z) = B_r(x)$.*

(ii) *$B_r(x)$ is a clopen set.*

(iii) *If $B_r(x) \cap B_q(y) \neq \emptyset$, then $B_r(x) \subseteq B_q(y)$ or $B_q(y) \subseteq B_r(x)$.*

(iv) *If $B_r(x) \cap B_q(y) = \emptyset$, then for all $z \in B_r(x)$ and $w \in B_q(y)$ we have $d(z, w) = d(x, y)$.*

Proof. (i) If $w \in B_r(x)$, then $d(w, z) \leq \max\{d(x, z), d(x, w)\} < r$, so $w \in B_r(z)$. The opposite containment holds by symmetry. (ii) Without loss of generality, assume $q < r$. If $z \in B_r(x) \cap B_q(y)$, then $B_r(x) \cap B_q(y) = B_q(z) = B_q(y)$. (iii) Let $z \in X \setminus B_r(x)$. Then $B_r(z) \cap B_r(x) = \emptyset$, so $B_r(x)$ is closed. (iv) If $z \in B_r(x)$ and $w \in B_q(y)$, then $d(z, w) \leq \max\{d(z, x), d(w, x)\} = d(w, x)$ and $d(w, x) \leq \max\{d(w, y), d(x, y)\} = d(x, y)$. By (i) and symmetry we have $d(x, y) \leq d(z, w)$. □

When restricted to the case of separable ultrametric spaces, we get the following result about the distance set.

Proposition 4.2. *Let X be a separable ultrametric space. Then the distance set of X is countable.*

Proof. Fix a countable dense subset D of X. Let $x, y \in X$ be distinct with $q < d(x, y)$. Then choose $z \in D \cap B_r(x)$ and $z' \in D \cap B_q(y)$. Then by Proposition 4.1(iv), the distance across disjoint balls is fixed and so $d(z, z') = d(x, y)$. It follows that $d(X \times X) = d(D \times D)$. □

To study the $\leq_B$ complexity of the isometry relation we must realize the class of all Polish ultrametric spaces as a standard Borel space, to do so we follow the method of Vershik [19, p. 59]. Recall that the space $\mathbb{R}^{\omega\times\omega}$ with the product topology is Polish. Since a dense subset of a metric space uniquely completes to the space itself, the description of a Polish ultrametric on a countable dense set is enough to recover the full space by the operation of completion.

Thus we define UM to be the set of all points $r = (r_{i,j})_{i,j\in\omega}$ in $\mathbb{R}^{\omega\times\omega}$ such that for all $i, j, k \in \omega$:

(i) $r_{i,i} = 0$,
(ii) If $i \neq j$, then $r_{i,j} > 0$,
(iii) $r_{i,j} = r_{j,i}$,
(iv) $r_{i,j} \leq \max\{r_{i,k}, r_{j,k}\}$.

Given a Polish ultrametric space (X, d), fix a countable dense subset $D_X \subseteq X$. Then any enumeration $\phi : \omega \to D_X$ selects a representation of (X, d) in UM, namely the point:

$$r = (d(\phi(i), \phi(j))_{i,j\in\omega}.$$

While it is possible to enumerate a countable dense subset of a Polish ultrametric space in a Borel manner, the choice of enumeration is not canonical, nor is the choice of countable dense set for that matter. As a result, a given Polish ultrametric space has many representations in UM.

Conversely, every code $r \in \mathrm{UM}$ describes naturally an ultrametric on ω, which is isometrically embedded into its (ultrametric) completion as a countable dense set. Therefore, each $r \in \mathrm{UM}$ indeed codes a unique (up to isometry) Polish ultrametric space.

Proposition 4.3. UM *is a standard Borel space.*

Proof. Let $\pi_{i,j} : \mathbb{R}^{\omega\times\omega} \to \mathbb{R}$ denote the (by definition) continuous projection map onto the (i, j)-th coordinate. We deal with each clause in the definition of UM separately. Define the sets

$$A = \{r \in \mathbb{R}^{\omega\times\omega} : \forall i \in \omega(r_{i,i} = 0)\} = \bigcap_{i\in\omega} \pi_{i,i}^{-1}(\{0\}),$$

$$B = \{r \in \mathbb{R}^{\omega\times\omega} : \forall i, j \in \omega(i \neq j \to r_{i,j} > 0)\} = \bigcap_{\substack{i,j\in\omega \\ i\neq j}} \pi_{i,j}^{-1}((0,\infty)),$$

$$C = \{r \in \mathbb{R}^{\omega\times\omega} : \forall i, j \in \omega(r_{i,j} = r_{j,i})\} = \bigcap_{i,j\in\omega} (\pi_{i,j} - \pi_{j,i})^{-1}(\{0\}),$$

$$D = \{r \in \mathbb{R}^{\omega\times\omega} : \forall i, j, k \in \omega(r_{i,j} \leq \max\{r_{i,j}, r_{j,i}\})\} = \bigcap_{i,j,k\in\omega} (\pi_{i,j} - \max\{\pi_{i,k}, \pi_{j,k}\})^{-1}(\{0\}).$$

We see that the sets A, C and D are closed, as they are intersections of continuous preimages of a closed set. By the same reasoning, the set B is G_δ. It follows that $\mathrm{UM} = A \cap B \cap C \cap D$ is G_δ, and hence is a standard Borel space by Theorem 2.2. □

We denote the isometry relation on UM by $\cong_i$. Gao and Kechris (see [7, Thm. 4.4]) proved that the isometry relation on the general class UM is Borel bireducible with isomorphism of countable structures, that is, universal among S_∞-orbit equivalence relations.

4.2 Compact ultrametric spaces

Recall that a compact metric space is automatically Polish (see [14, Prop. 4.2]) and that compactness is preserved under isometry. Denote by UM_{C} the invariant subclass of UM consisting of codes for compact ultrametric spaces.

Proposition 4.4. UM_{C} *is a standard Borel space.*

Proof. Recall that in the setting of complete metric spaces, compactness is equivalent to total-boundedness, hence we characterize UM_{C} as the Borel subspace:

$$\begin{aligned}\mathrm{UM}_{\mathrm{C}} &= \Big\{r \in \mathrm{UM} : \forall \varepsilon \in \mathbb{Q}^+\ \exists i_0, \dots i_n\ \forall j \Big(\min_{0\leq k\leq n} \{r_{i_k,j}\} < \varepsilon\Big)\Big\} \\ &= \bigcap_{\varepsilon\in\mathbb{Q}^+} \bigcup_{s\in[\omega]^{<\omega}} \bigcap_{j\in\omega} \bigcup_{i\in s} \big\{r \in \mathrm{UM} : r_{i,j} < \varepsilon\big\}.\end{aligned}$$

□

As a consequence of Proposition 4.1, any two balls of the same radius are either equal or disjoint – this implies that a covering by balls of a fixed radius is in fact a partition. Let X be a compact ultrametric space, then since the set of open balls of a fixed radius in X refines to a finite partition, it follows that for each $q > 0$, the set $\{B_q(x) : x \in X\}$ is finite. We obtain the following fact about the distance set.

Lemma 4.5. *The distance set of a compact ultrametric space can be enumerated in a strictly decreasing sequence with unique limit point* 0.

Proof. If $\varepsilon > 0$, X is finitely partitioned by disjoint open balls of radius ε. By Proposition 4.1(iv), those distances $\geq \varepsilon$ are exactly the distances between balls, of which there are only finitely many. Label these $r_1 > r_2 > \cdots$, and repeating with $\varepsilon 2^{-n}$ we eventually exhaust all distances. $\square$

Let $T \subseteq \omega^{<\omega}$ be a tree on ω and let $(q_n)_{n<\mathrm{lh}(T)}$ be a strictly decreasing sequence of positive real numbers, with $\lim_{n\to\infty} q_n = 0$ in the case that $\mathrm{lh}(T) = \omega$. We can define an ultrametric d_q on the set of branches of $[T]$ by:

$$d_q(u,v) = \begin{cases} q_n, & \text{if } n = \min\{k \in \omega : u(k) \neq v(k)\}, \\ 0, & \text{otherwise.} \end{cases}$$

Indeed d_q defines an ultrametric: symmetry and non-degeneracy are immediate and given $u, v, w \in [T]$ with $i < j$ least such that $u(i) \neq v(i)$ and $v(j) \neq w(j)$, if $d_q(u,w) = r_k$ with $k > i, j$, then $u(i) = w(i) = v(i)$, a contradiction.

We obtain a uniform representation of a compact ultrametric space in the following.

Proposition 4.6. *Every compact ultrametric space is isometric to a space* $([T], d_q)$ *for a finitely-branching tree* T *and strictly decreasing sequence* $(q_n)_{<\mathrm{lh}(T)}$.

Proof. Let X be a compact ultrametric space and let $(q_n)_n$ be a strictly decreasing enumeration of the distance set of X given by Lemma 4.5. For simplicity, we first define and work with T intuitively, then argue that T is isomorphic to a tree on ω.

For each $n \in \omega$, let the n-th level of T be made up of all open balls contained in X of radius q_n, with the extension relation defined by

$$B_{q_i}(z) \prec B_{q_j}(w) \iff j = i + 1 \text{ and } B_{q_i}(z) \subseteq B_{q_j}(w).$$

Since any two balls on a fixed level are either equal or disjoint, this is well-defined. Moreover, since each ball is closed as a set, each ball is compact and so contains only finitely many distinct subballs of any fixed smaller radius. As a result, T is finitely-branching.

The branches of T are then sequences of nested open balls, with each $x \in X$ defining a unique branch

$$P(x) = (B_{q_n}(x))_{n\in\omega} \in [T].$$

We define a map $\varphi : X \to [T]$ by

$$\varphi(x) \coloneqq P(x).$$

If $x, y \in X$ are distinct, then the balls around x and y of radius $d(x,y)$ are disjoint, making the branches $P(x)$ and $P(y)$ distinct. Using again that the open balls of X are closed as sets, since X is complete, the intersection over a branch is guaranteed to be a singleton, hence we get that φ is surjective. Finally, given $x, y \in X$, there is some $n \in \omega$ such that $q_n = d(x,y)$. It follows that n is least such that $B_{q_{n+1}}(x) = B_{q_{n+1}}(y)$, so

$$d(x,y) = d_q(P(x), P(y)) = d_q(\varphi(x), \varphi(y)),$$

and we have shown that φ is an isometry.

We can easily identify T as a tree on ω via the following. Fix an enumeration $\{x_n\}_n$ of a countable dense subset of X. Since the distance set of X is realized on $\{x_n\}_n$ and any point in a ball can be identified as its center, to each ball $B \subseteq X$ we may assign an index

$$\mathrm{rep}(B) \coloneqq \text{ the least } i \in \omega \text{ such that } x_i \in B.$$

Then

$$T \cong \{s \in \omega^{<\omega} : \exists z \in X \forall n \in \mathrm{dom}(s)(\mathrm{rep}(P(z)(n)) = s(n))\}.$$

□

It is originally a result of Gromov [8, p. 94, $3.27\frac{1}{2}$] that the isometry relation on compact metric spaces is smooth. The standard proof in the context of invariant descriptive set theory (see [6, Thm. 14.2.1]) uses a Borel reduction to equality on the Polish space $\prod_{n\in\omega} K(\mathbb{R}^{(n+1)^2})$, sequences of the range of the metric restricted to successively larger initial segments of a countable dense subset.

We get as a corollary to Proposition 4.6 the special case when restricted to compact ultrametric spaces, perhaps with more intuitive complete invariants.

Corollary 4.7. *Isometry on compact ultrametric spaces is smooth.*

Proof. Given compact ultrametric spaces X and Y, let (q_n) and (u_n) be strictly decreasing enumerations of their respective distance sets. Then by Proposition 4.6, there exist finitely-branching trees T_X and T_Y on ω such that

$$X \cong_i Y \iff (T_X, d_{(q_n)}) = (T_Y, d_{(u_n)}).$$

We define formally a reduction $f : \mathrm{UM_C} \to \mathrm{Tr_{FB}} \times \mathbb{R}^\omega$ and verify that it is Borel.

First, let $q_n : \mathrm{UM_C} \to \mathbb{R}$ be the sequence of recursively defined functions:

$$q_0(r) \coloneqq \max_{i,j\in\omega}\{r_{i,j}\},$$
$$q_{n+1}(r) \coloneqq \max_{i,j\in\omega}\{r_{i,j} : r_{i,j} < q_n(r)\}.$$

By Lemma 4.5 each is well-defined. Moreover, the map q_0 is Borel as the realized supremum over the projections $\pi_{i,j}$, from which it follows by induction that each q_n is Borel.

For each $n \in \omega$ let $\mathrm{rep}_n : \mathrm{UM_C} \times \omega \to \omega$ be the map

$$\mathrm{rep}_n(r, i) := \min\{j \in \omega \setminus \{i\} : r_{i,j} < q_n(r)\}.$$

By an analogous argument as for the q_n, each map rep_n is Borel.

Fix a Borel selector Θ for isomorphism on $\mathrm{Tr_{FB}}$ and define $f_1 : \mathrm{UM_C} \to \mathrm{Tr_{FB}}$ and $f_2 : \mathrm{UM_C} \to \mathbb{R}^\omega$ by

$$f_1(r) := \Theta\left(\{(\mathrm{rep}_n(r, i))_{n\in\omega} : i \in \omega\}\right),$$
$$f_2(r) := (q_n(r))_{n\in\omega}.$$

We check Borel measurability of each.

Let $s \in \omega^{<\omega}$ and $b \in \{0, 1\}$ and write $N_s^b = \{T \in \mathrm{Tr_{FB}} : T(s) = b\}$. Then in the case that $b = 1$, we have

$$\begin{aligned} f^{-1}(N_s^1) &= \{r \in \mathrm{UM_C} : \exists i \forall n \in \mathrm{dom}(s)(\mathrm{rep}_n(r, i) = s(n))\} \\ &= \bigcup_{i\in\omega} \bigcap_{n\in\mathrm{dom}(s)} \{r \in \mathrm{UM_C} : \mathrm{rep}_n(r, i) = s(n)\} \end{aligned}$$

is Borel. The case for $b = 0$ follows immediately by taking a complement.

For f_2, let $V = \prod_{n\in\omega} U_n \subseteq \mathbb{R}^\omega$ a basic open subset of $\mathbb{R}^\omega$. Then there is a finite set $F \subset \omega$ such that $U_n = \mathbb{R}$ for all $n \notin F$. We have that

$$\begin{aligned} f_2^{-1}(V) &= \{r \in \mathrm{UM_C} : \forall n \in F(q_n(r) \in U_n)\} \\ &= \bigcap_{n\in F} \{r \in \mathrm{UM_C} : r \in q_n^{-1}(U_n)\}, \end{aligned}$$

hence is Borel. It follows that setting $f = (f_1, f_2)$ gives the desired Borel reduction. □

4.3 Heine-Borel ultrametric spaces

Recall that a metric space is called *Heine-Borel* (also *proper*) if every closed, bounded subset is compact. It's standard (e.g. see [6, Ex. 15.1.1]) that Heine-Borel implies complete and separable. Likewise, is the fact that the Heine-Borel property is preserved under isometry. We denote by $\mathrm{UM_{HB}}$ the invariant subclass of UM consisting of codes for Heine-Borel ultrametric spaces.

Proposition 4.8. $\mathrm{UM_{HB}}$ *is a standard Borel space.*

Proof. Notice that we can characterize the property of being Heine-Borel by the existence of an $x \in X$ such that $B_r(x)$ is compact for all $r > 0$. Further, this property holds for a point in X if and only if it holds for every point in X, hence the characterization:

$$\mathrm{UM_{HB}} = \bigcap_{q\in\mathbb{Q}^+} \{r \in \mathrm{UM} : \{j : r_{0,j} < q\} \in \mathrm{UM_C}\}.$$

By Proposition. 4.4, the predicate which checks membership in $\mathrm{UM_C}$ is Borel, hence the result. $\square$

It was shown by Gao and Kechris [7, Thm. 8.2] that the isometry relation on the class of Heine–Borel ultrametric spaces is Borel bireducible with E_0. In their proof, the upper bound E_0 is obtained via Theorem 2.10, together with a result of Hjorth (see [7, Thm. 7.1]) which specifies some conditions on a fairly general class of Polish metric spaces which are sufficient for concluding that the isometry relation is essentially countable.

The proof of Hjorth's result is deep. We present an alternative route to the special case for Heine-Borel ultrametric spaces via representing a Heine-Borel ultrametric space as a locally-finite (graph-theoretic) tree based on the following observation.

Notice that the requirements on the distance set of a Heine-Borel ultrametric space are weaker than those in the compact case, there is no guarantee of a $\geq$–well-ordering. Nevertheless, by compactness of the balls, this property persists locally: any open ball is contained in a larger compact ball, in which we can perform the argument from Lemma 4.5. As a result, each ball has a unique minimum radius 'parent' ball in which it is properly contained.

Proposition 4.9. *Isometry of Heine-Borel ultrametric spaces is Borel reducible to isomorphism on locally-finite trees.*

Proof. Let X be a Heine-Borel ultrametric space. We associate to X a locally finite (graph-theoretic) tree $T_X = (V_X, E_X)$ given by

$$V_X = \{B \subseteq X : B \text{ is an open ball}\},$$

$$E_X = \{\{B, B'\} \in V_X^2 : B \subset B' \wedge \neg\exists B'' \in V_X(B \subset B'' \subset B')\},$$

along with vertex labels $\lambda_X(B) \coloneqq \mathrm{diam}(B)$ for all $B \in V_X$.

With respect to Remark 2.16, for simplicity we may treat the resulting locally finite labelled tree as an object (T_X, λ_X) in $\mathrm{Gr_{LF}}$ and claim that $X \mapsto (T_X, \lambda_X)$ is a reduction of isometry to label preserving isomorphism of locally-finite trees.

First, we make two observations. Similar to the case in Proposition 4.6, each point in $x \in X$ defines a unique (now possibly bi-infinite) path

$$P(x) = \{B \in V_X : x \in B\}.$$

Further, if $z \in X$, we have that

$$d(x, z) = \min\{\mathrm{diam}(B) : B \in P(x) \cap P(y)\}.$$

Now, let X and Y be Heine-Borel ultrametric spaces.

If $\varphi : X \to Y$ is an isometry, let ψ be the image map of φ restricted to V_X. If $B_r(x) \in V_X$ for some $x \in X$ and $r > 0$, then $\psi(B_r(x)) = B_r(\varphi(x))$, so indeed ψ is a map of V_X into V_Y. Likewise, for each $B_r(y) \in V_X$, we have $\psi(B_r(\varphi^{-1}(y))) = B_r(y)$, so ψ is surjective.

We show ψ preserves adjacency. Let $B, B' \in V_X$ and suppose $B \subset B'$. It's clear that $B \subset B'$ holds if and only if $\psi(B) \subset \psi(B')$ holds so suppose for contrapositive that there is some $C \in V_Y$ such that $\psi(B) \subset C \subset \psi(B')$. It follows that $B \subset \psi^{-1}(C) \subset B'$ and therefore $\{B, B'\} \notin E_X$.

To see that ψ preserves the vertex labels, it is enough to note that since φ is an isometry, for each $B \in B_X$ we have

$$\lambda_X(B) = \operatorname{diam}(B) = \operatorname{diam}(\psi(B)) = \lambda_Y(\psi(B)).$$

For the converse, suppose that $\psi : V_X \to V_Y$ is an isomorphism preserving the labels λ_X. Define a map $\varphi : X \to Y$ by

$$\varphi(x) \coloneqq \text{the unique point contained in the singleton } \bigcap\{\psi(B) : B \in P(x)\}.$$

If $y \in Y$, then by the surjectivity of ψ, there is a unique path $\{\psi^{-1}(C) : C \in P(y)\}$ whose intersection consists of a single point which φ maps back onto y, hence the surjectivity of φ. Now, let $z, w \in X$. Then

$$\begin{aligned}
d(z, w) &= \min\{\operatorname{diam}(B) : B \in P(z) \cap P(w)\} \\
&= \min\{\lambda_X(B) : B \in P(z) \cap P(w)\} \\
&= \min\{\lambda_X(\psi(B)) : \psi(B) \in P(\varphi(z)) \cap P(\varphi(w))\} \\
&= \min\{\operatorname{diam}(C) : C \in P(\varphi(z)) \cap P(\varphi(w))\} \\
&= d(\varphi(z), \varphi(w))
\end{aligned}$$

and it follows that φ is an isometry.

To verify Borel measurability, we construct the above reduction formally between the spaces $\mathrm{UM_{HB}}$ and $\mathrm{Gr_{LF}}$. Fix a Borel pairing function $\langle \cdot, \cdot \rangle : \omega^2 \to \omega$ and let $(i, j) \in \omega^2$ represent the ball

$$B_{r_{i,j}}(i) = \{k \in \omega : r_{i,k} < r_{i,j}\}.$$

Recall that two balls are adjacent if one is a maximal proper subball of the other, which we may express as the predicate:

$$\mathrm{MPC}\left((i, j), (i', j')\right) \iff (r_{i,j} < r_{i',j'}) \wedge (r_{j,j'} < r_{j,j'}) \wedge \neg\exists k (r_{i,j} < r_{j',k} < r_{j',i'}).$$

Thus we define the reduction $f : \mathrm{UM_{HB}} \to \mathrm{Gr_{LF}}$ by

$$f(r) \coloneqq \{(\langle i, j\rangle, \langle i', j'\rangle) : \mathrm{MPC}\left((i, j), (i', j')\right)\} \in \mathrm{Gr_{LF}}.$$

We check that f maps basic open subsets of $\mathrm{Gr_{LF}}$ to Borel subsets of $\mathrm{UM_{HB}}$. Let $n, m \in \omega$, $b \in \{0, 1\}$ and consider the basic open cylinder of $2^{\omega\times\omega}$ given by

$$C = \{z \in 2^{\omega\times\omega} : z(n, m) = b\}.$$

It suffices to show that $f^{-1}(C)$ is Borel. Let $(i,j) = \langle a \rangle^{-1}$ and $(i',j') = \langle b \rangle^{-1}$. Without loss of generality, assume $b = 1$, since the case for $b = 0$ follows by taking complements. Then

$$\begin{aligned} f^{-1}(C) = &\{r \in \mathrm{UM_{HB}} : \mathrm{MPC}\left((i,j),(i',j')\right)\} \\ =&\{r \in \mathrm{UM_{HB}} : r_{i,j} < r_{i',j'}\} \cap \{r \in \mathrm{UM_{HB}} : r_{i,i'} < r_{i',j'}\} \\ &\cap \bigcap_{k\in\omega} \{r \in \mathrm{UM_{HB}} : (r_{i,j} \geq r_{i',k}) \vee (r_{i',j'} \leq r_{i',k})\}. \end{aligned}$$

Via the usual argument with the projection maps for $\mathbb{R}^{\omega\times\omega}$, this preimage is seen to be Borel, thus completing the proof. □

Since the isomorphism relation on locally finite trees is essentially countable (Theorem 2.15) we get have obtained the special Heine-Borel case of Hjorth's result for essential countability of isometry.

Corollary 4.10. *Isometry on* $\mathrm{UM_{HB}}$ *is essentially countable.*

We are now ready to prove the following.

Theorem 4.11. *Isometry of Heine-Borel ultrametric spaces is Borel bireducible to* E_0*.*

We proceed in two parts, first proving that $\cong_i \leq_B E_0$. Since we have proved that isometry on Heine-Borel ultrametric spaces is essentially countable, recall that by Theorem 2.10, it suffices to show that $\cong_i$ is hypersmooth.

Having completed the above, by the Harrington–Kechris–Louveau dichotomy (Theorem 2.8), it is sufficient to show that $\cong_i$ is not smooth in order to conclude that $E_0 \leq_B \cong_i$. We demonstrate this by constructing a Borel reduction from a non-smooth hyperfinite equivalence relation to $\cong_i$.

Claim 4.12. *Isometry on Heine-Borel ultrametric spaces is hypersmooth.*

Proof. On the standard Borel space $\mathrm{UM}_{\mathrm{C}}^{\omega}$, let E be the hypersmooth equivalence relation defined by

$$(X_n)_{n\in\omega}\, E\, (Y_n)_{n\in\omega} \iff \exists k \exists m \forall n (X_{k+n} \cong_i Y_{m+n}).$$

Given a Heine-Borel ultrametric space X, fix an arbitrary point $x \in X$. Using the notation from the proof of Proposition 4.9, consider the path $P(x)$ in V_X of balls containing the point x. Fix some ball $B_0 \in P(x)$, then for each n, let $B_{n+1} \in P(x)$ be the unique parent of B_n. We claim that the map defined by

$$f(X) = (B_n)_{n\in\omega}$$

is a Borel reduction of $\cong_i$ to E on $\mathrm{UM}_{\mathrm{C}}^{\omega}$. Let X, Y be Heine-Borel ultrametric spaces with $x \in X$ and $y \in Y$ fixed. For each $n \in \omega$ write

$$r_n = \mathrm{diam}(f(X)(n)) \quad \text{and} \quad q_n = \mathrm{diam}(f(Y)(n)).$$

Suppose $\varphi : X \to Y$ is an isometry. Then there is some $k \in \omega$ such that $B_{q_k}(\varphi(x)) = B_{q_k}(y)$. Since the isometric preimage of an open ball is an open ball, there is some $m \in \omega$ such that

$$r_m = \operatorname{diam}(\varphi^{-1}(B_{q_k}(y)))$$

and it follows that $B_{r_{k+n}}(x) \cong_i B_{q_{m+n}}(y)$ for all $n \in \omega$, proving that $f(X)\ E\ f(Y)$.

For the converse, suppose that there are $m, k \in \omega$ such that

$$f(X)(k+n)\ E\ f(Y)(m+n) \quad \forall n \in \omega.$$

It follows immediately that $r_{m+n} = q_{k+n}$ for all $n \in \omega$, and so we may perform the relabelling $R_n = r_{m+n} = q_{k+n}$ for all $n \in \omega$, after which the hypothesis becomes $B_{R_n}(x) \cong_i B_{R_n}(y)$. For each $n \in \omega$, let

$$\mathcal{P}^x_{n+1} = \{\, B_{R_n}(z) : z \in B_{R_{n+1}}(x) \,\}, \quad \mathcal{P}^y_{n+1} = \{\, B_{R_n}(z) : z \in B_{R_{n+1}}(y) \,\},$$

be the respective partitions of $B_{R_{n+1}}(x)$ and $B_{R_{n+1}}(y)$ into subballs of radius R_n. We define a tree T whose n-th level consists of all bijections $\phi_n : \mathcal{P}^x_n \to \mathcal{P}^y_n$ which satisfy:

(i) $\phi_n(B) \cong_i B$ for all $B \in \mathcal{P}^x_n$,

(ii) $\phi_n\big(B_{R_n}(x)\big) = B_{R_n}(y)$,

with the extension relation defined naturally by

$$\phi_i \prec \phi_j \iff i+1 = j \text{ and } \phi_j \restriction \mathcal{P}^x_i = \phi_i.$$

By compactness, for each $n \in \omega$ the sets $\mathcal{P}^x_n$ and $\mathcal{P}^y_n$ are finite, and it follows that T is finitely-branching. We prove that each level of T contains at least one node.

Fix an $n \in \omega$ and let $\varphi_{n+1} : B_{R_{n+1}}(x) \to B_{R_{n+1}}(y)$ be an isometry. The image map of φ_{n+1} induces a bijection $\phi_{n+1} : \mathcal{P}^x_{n+1} \to \mathcal{P}^y_{n+1}$ which satisfies (i). If $\phi_{n+1}\big(B_{R_n}(x)\big) \neq B_{R_n}(y)$, let $\theta : \mathcal{P}^x_{n+1} \to \mathcal{P}^x_{n+1}$ be defined by

$$\theta(B) = \begin{cases} \phi^{-1}_{n+1}(B_{R_n}(y)), & \text{if } B = B_{R_n}(x), \\ B, & \text{otherwise.} \end{cases}$$

Then $\phi_{n+1} \circ \theta$ satisfies both (i) and (ii), hence is in the $(n+1)$–th level of T. It follows from König's Lemma (see [14, Ex. 4.12]) that T has an infinite branch, which we denote

$$(\phi_n)_{n \in \omega}.$$

Now we construct a coherent chain of isometries recursively. Fix an isometry $\varphi_0 : B_{R_{n+1}}(x) \to B_{R_{n+1}}(y)$. Now, given an isometry $\varphi_n : B_{R_n}(x) \to B_{R_n}(y)$, for each $B \in \mathcal{P}^x_{n+1}$, let $\pi_B : B \to \phi_{n+1}(B)$ be an isometry which witnesses condition (i) for ϕ_{n+1}. In particular, for $B = B_{R_n}(x)$ we choose $\pi_B = \varphi_n$. Now define $\varphi_{n+1} : B_{R_{n+1}}(x) \to B_{R_{n+1}}(y)$ by

$$\varphi_{n+1} := \bigcup_{B \in \mathcal{P}^x_{n+1}} \pi_B.$$

Since the union is disjoint, this is well-defined. We check that φ_{n+1} is an isometry. Let $z, w \in B_{R_{n+1}}(x)$ with $B, B' \in \mathcal{P}^x_{n+1}$ such that $z \in B$ and $w \in B'$. If $B = B'$, then

$$\begin{aligned} d(\varphi_{n+1}(z), \varphi_{n+1}(w)) &= d(\pi_B(z), \pi_B(w)) \\ &= d(z, w). \end{aligned}$$

If $B \neq B'$, then

$$\begin{aligned} d(z, w) &= d(B, B') \\ &= d(\phi_{n+1}(B), \phi_{n+1}(B')) \\ &= d(\pi_B(z), \pi_{B'}(w)) \\ &= d(\varphi_{n+1}(z), \varphi_{n+1}(w)). \end{aligned}$$

To see that φ_{n+1} is surjective, given $u \in B_{R_{n+1}}(y)$, let $B \in \mathcal{P}^x_{n+1}$ such that

$$u \in \phi_{n+1}(B) = \pi_B(B) = \varphi_{n+1}(B).$$

Then surjectivity of φ_{n+1} follows from the surjectivity of π_B. Therefore, φ_{n+1} is an isometry extending φ_n and as a result, the global map $\varphi : X \to Y$ defined by

$$\varphi := \bigcup_{n \in \omega} \varphi_n : X \to Y$$

is a coherent isometry.

Finally, working over the formal space $\mathrm{UM}_{\mathrm{HB}}$, we verify that the reduction f is Borel. Define recursively the functions $q_n : \mathrm{UM}_{\mathrm{HB}} \to (0, \infty)$ by

$$q_0(r) := r_{0,1},$$

$$q_{n+1}(r) := \min_{i \in \omega} \{r_{0,i} : r_{0,i} > q_n(r)\}.$$

Each is Borel by a similar argument to the one in the proof of Corollary 4.7: each q_{n+1} is the realized infimum of projection maps while $q_0 = \pi_{0,1}$.

Thus for each $n \in \omega$ define the Borel map $\mathcal{B}_n : \mathrm{UM}_{\mathrm{HB}} \to \mathrm{UM}_{\mathrm{C}}$ by

$$(\mathcal{B}_n(r))_{j,i} = \begin{cases} r_{0,i} & \text{if } j = 0 \text{ and } r_{0,i} < q_n(r) \\ 0 & \text{otherwise} \end{cases} \qquad \forall (j, i) \in \omega^2 \ \forall r \in \mathrm{UM}_{\mathrm{HB}} .$$

It follows that the reduction $f : \mathrm{UM}_{\mathrm{HB}} \to \mathrm{UM}_{\mathrm{C}}^{\omega}$ defined by

$$f(r) = (\mathcal{B}_n(r))_{n \in \omega}$$

is Borel, completing the proof. □

prove that E_0 is a lower bound for isometry on Heine-Borel ultrametric spaces

Based on the observation that for each fixed prime p, the of p-adic numbers form a Heine-Borel ultrametric space with the usual p-adic absolute value,

We proceed by Borel reducing a non-smooth $\mathbb{Z}$-shift

Claim 4.13. $E_0 \leq_B \cong_i$.

Proof. Let F_0 be a transversal for isomorphism of non-empty finite trees on ω. Consider the space of injective maps from $\mathbb{Z} \to F_0$, which itself is Polish when viewed as the G_δ subset of product space $F_0^{\mathbb{Z}}$ given by

$$\begin{aligned} A &= \{x : \mathbb{Z} \to F_0 : x \text{ is injective}\} \\ &= \bigcap_{\substack{m,n \in \mathbb{Z} \\ m \neq n}} \{x \in F_0^{\mathbb{Z}} : x(m) \neq x(n)\}. \end{aligned}$$

Fix a Borel well-ordering $\leq^*$ of F_0, and to each finite tree $T \in F_0$, assign a finite ultrametric d_T given by

$$d_T(u,v) = \begin{cases} 2^{-\max\{n : u(n) = v(n)\}}, & \text{if } u \neq v, \\ 0, & \text{otherwise.} \end{cases}$$

Since T is non-empty, the resulting space (T, d_T) has diameter exactly 1.

Let $\theta : A \to \mathbb{Z}$ be a map such that for any $x \in A$,

$$x(\theta(x)) = \text{ the } \leq^*\text{-least tree in } \operatorname{ran}(x).$$

For each $x \in A$ we define a level function $L_x : \mathbb{Z} \to \mathbb{Z}$ by

$$L_x(n) = \begin{cases} 2(n - \theta(x)) & \text{if } n \geq \theta(x) \\ 2(\theta(x) - n) - 1 & \text{if } n < \theta(x) \end{cases}$$

Finally, let $(U(x), d_x)$ be the ultrametric space consisting of points

$$U(x) = \bigcup_{n \in \mathbb{Z}} \{(n,u) : u \in x(n)\},$$

with d_x defined by

$$d_x((n,u),(m,w)) = \begin{cases} 2^{L_x(n)} d_{x(n)}(u,w) & \text{if } n = m \\ 2^{\max\{L_x(n), L_x(m)\}} & \text{if } n \neq m \end{cases} \quad \forall (n,u),(m,v) \in U(x).$$

We show d_x satisfies the strong triangle inequality. Let $(n,u),(m,w),(l,t) \in U(x)$. Then there are three cases to consider:
Case 1. If $n = m = l$, then all three points lie in the finite tree $x(n)$, on which d_x is equal to the scaled ultrametric $2^{L_x(n)} d_{x(n)}$.
Case 2. If $n = m$ and $n \neq l$, then we have

$$d_x((n,u),(m,w)) = 2^{L_x(n)} d_{x(n)}(u,w)$$

and

$$d_x((n,u),(l,t)) = 2^{\max\{L_x(n),L_x(l)\}} = 2^{\max\{L_x(m),L_x(l)\}} = d_x((m,w),(l,t)).$$

Case 3. If $n \neq m \neq l$ and $n \neq l$, then

$$d_x((n,u),(l,t)) = 2^{\max\{L_x(n),L_x(l)\}} \quad \text{and} \quad d_x((m,w),(l,t)) = 2^{\max\{L_x(m),L_x(l)\}}.$$

Notice that

$$\begin{aligned}
& d_x((n,u),(m,w)) && \leq \max\{d_x((n,u),(l,t)), d_x((m,w),(l,t))\} \\
\iff & 2^{\max\{L_x(n),L_x(m)} && \leq \max\{2^{\max\{L_x(n),L_x(l)\}}, 2^{\max\{L_x(m),L_x(l)\}}\} \\
\iff & \max\{L_x(n),L_x(m) && \leq \max\{\max\{L_x(n),L_x(l)\}, \max\{L_x(m),L_x(l)\}\} \\
\iff & \max\{L_x(n),L_x(m)\} && \leq \max\{L_x(n),L_x(l),L_x(m)\}
\end{aligned}$$

which holds.

Fix an $(n,u) \in U(x)$ and an $N \in \omega$ and consider $B = B_{2^N}((n,u))$. Notice, if $(m,k) \in B$ then $L_x(m) < N$, therefore, there are only finitely many $m \in \mathbb{Z}$ satisfying this. Since each tree $x(n)$ consists of only finitely points, we conclude that B is finite, and therefore compact. It follows that $(U(x), d_x)$ is Heine-Borel. On the other hand, since d_x is unbounded, the space $U(x)$ itself is not compact.

Claim that

$$\exists m \in \mathbb{Z} \forall n \in \mathbb{Z}(x(n) = y(m+n)) \iff U(x) \cong_i U(y) \quad \forall x,y \in A.$$

If $x, y \in A$ and $m \in \mathbb{Z}$ such that $x(n) = y(m+n)$ for all $n \in \mathbb{Z}$, then $\text{ran}(x) = \text{ran}(y)$ so $x(\theta(x)) = y(\theta(y))$. In fact, we have that $x(\theta(x)) = y(\theta(x)+m)$.

Let $n \in \mathbb{Z}$. If $n \geq \theta(x)$, then $n + m \geq \theta(x) + m = \theta(y)$ and we have

$$L_y(m+n) = 2(m+n-\theta(y)) = 2(n-\theta(x)) = L_x(n).$$

If $n < \theta(x)$, then $n+m < \theta(x)+m = \theta(y)$ and we have

$$L_y(m+n) = 2(\theta(y)-m-n)-1 = 2(\theta(x)-n)-1 = L_x(n).$$

Now, define a map $\varphi : U(x) \to U(y)$ by

$$\varphi((n,u)) = (n+m,u) \quad \forall (n,u) \in U(x).$$

Indeed φ is well-defined since we have a direct equality of the finite trees $x(n) = y(m+n)$ for all $n \in \mathbb{Z}$. It follows immediately that φ is surjective.

Fix $(n,u),(k,v) \in U(x)$. We check that φ preserves distances by cases:
Case 1. If $n = k$, then

$$\begin{aligned}
d_x((n,u),(k,v)) &= 2^{L_x(n)} d_{x(n)}(u,v) \\
&= 2^{L_y(m+n)} d_{y(m+n)}(u,v) \\
&= d_y((m+n,u),(m+k,v)) \\
&= d_y(\varphi((n,u)),\varphi((k,v))).
\end{aligned}$$

Case 2. If $n \neq k$, then

$$\begin{aligned} d_x((n,u),(k,v)) &= 2^{\max\{L_x(n),L_x(k)\}} \\ &= 2^{\max\{L_y(m+n),L_y(m+k)\}} \\ &= d_y((m+n,u),(m+k,v)) \\ &= d_y(\varphi((n,u)),\varphi(k,v)). \end{aligned}$$

Therefore φ is an isometry.

For the converse, assume that $\varphi : U(x) \to U(y)$ is an isometry. First, we can recover the value of $\theta(x)$ by identifying the $n \in \mathbb{Z}$ such that the unique subspace $\{(n,u) : u \in x(n)\} \subset U(x)$ has diameter exactly 1. Let $n, k \in \mathbb{Z}$ be such $n = \theta(x)$ and $k = \theta(y)$. It follows that

$$\varphi(\{(n,u) : u \in x(n)\}) = \{(m+n,u) : u \in y(m+n)\} \quad \text{for some } m \in \mathbb{Z}.$$

Now fix any $j \in \mathbb{Z}$. Pick $v \in x(j)$ and $u \in x(n)$. Since $n = \theta(x)$ we have $L_x(n) = 0$, so

$$d_x\big((j,v),(n,u)\big) = 2^{\max\{L_x(j),L_x(n)\}} = 2^{L_x(j)}.$$

Write $\varphi((j,v)) = (i,t)$ and $\varphi((n,u)) = (m+n,w)$. Then

$$d_y\big(\varphi((j,v)),\varphi((n,u))\big) = d_y\big((i,t),(m+n,w)\big) = 2^{\max\{L_y(i),L_y(m+n)\}} = 2^{L_y(i)}.$$

It follows that $L_y(i) = L_x(j)$, from which we get

$$i = \begin{cases} \theta(y) + (j - \theta(x)) = j+m, & \text{if } j \geq \theta(x), \\ \theta(y) - (\theta(x) - j) = j+m, & \text{if } j < \theta(x). \end{cases}$$

Hence for every $j \in \mathbb{Z}$,

$$\varphi\big(\{(j,u) : u \in x(j)\}\big) = \{(j+m,u) : u \in y(j+m)\}.$$

Since F_0 consists only of pairwise non-isomorphic trees and each $x \in A$ is injective, for each $k \in \mathbb{Z}$ the finite ultrametric space $(x(k), d_{x(k)})$ is unique up to equality. We conclude that $x(k) = y(m+k)$ for all $m \in \mathbb{Z}$.

$\square$

In the proof of Claim 4.12, non-smoothness enters via our inability to uniformly select a canonical point from a given Heine-Borel ultrametric space. We instead rely on the fact that isometric equivalence is determined by an eventual agreement of large scale structure around two arbitrarily chosen points. This uncertainty is eliminated in the case that each space has a pre-designated point which is preserved by isometry.

Corollary 4.14. *Isometry of pointed, homogeneous, ultrahomogeneous Heine-Borel ultrametric spaces is smooth.*

Proof. The case for ultrahomogeneous and homogeneous follow from pointed, thus, let X and Y be Heine-Borel ultrametric spaces, with distinguished points $x \in X$ and $y \in Y$ respectively. We have that $X \cong_i Y$ as pointed spaces if there exists an isometry $\varphi : X \to Y$ such that $\varphi(x) = y$.

Let $B_0 \subseteq X$ be the open ball containing x such that $\mathrm{diam}(B_0) < 1$ is least, and let $C_0 \subseteq Y$ the analogously defined open ball around y. Then if $\varphi : X \cong_i Y$ with $\varphi(x) = y$, we have $\varphi(B_0) = C_0$. Thus the Borel reduction $f : \mathrm{UM}_{\mathrm{HB}} \to \mathrm{UM}_{\mathrm{C}}^{\omega}$ from Claim 4.12 defined by

$$f(X) = (B_n)_{n\in\omega}$$

Borel reduces the isometric equivalence of the pointed spaces X and Y to equality on $\mathrm{UM}_{\mathrm{C}}^{\omega}$, hence proving the result. □

4.4 Isometry on wider classes of Polish ultrametric spaces

Naturally one asks whether a similar combinatorical representation to those in Proposition 4.6 and Proposition 4.9 can be obtained for locally compact Polish ultrametric spaces.

The exact $\leq_B$ complexity of isometry of locally compact Polish ultrametric spaces remained unknown following the original Gao and Kechris [7] paper. In 2018, an exact lower bound was obtained in by Camerlo, Marcone, and Ros [2, Cor. 5.4].

Theorem 4.15. *Isometry of locally compact Polish ultrametric spaces is Borel bireducible with isomorphism of countable structures.*

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