# AN INTRODUCTION TO NONCOMMUTATIVE GEOMETRY AND HOPF ALGEBRAS

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Abstract

The duality between geometry and commutative algebra is pervasive throughout mathematics. However, the commutativity of the algebra restricts the range of geometries we can study and prohibits the geometrical interpretation of noncommutative algebras such as those present in quantum physics. Here we give a gentle introduction to the study of noncommutative geometry, with an overview of some of the main concepts and constructions.

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## 1. Introduction

The study of non-commutative geometry began with the development of quantum mechanics in the early 20th century where noncommutative operator algebras replaced the typically commutative setting of classical physics. Techniques studying the geometry of commutative algebras existed - algebraic geometry studies the space of prime ideals of a ring and can reason geometrically about algebraic structures. However, a noncommutative algebra may have no two-sided prime ideals and so the extension is non-trivial.

The main approach is outlined in the authoritative text on the subject Noncommutative Geometry by Alain Connes [4]. The approach is as follows: firstly associate to a geometrical space a not-necessarily commutative algebra. Secondly extend a geometrical concept so that the geometrical properties of the underlying space can be recovered from the algebra. Thirdly use these techniques to recover geometrical properties of an abstract space represented by some algebra. The significance (and difficulty) of this process rests on the potential noncommutativity of the algebra.

In his text, Connes extends several notions to noncommutative algebras including:

I. Measure Theory;
II. Topology and K-Theory;
III. Differential Geometry;
IV. Metric Spaces.

After Connes' theory was published, another approach to noncommutative geometry arose as the theory of quantum groups. This theory made use of Hopf algebras to generalise the notion of symmetry to noncommutative spaces by deforming Lie groups and Lie algebras into Hopf algebras.

In Section 2 we discuss the geometry of the quotients of topological spaces and construct a more general notion of a quotient which is achieved through the use of groupoids. The last portion of that section is then spent discussing an equivalence of this new quotient.

Section 3 describes two approaches to the differential geometry of noncommutative algebras. We describe briefly, the de Rham cohomology of smooth manifolds and then the construction of a cohomology for associative algebras that serves as the first steps towards an algebraic analogue. We then shift to another approach which makes use of differential algebras.

Finally in Section 4 we use the more recent quantum groups approach to noncommutative geometry in order to introduce some results in the theory of Hopf algebras and provide some examples of where quantum groups arise.

Noncommutative geometry is a relatively new field. However, it spans a wide range of mathematics and different approaches. Here we collate some approaches and palatable examples to convey an understanding of the general direction of the field.

## 2. Noncommutative Quotients

One can associate to a locally compact Hausdorff space $X$, a commutative $C^*$-algebra that allows one to recover $X$. However, it is not always reasonable to assume that a quotient of $X$ even be Hausdorff. The canonical example is the line with two origins:

*Example* 2.1. Consider the space of two copies of the real line $X = \mathbb{R} \times \{0, 1\}$ with the standard topology on $\mathbb{R}$. Now consider the equivalence relation $\sim$ that equates $(x, 0) \sim (x, 1) \iff x \neq 0$. The space $X$ is the product of two Hausdorff spaces and is thus Hausdorff. However the quotient $X/\sim$, visualised as the line with two origins, is not Hausdorff as the two origins cannot be separated by disjoint open sets.

As a result, quotients of topological spaces that can no longer be described by commutative $C^*$-algebras provide a large source of noncommutative spaces. The goal is to associate to such quotients a not necessarily commutative algebra that agrees in some sense with the classical algebra-geometry correspondence, and construct some topological tools. In this chapter we describe the machinery to define a non-commutative quotient, and define a notion of equivalence between algebras that agrees with the geometric structure they describe. We approach the subject as done by both Connes in Chapter 2 of [4] and by Khalkhali in Chapter 3 of [6, 7].

The classical way to approach the algebra of the quotient space is to apply a quotient construction to the corresponding $C^*$ algebra. Suppose that $X$ is a locally compact Hausdorff space with equivalence relation $\sim$. The space $X$ has the corresponding $C^*$-algebra $C(X) = \{f : X \to \mathbb{C}\}$ and the classical quotient is

$$\{f : X \to \mathbb{C} \mid \forall a \sim b \; f(a) = f(b)\}$$

The classical quotient construction allows us to apply topological tools when the quotient is "nice" i.e remains Hausdorff etc. If not, the classical quotient becomes too small and only constant functions remain. Thus we wish to define a noncommutative quotient whose K-theory and cohomology agrees with the classical quotient in the nice case. The noncommutative quotient will be larger, allowing us to obtain information from spaces even if the classical quotient becomes trivial.

The construction of a noncommutative quotient requires the notions of a groupoid and a groupoid algebra, which we define in the following subsections.

## 2.1. **Groupoids.**

**Definition 2.1.** *A groupoid is a small category where all the morphisms are isomorphisms.*

By small category we mean a category in which the objects form a set. We refer to the objects of a groupoid $\mathcal{G}$ by Obj $\mathcal{G}$ and the morphisms simply by $\mathcal{G}$. One can think of a groupoid as a generalisation of a group. By Cayley's theorem, every group is isomorphic to a subgroup of a symmetric group: the group of permutations on a set. Thus a groupoid with one object - as a single object and a set of isomorphisms - forms a group. The set of isomorphisms under composition form the group and thus we refer to the isomorphisms by $\mathcal{G}$.

*Example* 2.2. A natural occurrence of a groupoid is the fundamental groupoid $\mathcal{G}$ of a topological space $X$. The objects of the fundamental groupoid are the points of $X$, and the morphisms between two points are the equivalence classes of paths between the points where two paths are equivalent if they are homotopic. The restriction of the fundamental groupoid to a point $x \in X$ is the fundamental group at that point.

The canonical way to form an algebra from a group is to construct its group algebra by taking a vector space with group elements as basis elements and define a product by group multiplication. To extend this notion we take the basis elements to be the morphisms of the groupoid and the multiplication defined by composition of morphisms.

**Definition 2.2.** *The groupoid algebra $\mathbb{C}\mathcal{G}$ of a groupoid $\mathcal{G}$ over $\mathbb{C}$ is a generalisation of a group algebra defined by*

$$\mathbb{C}\mathcal{G} = \bigoplus_{\gamma \in \mathcal{G}} \mathbb{C}\gamma,$$

*that is the finite sums of elements of the form $\lambda\gamma$ where $\lambda \in \mathbb{C}$ and $\gamma \in \mathcal{G}$. We define multiplication between the basis elements as the composition of morphisms if they exist, and 0 if they don't.*

The elements of $\mathbb{C}\mathcal{G}$ are finite linear combinations of morphisms over $\mathbb{C}$. Thus a useful alternative definition is the set of complex-valued functions from $\mathcal{G}$ with finite support

$$\mathbb{C}\mathcal{G} \simeq \{f : \mathcal{G} \to \mathbb{C} \mid f \text{ has finite support}\}$$

endowed with the convolution product

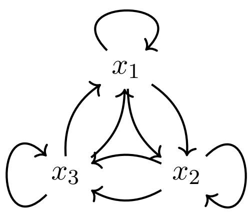
$$f * g = \sum_{\gamma = \gamma_1 \circ \gamma_2} f(\gamma_1) g(\gamma_2).$$

2.2. **Quotients.** With the description of a groupoid algebra, the noncommutative quotient will be the groupoid algebra of a space given some equivalence relation. To construct this groupoid, suppose we have space $X$ with equivalence relation $\sim$. We define the corresponding groupoid $\mathcal{G}$ by taking the objects to be the elements of $X$, i.e Obj $\mathcal{G} = X$. For all $x, y \in X$ we define an isomorphism from $x$ to $y$ if and only if $x \sim y$.

**Definition 2.3.** *The noncommutative quotient of $X$ with equivalence relation $\sim$ is the groupoid algebra $\mathbb{C}\mathcal{G}$.*

We wish to see what similarities this definition has with the classical quotient construction.

*Example* 2.3. Consider the set $X = \{x_1, x_2, x_3\}$ with equivalence relation $\sim$ equating every point. The objects of the groupoid $\mathcal{G}$ are the elements of $X$, i.e Obj $\mathcal{G} = X$. The isomorphisms are given by the following graph.



Thus taking the groupoid algebra we find

$$\mathbb{C}\mathcal{G} \simeq M_3(\mathbb{C})$$

by associating to the morphism taking $x_i \mapsto x_j$ the matrix $E_{ij}$. Whereas the equivalence relation equates every element, the classical quotient is

$$C(X/\sim) = \{f : X \to \mathbb{C} \mid f(x_1) = f(x_2) = f(x_3)\} \simeq \mathbb{C}.$$

For a slightly less trivial example, we keep the set $X$ but take the equivalence relation generated by the action of $S_3$.

*Example* 2.4. Let $S_3$ act on $X = \{x_1, x_2, x_3\}$ by $\sigma \cdot x_i = x_{\sigma(i)}$ for all $1 \leq i \leq 3$ and $\sigma \in S_3$. This induces the equivalence relation $x_i \sim x_j$ if there exists $\sigma \in S_3$ such that $\sigma \cdot x_i = x_j$. Clearly this equates every element of $X$ and induces the same equivalence relation as in the previous example. Now to generate the groupoid $\mathcal{G}$ we take $\mathrm{Obj}\,\mathcal{G} = X$ and

$$\mathrm{Hom}_{\mathcal{G}}(x_i, x_j) = \{\sigma \in S_3 \mid \sigma(i) = j\}.$$

Here the equivalence relation equates each pair of points by two distinct elements of $S_3$ and this is reflected in the construction of the groupoid. Therefore the classical quotient remains the same as the equivalence relation hasn't changed but the groupoid encodes the entirety of the equivalence.

The groupoid algebra in this case is

$$\mathbb{C}\mathcal{G} \simeq \mathbb{C}\mathbb{Z}_2 \otimes M_3(\mathbb{C}).$$

To show this we denote

$$\begin{aligned} \mathrm{Hom}_{\mathcal{G}}(x_i, x_i) &= \{\gamma_{ii}^0 = \varepsilon,\ \gamma_{ii}^1 = (j\ k)\}, \\ \mathrm{Hom}_{\mathcal{G}}(x_i, x_j) &= \{\gamma_{ij}^0 = (i\ j\ k),\ \gamma_{ij}^1 = (i\ j)\}, \end{aligned}$$

Then we construct a map from $\mathbb{C}\mathcal{G}$ to $\mathbb{C}\mathbb{Z}_2 \otimes M_3(\mathbb{C})$ by

$$\gamma_{ij}^0 \mapsto 0 \otimes E_{ij}, \quad \gamma_{ij}^1 \mapsto 1 \otimes E_{ij}.$$

In both examples the two quotients are not isomorphic and moreover share neither commutativity nor dimension.

2.3. **Morita Equivalence.** To describe the relation between the two quotients we define a weaker notion of equivalence which we do with the notion of an equivalence of categories.

**Definition 2.4.** *Two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there exist functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ such that their compositions are isomorphic to the identity functors:*

$$\mathcal{F} \circ \mathcal{G} \simeq 1_{\mathcal{D}}, \quad \mathcal{G} \circ \mathcal{F} \simeq 1_{\mathcal{C}}$$

*in the sense that there is a natural transformation from one to the other.*

We now define the equivalence between two quotients:

**Definition 2.5.** *Algebras $A$ and $B$ over a field are Morita equivalent if they have equivalent bimodule categories. That is*

$${}_A\mathcal{M}_A \simeq {}_B\mathcal{M}_B.$$

To construct a more computationally convenient definition of Morita equivalence we first need to describe the tensor products of bimodules . Suppose $A$ and $B$ are algebras, $X$ is a $(A, B)$-bimodule and $Y$ is a $(B, A)$-bimodule. Thus $X$ is an abelian group on which $A$ acts on the left and $B$ acts on the right. Moreover the actions of $A$ and $B$ on $X$ agree so for all $x \in X, a \in A, b \in B$ we have $(a \cdot x) \cdot b = a \cdot (x \cdot b)$. The tensor product of $X$ and $Y$ over $A$

$$X \otimes_B Y$$

is the tensor product of $X$ as a right $B$-module and $Y$ as a left $B$-module. One can then place a $(A, A)$-bimodule structure on $X \otimes_B Y$ by $a \cdot (x \otimes_B y) = (a \cdot x) \otimes_B y$ and $(x \otimes_B y) \cdot a = x \otimes_B (y \cdot a)$ for all $a \in A$ and $x \otimes_B y \in X \otimes_B Y$ and then extend the structure linearly.

With this we can provide an equivalent definition of Morita equivalence.

**Definition 2.6.** *Two algebras $A$ and $B$ are Morita equivalent if there exists an $(A, B)$-bimodule $X$ and $(B, A)$-bimodule $Y$ such that*

$$X \otimes_B Y \simeq A, \quad Y \otimes_A X \simeq B.$$

*where $A$ and $B$ are taken to be bimodules over themselves and the equivalence is module isomorphism.*

With this definition we can immediately verify some Morita equivalences.

**Lemma 2.1.** *Let $A$ be a unital algebra. The algebra of $A$-valued $n \times n$ matrices $B = M_n(A)$ is Morita equivalent to $A$.*

*Proof.* Take $X = A^n$ as an $(A, B)$-bimodule by the canonical multiplication on either side. Similarly take $Y = A^n$ to be a $(B, A)$-bimodule. Now define a map $\varphi$ from $X \otimes_B Y$ to $A$ by

$$(a_1, \ldots, a_n)^T \otimes_B (b_1, \ldots, b_n)^T \mapsto \sum_{i=1}^{n} a_i b_i = a^T b.$$

The mapping is surjective as for all $x \in A$ we have

$$(x, 0, \ldots, 0) \otimes (1_A, 0, \ldots, 0) \mapsto x.$$

To show that the mapping is injective suppose $a \otimes b \mapsto 0$ for some $a \otimes b \in X \otimes_B Y$. Then $a^T b = 0$. Thus the two elements of $A^n$ are orthogonal so take the projection matrix $P = \frac{bb^T}{b^T b}$ onto the subspace generated by $b$. We show that $P$ will fix $b$ and annihilate $a$. Thus

$$a \otimes b = a \otimes P \cdot b = a \cdot P \otimes b = a^T \frac{bb^T}{b^T b} \otimes b = 0 \otimes b = 0.$$

Therefore $\varphi$ is injective on elements of the form $a \otimes b \in X \otimes_B Y$. Now suppose $x = \sum_{i=1}^{m} a_i \otimes b_i \in X \otimes_B Y$ such that $x \mapsto 0$. Without loss of generality we can assume that no $a_i$ or $b_i$ is 0. Now for each $i \neq 1$ there is an invertible matrix $P_i$ such that $P_i \cdot b_i = b_1$. We then have

$$x = \sum_{i=1}^{m} a_i \otimes P_i^{-1} P_i \cdot b_i = \sum_{i=1}^{m} a_i \cdot P_i^{-1} \otimes b_1 = \left( \sum_{i=1}^{m} a_i \cdot P_i^{-1} \right) \otimes b_1$$

By the previous part we have that $x = 0$. Hence $\varphi$ is injective on $X \otimes_B Y$.

To show $\varphi$ is a $(A, A)$-bimodule isomorphism, suppose $x \in A$ and $a \otimes b \in X \otimes_B Y$.

$$x \cdot \varphi(a \otimes b) = x \sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} (x a_i) b_i = \varphi(x \cdot a \otimes b).$$

and similarly for the right action. Therefore since $\varphi$ is a linear map, it is an $(A, A)$-bimodule isomorphism.

Similarly define a linear map $\psi$ from $Y \otimes_A X$ to $M_n(A)$ by

$$(a_1, \ldots, a_n)^T \otimes_A (b_1, \ldots, b_n)^T \mapsto (a_i b_j)_{ij} = ab^T.$$

First for all $1 \leq i \leq n$ define $e_i \in A^n$ to be zero everywhere except for in the $i$-th position where it is $1_A$.

To show $\psi$ is surjective, suppose $M = (m_{ij})_{ij} \in M_n(A)$. Then

$$\sum_{i=1}^{n} e_i \otimes (m_{i1}, \ldots, m_{in})^T \mapsto M$$

For injectivity, suppose $\sum_{\alpha=1}^{m} a_\alpha \otimes b_\alpha \mapsto 0$ for some $\sum_{\alpha=1}^{m} a_\alpha \otimes b_\alpha \in Y \otimes_A X$. Then $\sum_{\alpha=1}^{m} a_{\alpha_i} b_{\alpha_j} = 0$ for all $1 \leq i, j \leq n$. Thus

$$\begin{aligned}
\sum_{\alpha=1}^{m} a_\alpha \otimes b_\alpha &= \sum_{\alpha=1}^{m} \left( \sum_{i=1}^{n} e_i \cdot a_{\alpha_i} \right) \otimes b_\alpha \\
&= \sum_{\alpha=1}^{m} \left( \sum_{i=1}^{n} e_i \cdot a_{\alpha_i} \otimes b_\alpha \right) \\
&= \sum_{\alpha=1}^{m} \left( \sum_{i=1}^{n} e_i \otimes a_{\alpha_i} \cdot b_\alpha \right) \\
&= \sum_{i=1}^{n} e_i \otimes \left( \sum_{\alpha=1}^{m} a_{\alpha_i} \cdot b_\alpha \right) = 0
\end{aligned}$$

To see that $\psi$ is a homomorphism, note that for all $M \in M_n(A)$

$$M \cdot \psi(a \otimes b) = M \cdot ab^t = Ma \cdot b^T = \psi(Ma \otimes b).$$

by the same reasoning, $\psi$ is a homomorphism on the right as well.

Therefore both $\varphi$ and $\psi$ are bimodule isomorphisms and thus $A$ is Morita equivalent to $M_n(A)$ for any unital algebra $A$. □

Looking back at example 2.3 we see now that the two quotients are Morita equivalent. Thus we can note that commutativity is not preserved under Morita equivalence.

However, what is preserved under Morita equivalence is important to the study of noncommutative geometry. If two algebras are Morita equivalent then they have isomorphic Hochschild cohomology groups. We define these notions in section 3.

A Morita equivalence describes an equivalence of geometry and in the commutative case isomorphic algebras yield the same geometry. We show that commutative algebras are isomorphic if and only if they are Morita equivalent. To do this we first need an important result from category theory

**Lemma 2.2** (Yoneda Lemma)**.** *Let $\mathcal{C}$ be a category, $X$ an object of $C$, and consider the contravariant functor $h_X := \mathrm{Hom}_{\mathcal{C}}(X, \_)$. For every functor $\mathcal{F} : \mathcal{C} \to \mathrm{Set}$ whose domain is locally small, there is a bijection between $\mathcal{F}_X$ and the natural transformations from $\mathrm{Hom}_{\mathcal{C}}(X, \_)$ to $\mathcal{F}_X$*

$$\mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(X, \_), \mathcal{F}) \cong \mathcal{F}_X$$

*that associates a natural transformation $\alpha \in \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(X, \_), \mathcal{F})$ to the element $\alpha_X(\mathrm{id}_X)$.*

We present it here without proof but the interested reader can find a proof in Section 2.2 of [14].

We use the Yoneda lemma to first prove a more general result about the center of algebras.

**Lemma 2.3.** *Suppose $A$ is a unital algebra. Then the endomorphism ring of the hom functor $\mathrm{Nat}_{{}_A\mathcal{M}_A}(h_A, h_A)$ is isomorphic to $Z(A)$ as algebras.*

*Proof.* We first note that ${}_A\mathcal{M}_A$ is a small category as the collection of module homomorphisms from one module to another form a set. Thus from the Yoneda lemma we obtain

$$\mathrm{Nat}(h_A, h_A) \simeq \mathrm{Hom}(A, A).$$

Now define a function $f : Z(A) \to \mathrm{Hom}(A, A)$ for all $a \in Z(A)$ by $a \mapsto \varphi_a$ where $\varphi_a(x) = ax$ for all $x \in A$. Now note that for any $\varphi \in \mathrm{Hom}(A, A)$ we have

$$a\varphi(1) = \varphi(a) = \varphi(1)a$$

for all $a \in A$. Thus $\varphi(1)$ is central. Moreover since $\varphi$ is uniquely determined by $\varphi(1)$ we see that $f$ is a bijection. Moreover $f$ is linear as

$$f(\lambda(a+b))(x) = \varphi_{\lambda(a+b)} = \lambda(a+b)x = (\lambda a)x + (\lambda b)x = \lambda f(a)(x) + \lambda f(b)(x)$$

for all $\lambda \in \mathbb{C}$, $a, b \in Z(A)$, and $x \in A$. And finally $f$ is multiplicative as

$$f(ab)(x) = \varphi_{ab}(x) = abx = a(bx) = \varphi_a \circ \varphi_b(x) = f(a)f(b)(x).$$

□

**Theorem 2.1.** *If $A$ and $B$ are commutative unital algebras then $A$ and $B$ are Morita equivalent if and only if they are isomorphic.*

*Proof.* Suppose that $A \simeq B$ and $\varphi : A \to B$ is an isomorphism. Define an $A$-action on $B$ by $a \cdot b = \varphi(a)b$ for all $a \in A$ and $b \in B$. Note that this defines both a left and a right action on $B$ and they agree. Similarly define a $B$-action on $A$ with $\varphi^{-1}$. Define $f : A \otimes_A B \to B$ by

$$a \otimes b \mapsto \varphi(a)b$$

for all $a \otimes b \in A \otimes_A B$. The mapping is surjective as $1 \otimes b \mapsto b$ for all $b \in B$. And $f$ is injective as if $a \otimes b \mapsto 0$ then $\varphi(a)b = 0$ and thus

$$a \otimes b = 1_A \cdot a \otimes b = 1_A \otimes a \cdot b = 1_A \otimes \varphi(a)b = 0.$$

Moreover $f$ is linear by the linearity of $\varphi$ and $f$ is a homomorphism as if $a \in A$ and $x \otimes y \in A \otimes_A B$

$$a \cdot f(x \otimes y) = \varphi(a)\varphi(x)y = \varphi(ax)y = f(ax \otimes y) = f(a \cdot x \otimes y).$$

Therefore $A \otimes_A B \simeq B$ and by symmetry $B \otimes_B A \simeq A$. Thus $A$ is Morita equivalent to $B$.

Now suppose ${}_A\mathcal{M}_A$ and ${}_B\mathcal{M}_B$ are equivalent categories. Note that $h_A = 1_{{}_A\mathcal{M}_A}$ as $\mathrm{Hom}(A, M) = M$ for any $A$-bimodule. Similarly for $h_B$. Since the endomorphism rings of the identity functors of equivalent categories are isomorphic, the centers of $A$ and $B$ are isomorphic by Lemma 2.3. Because both $A$ and $B$ are commutative we have

$$A = Z(A) \simeq Z(B) = B.$$

□

## 3. Noncommutative Homology and Cohomology

Homology and cohomology are a wide class of mathematical tools that associate to mathematical objects, a sequence of abelian groups. Famously in algebraic topology, simplicial and singular homology theories are used to reason algebraically about topological properties. The de Rham cohomology is another such theory used in the study of smooth manifolds and is a topological and smooth invariant. To describe the differential geometry of algebras we wish to generalise the de Rham cohomology.

### 3.1. De Rham Cohomology.

Before we discuss the Hochschild cohomology, to appreciate the similarities to the de Rham cohomology, we give a brief summary of the construction which we take from [12].

To describe the de Rham cohomology, we first need the notion of an alternating map.

**Definition 3.1.** *Let $V$ and $W$ be vector spaces and let $S_n$ denote the symmetric group on $n$ elements. An alternating map is a $k$-linear map $f : V^k \to W$ such that for all $(v_1, \dots, v_k) \in V^k$ and $\sigma \in S_k$*

$$f(v_{\sigma(1)} \times \cdots \times v_{\sigma(k)}) = \mathrm{sign}(\sigma) f(v_1 \times \cdots \times v_k).$$

We denote the set of alternating functions into $\mathbb{R}$ by

$$\mathrm{Alt}^k(V) = \{f : V^k \to \mathbb{R} \mid f \text{ is alternating}\}.$$

which forms a $\mathbb{R}$-vector space under pointwise addition and scalar multiplication by $\mathbb{R}$.

*Example* 3.1. The canonical example is the determinant viewed as a function $\det : (\mathbb{R}^n)^n \to \mathbb{R}$ that takes $n$ elements $v_1, \dots, v_n$ of $\mathbb{R}^n$ to the determinant of the matrix $[v_1, \dots, v_n]$. Since the determinant is $n$-linear and alternating under permutation of the columns, $\det \in \mathrm{Alt}(\mathbb{R}^n)$.

However, we wish to be able to combine arbitrary alternating functions. We achieve this by specific permutations of the inputs of the products of alternating functions. Such permutations are called shuffles. They are described as follows:

**Definition 3.2.** *Let $k, l \geq 1$. A $(k, l)$-shuffle is a permutation $\sigma \in S_{k+l}$ such that*

$$\sigma(1) < \sigma(2) < \cdots < \sigma(k) \text{ and } \sigma(k+1) < \cdots < \sigma(k+l).$$

*We denote the set of all $(k.l)$-shuffles by $S(k, l)$.*

We combine alternating functions with the notion of an exterior product.

**Definition 3.3.** *Let $k, l \geq 1$ and $\omega \in \mathrm{Alt}^k(V), \tau \in \mathrm{Alt}^l(V)$. The exterior product is a map $\wedge : \mathrm{Alt}^k \oplus \mathrm{Alt}^l \to \mathrm{Alt}^{k+l}$ defined by*

$$\omega \wedge \tau = \sum_{\sigma \in S(k,l)} \mathrm{sign}(\sigma)\omega(v_{\sigma(1)}, \dots, v_{\sigma(k)})\tau(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)})$$

We now shift our attention towards functions that map into the set of alternating functions and these will ultimately be the objects of our attention - forms. But we require that such functions be smooth.

**Definition 3.4.** *Let $U$ be an open subset of $\mathbb{R}^n$. A function $f : U \to \mathrm{Alt}^k(\mathbb{R}^n)$ is said to be $C^\infty$-smooth if for all $k \leq n$ and $\sigma \in S(k, n-k)$ the mapping from $U$ to $\mathbb{R}$ given by*

$$x \mapsto (f(x))(e_{\sigma(1)}, \dots, e_{\sigma(k)})$$

*is smooth.*

And finally we can define the notion of a differential form.

**Definition 3.5.** *Let $U$ be an open subset of $\mathbb{R}^n$ and $k \geq 0$. A differential $k$-form on $U$ is a $C^\infty$-smooth function $f : U \to \mathrm{Alt}^k(\mathbb{R}^n)$. The set of $k$-forms on $U$ is denoted by $\Omega^k(U)$.*

Note in particular that, since $\mathrm{Alt}^0(U) = \mathbb{R}$, 0-forms are smooth functions from $U$ to $\mathbb{R}$.

In the de Rham cohomology the spaces of $k$-forms will form the sequence of modules. The homomorphism between them will be the exterior derivative that takes a $k$-form to a $(k+1)$-form. The exterior derivative of a 0-form $f : U \to \mathbb{R}$ is the 1-form $df : U \to \mathrm{Alt}^1(U)$ given by

$$(df)(x)(v) = \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}$$

for all $x, v \in U$. Note here that $(df)(x) \in \mathrm{Alt}^1(U)$. The exterior derivative of a $k$-form is as follows:

**Definition 3.6.** *Suppose $\omega \in \Omega^k(U)$ for some open set $U \subseteq \mathbb{R}^n$. Let $x_i : \mathbb{R}^n \to \mathbb{R}$ denote the projection onto the $i$-th coordinate and let $\omega_\sigma : U \to \mathbb{R}$ be defined by*

$$\omega_\sigma(x) = w(x)(e_{\sigma(1)}, \dots, e_{\sigma(k)}).$$

*for all $\sigma \in S(k, n-k)$. The exterior derivative $d\omega$ of $\omega$ is given by*

$$d\omega = \sum_{\sigma \in S(k,n-k)} d\omega_\sigma \wedge dx_{\sigma(1)} \wedge \cdots \wedge dx_{\sigma(k)}.$$

*Example* 3.2. Let $U \subseteq \mathbb{R}$ and $f : U \to \mathbb{R}$ be the smooth function taking $x \mapsto x^2$. Then the exterior derivative is given by

$$(df)(x)(v) = 2xv$$

for all $x, v \in \mathbb{R}$. Now to compute the second exterior derivative of $f$ we want so sum over all $(n, n-k)$-shuffles but since $n = k = 1$ we only have the trivial permutation. Thus

$$d^2 f = d\omega \wedge dx$$

where $x$ is the identity function on $U$ and $\omega(x) = df(x)(1) = 2x$. Thus we obtain

$$(d^2 f)(x)(v_1, v_2) = \sum_{\sigma \in S(1,1)} 2(v_{\sigma(1)}) \cdot 1(v_{\sigma(2)}) = 2(v_1) \cdot 1(v_2) - 2(v_2) \cdot 1(v_1) = 0$$

for all $x \in \mathbb{R}$ and $(v_1, v_2) \in \mathbb{R}^2$.

With the definition of $k$-forms and the exterior derivative, we can now construct the de Rham complex and cohomology.

**Definition 3.7.** *The de Rham cohomology of an open subset $U$ of $\mathbb{R}^n$ is the cohomology of the cochain complex*

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \cdots$$

*where we denote the $k$-th cohomology group at $U$ by*

$$H^k(U) = \frac{\ker(d : \Omega^k(U) \to \Omega^{k+1}(U))}{\operatorname{Im}(d : \Omega^{k-1}(U) \to \Omega^k(U))}$$

We summarise an example given in Section 15.10 of [1].

*Example* 3.3. To compute the de Rham cohomology of $S^1$ we require the Mayer-Vietoris sequence as described in Section 15.8 of [1] which allows us to compute the higher order cohomology groups of a space $M$ by decomposing $M$ into a union of open sets. Suppose $M$ is the union of two open sets $U$ and $V$. The Mayer-Vietoris sequence is the long exact sequence of cohomology groups given by:

$$H^0(M) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to H^1(M) \to \cdots.$$

To determine cohomology groups of $S^1$ we need two observations. The first is that the 0-th cohomology group of a manifold $M$ is given by $H^0(M) = \mathbb{R}^n$ where $n$ is the number of connected components of $n$. This is because the 0-th cohomology group is the set of smooth functions that are locally constant. The second observation is that $S^1$ is a 1-dimensional manifold and thus the higher order cohomology groups will be 0. We now decompose $S^1$ into two open intervals $U$ and $V$ such that the intersection of $U$ and $V$ is a pair of disjoint open intervals. Since $S^1$, $U$, and $V$ are connected, they each have 0-th cohomology group $\mathbb{R}$. However $U \cap V$ has two connected components. Thus the Mayer-Vietoris sequence for $S^1$ is

$$0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R}^2 \to H^1(S^1) \to 0.$$

Since the sequence is long exact we obtain

$$H^1(S^1) = \mathbb{R}$$

and therefore

$$H^n(S^1) = \begin{cases} \mathbb{R} & n \leq 1 \\ 0 & n > 1 \end{cases}$$

3.2. **Hochschild Cohomology.** With the classical description of the de Rham cohomology done we now wish to associate an analogue cohomology to a not necessarily commutative algebra. We can begin to do this with the Hochschild cohomology of associative algebras.

**Definition 3.8.** *Let $A$ be an algebra and $M$ an $A$-bimodule. The Hochschild cohomology of an algebra $A$ with coefficients in $M$ is the cohomology of the cochain complex*

$$C^0(A,M) \xrightarrow{\delta_0} C^1(A,M) \xrightarrow{\delta_1} \cdots$$

*where $C^0(A,M) = M$ and $C^n(A,M) = \mathrm{Hom}_A(A^{\otimes n}, M)$ for $n > 0$. And $\delta_0$ takes elements to the ring commutator of $A$.*

$$\delta_0(m) = [m, \cdot] \quad \forall f \in A$$

*and otherwise*

$$\begin{aligned}\delta_n(f)(a_1, \dots a_{n+1}) &= a_1 f(a_2, \dots, a_{n+1}) + (-1)^{n+1} f(a_1, \dots, a_n) a_{n+1} \\ &\quad + \sum_{i=1}^{n} (-1)^i f(a_1, \dots, a_i a_{i+1}, \dots, a_{n+1}).\end{aligned}$$

*for $f \in C^n(A,M)$, $n > 0$. We denote the $n$-th Hochschild cohomology group by $H^n(A,M)$.*

As an example we compute first two cohomology groups of $\mathbb{C}$ over itself.

*Example* 3.4. The cochain complex is defined to be

$$0 \to \mathbb{C} \xrightarrow{\delta_0} \mathrm{Hom}(\mathbb{C},\mathbb{C}) \xrightarrow{\delta_1} \mathrm{Hom}(\mathbb{C}^2,\mathbb{C}) \xrightarrow{\delta_2} \cdots$$

The 0-th cohomology group is then the kernel of $\delta_0$. However, since the image of $\delta_0$ takes an element of $\mathbb{C}$ to is commutator with another element, we know that $\ker(\delta_0) = \mathbb{C}$ by the commutativity of $\mathbb{C}$ and therefore.

$$H^0(\mathbb{C},\mathbb{C}) = \mathbb{C}.$$

To compute the first cohomology group now suppose $f \in \ker(\delta_1)$ and $a_1, a_2 \in \mathbb{C}$. Then

$$\delta_1(f)(a_1, a_2) = a_1 f(a_2) + f(a_1) a_2 - f(a_1 a_2) = 0.$$

and therefore

$$f(a_1 a_2) = a_1 f(a_2) + f(a_1) a_2.$$

We now take $a_2 = 1$ and note that the $\mathbb{C}$ acts on $\mathbb{C}$ by multiplication to obtain

$$f(a_1) = a_1 f(1) + f(a_1) = 2f(a_1)$$

for all $a_1 \in \mathbb{C}$. Therefore $f = 0$ and $\ker(\delta_1) = 0$. Thus the first cohomology group is

$$H^1(\mathbb{C},\mathbb{C}) = 0.$$

We now consider the Hochschild cohomology groups of a noncommutative algebra.

*Example* 3.5. Let $A = M_n(\mathbb{C})$. The 0-th cohomology group of $A$ over $A$ is given by

$$H^0(A, A) = \ker(\delta_0) = \{M \in A \mid \forall N \in A\ [M, N] = 0\}.$$

The only such elements of $A$ are scalar multiples of the identity and thus we obtain

$$H^0(A, A) = \mathbb{C}.$$

The first cohomology group of $A$ is given by

$$H^1(A, A) = \frac{\ker(\delta_1)}{\operatorname{Im}(\delta_0)}.$$

As in Example 3.4 $\ker(\delta_1)$ is the set of all derivations on $A$. The image of $\delta_0$ is given by

$$\operatorname{Im}(\delta_0) = \{f \in \operatorname{Hom}(A, A) \mid f = [M, \cdot] \text{ for some } M \in A\}.$$

Such maps are called inner. It is well known that all derivations on $M_n(\mathbb{C})$ are inner and the interested reader may see [5] by Dirac for a proof. Thus the first cohomology group is trivial.

$$H^1(A, A) = 0.$$

3.3. **Differential Algebras.** Another approach to a noncommutative differential geometry is outlined in [3] by Beggs and Majid where a differential structure is placed on arbitrary algebras in the form of modules acting as $n$-forms.

**Definition 3.9.** *If $A$ is an algebra and $\Omega^1$ an $A$-bimodule then a map $d : A \to \Omega^1$ is a derivation if it is linear and satisfies the Leibniz rule:*

$$d(ab) = ad(b) + d(a)b$$

*for all $a, b \in A$.*

**Definition 3.10.** *Let $A$ be an algebra and $\Omega^1$ an $A$-bimodule. A first order differential calculus is a 3-tuple $(A, \Omega^1, d)$ where $d$ is a derivation*

$$d : A \to \Omega^1$$

*satisfying $\Omega^1 = \operatorname{span}\{ad(b) \mid a, b \in A\}$.*

Here $\Omega^1$ generalises the notion of 1-forms. We say that a differential calculus is inner if there is some element $\theta \in \Omega^1$ such that $d(a) = \theta a - a\theta$ for all $a \in A$.

We present an example from [3] with the details filled in.

*Example* 3.6. Let $A = \mathbb{C}[x]/\langle x^2 \rangle$ with

$$\Omega^1 = \left\{ \sum_{i=1}^{n} a_i \otimes b_i \in A \otimes A \mid \sum_{i=1}^{n} a_i b_i = 0 \right\}$$

and derivation $d : A \to \Omega^1$ given by

$$d(a) = 1 \otimes a - a \otimes 1$$

for all $a \in A$. Here $\Omega^1$ inherits the canonical bimodule structure of $A \otimes A$. Then $(A, \Omega^1, d)$ forms a first order differential calculus that is not inner. We can see that $d$ is a derivation because if $a, b \in A$ then

$$\begin{aligned} ad(b) + d(a)b &= a(1 \otimes b - b \otimes 1) + (1 \otimes a - a \otimes b)a \\ &= a \otimes b - ab \otimes 1 + 1 \otimes ab - a \otimes b \\ &= 1 \otimes ab - ab \otimes 1 \\ &= d(ab) \end{aligned}$$

It is easy to see that as a vector space we have

$$A \otimes A = \langle 1 \otimes 1, 1 \otimes x, x \otimes 1, x \otimes x \rangle$$

Now to find a basis for $\Omega^1$ suppose

$$a(1 \otimes 1) + b(1 \otimes x) + c(x \otimes 1) + d(x \otimes x) \in \Omega^1$$

Then $a + bx + cx + dx^2 = a + bx + cx = 0$ so $a = 0$ and $b = -c$. We then obtain a basis

$$\Omega^1 = \langle 1 \otimes x - x \otimes 1, x \otimes x \rangle$$

Now notice that

$$d(x) = 1 \otimes x - x \otimes 1 \quad \text{and} \quad xd(x) = x(1 \otimes x - x \otimes 1) = x \otimes x - x^2 \otimes 1 = x \otimes x$$

so in fact $\{d(x), xd(x)\}$ forms a basis for $\Omega^1$. In this basis the left module structure is obvious and the right module structure is given by

$$d(x)x = (1 \otimes x - x \otimes 1)x = 1 \otimes x^2 - x \otimes x = -x \otimes x = -xd(x)$$

and

$$xd(x)x = (x \otimes x)x = x \otimes x^2 = 0.$$

Now suppose for a contradiction that $(A, \Omega^1, d)$ is inner with some element $\theta = \lambda d(x) + \mu x d(x)$ for some $\lambda, \mu \in \mathbb{C}$. Then

$$\begin{aligned} d(x) &= \theta x - x\theta \\ &= (\lambda d(x) + \mu x d(x))x - x(\lambda d(x) + \mu x d(x)) \\ &= -\lambda x d(x) - \lambda x d(x) \\ &= -2\lambda x d(x). \end{aligned}$$

However since $d(x)$ and $xd(x)$ are linearly independent we have a contradiction. Therefore $(A, \Omega^1, d)$ is not inner.

3.3.1. *Differential Calculus of Finite Sets.* To ground the notion of differential calculi we describe the first order differential calculus of the algebra of functions on a finite set. Let $X$ be a finite set of size $n$. The algebra is given by the set of complex valued functions on $X$ so we set $A = \mathbb{C}(X)$ and let $\Omega^1$ be an $A$-bimodule with derivation $d : A \to \Omega^1$ such that $(A, \Omega^1, d)$ is a first order differential calculus. Proposition 1.24 from Beggs and Majid [3], which we state without proof, describes all such differential calculi

**Proposition 3.1.** *The differential calculus $(A, \Omega^1, d)$ is inner and corresponds to a directed graph $G$ on $X$. Here we denote a directed edge from $x$ to $y$ by $\omega_{x \to y}$. The correspondence is given by*

$$\Omega^1 = \mathrm{span}_{\mathbb{C}}\{\omega_{x \to y}\}$$

*with left and right actions given by*

$$f \cdot \omega_{x \to y} = f(x)\omega_{x \to y}, \quad \omega_{x \to y} \cdot f = f(y)\omega_{x \to y}$$

*for all $f \in A$ and differential operator given by*

$$d(f) = \sum_{x \to y} (f(x) - f(y))\omega_{x \to y}$$

*where $f \in A$ and the summation runs over all directed edges in $G$.*

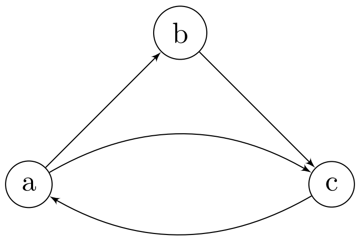
In this case by setting $\theta = \sum_{x \to y} \omega_{x \to y}$ we see that the calculus is inner as

$$d(f) = \sum_{x \to y} (f(x) - f(y))\omega_{x \to y} = \sum_{x \to y} f \cdot \omega_{x \to y} - \omega_{x \to y} \cdot f = f \cdot \theta - \theta \cdot f$$

for all $f \in A$.

Consider, for example, the first order calculus on a set of three elements.

*Example* 3.7. Let $X = \{a, b, c\}$ with first order differential calculus described by the following directed graph:




Here $\Omega^1$ is a four-dimensional vector space given by

$$\Omega^1 = \mathrm{span}_{\mathbb{C}}\{\omega_{a \to b}, \omega_{b \to c}, \omega_{a \to c}, \omega_{c \to a}\}.$$

Let $f$ be the function that is 1 on $a$ and 0 on $b$ and $c$. The differential of $f$ is then given by

$$\begin{aligned} d(f) &= \sum_{x \to y} f \cdot \omega_{x \to y} - \omega_{x \to y} \cdot f \\ &= (f(a) - f(b))\omega_{a \to b} + (f(b) - f(c))\omega_{b \to c} \\ &+ (f(a) - f(c))\omega_{a \to c} + (f(c) - f(a))\omega_{c \to a} \\ &= \omega_{a \to b} + \omega_{a \to c} + \omega_{c \to a}. \end{aligned}$$

3.3.2. *Differential Graded Algebras.* To extend $d$ to the notion of an exterior derivative, Beggs and Majid construct an algebra as a sum of $A$-modules $\Omega^i$ for $i \in \mathbb{N}$ with $\Omega^0 = A$. In this setting $A$ serves as the generalisation of 0-forms - the set of smooth functions on some abstract space and $\Omega^k$ as the space of $k$-forms.

**Definition 3.11.** *A differential graded algebra of an algebra $A$ is a graded algebra $\Omega = \oplus_{i=0}^{\infty} \Omega^i$ with product $\wedge$. Here $\Omega^0 = A$ and $\Omega$ is equipped with derivation $d : \Omega^n \to \Omega^{n+1}$ for all $n \in \mathbb{N}$ such that $d^2 = 0$ and*

$$d(\omega \wedge \rho) = d(\omega) \wedge \rho + (-1)^n \omega \wedge d(\rho)$$

*for all $\rho \in \Omega$, $\omega \in \Omega^n$.*

The noncommutative analogue of the de Rham complex is the complex

$$\Omega^0 \xrightarrow{\delta} \Omega^1 \xrightarrow{\delta} \Omega^2 \xrightarrow{\delta} \cdots$$

with cohomology groups

$$H^n_{\mathrm{dR}}(A) = \frac{\ker(d|_{\Omega^n})}{d(\Omega^{n-1})}$$

## 4. Quantum Groups

The universal enveloping algebra of a Lie group, and the set of functions on a group can both be shown to have the structure of a Hopf algebra. A quantum group is a class of (generally non-commutative) Hopf algebra parameterised by some value $q$ that recovers the structure of a well-known algebra in the limit $q \to 1$. In this sense a quantum group can be seen as a deformation of some algebras. It also explains the name "Quantum group" where $q \to 1$ approaches a "classical" case in the same way a quantum system approaches a classical one in the limit $\hbar \to 0$.

4.1. **Hopf Algebras.** Hopf algebras combine the algebraic structure of both algebras and co-algebras. Thus they result in algebraic objects with a lot of structure. This means that Hopf algebras become useful in generalising some structures such as groups and Lie algebras. We describe some explicit constructions in later sections. Note that we consider only finite dimensional Hopf algebras.

Firstly, a Hopf algebra has the structure of an unital associative algebra. We give the definition but note that that unital condition is enforced by the construction of unit map. This allows us to more neatly construct a counit when we define a coalgebra.

**Definition 4.1.** *A unital associative algebra* $(H, \nabla, \eta)$ *over* $\mathbb{C}$ *is a vector space* $H$ *with linear product and unit maps*

$$\nabla : H \otimes H \to H$$
$$\eta : \mathbb{C} \to H$$

*satisfying associativity and the commutativity of field elements:*

$$\nabla \circ (\nabla \otimes \mathrm{id}_H) = \nabla \circ (\mathrm{id}_H \otimes \nabla) \tag{1}$$

$$\nabla \circ (\eta \otimes \mathrm{id}_H) = \nabla \circ (\mathrm{id}_H \otimes \eta) = \mathrm{id}_H \tag{2}$$

*with* $\eta(1_{\mathbb{C}}) = 1_H$ *an identity element.*

We can represent the associativity as satisfying the following commuting diagram:

$$\begin{tikzcd}
& H \otimes H \otimes H \arrow[dl, "\nabla\otimes\mathrm{id_H}"'] \arrow[dr, "\mathrm{id}_H\otimes\nabla"] & \\
H \otimes H \arrow[dr, "\nabla"'] & & H \otimes H \arrow[dl, "\nabla"] \\
& H &
\end{tikzcd}$$

Figure 1. Associativity of Algebra Product.

And the commutativity of the unit map as satisfying:

$$\begin{tikzcd}
 & H & \\
H\otimes H \arrow[ur, "\nabla"] & & H\otimes H \arrow[ul, "\nabla"'] \\
 & \mathbb{C}\otimes H \simeq H \simeq H\otimes\mathbb{C} \arrow[uu, "\mathrm{id}_H"] \arrow[ul, "\eta\otimes\mathrm{id}_H"] \arrow[ur, "\mathrm{id}_H\otimes\eta"'] &
\end{tikzcd}$$

FIGURE 2. Commutativity of Algebra Unit Map.

These diagrams serve little purpose when describing solely an algebra. However, they help illustrate the natural construction of a coalgebra which we describe next.

We also wish the Hopf algebra to have the structure of a coalgebra. Where an algebra is a vector space with product and unit map, a coalgebra is a vector space with coproduct and counit.

**Definition 4.2.** *A coalgebra $(H, \Delta, \varepsilon)$ over $\mathbb{C}$ is a vector space $H$ with coproduct $\Delta$ and counit $\varepsilon$:*

$$\Delta : H \to H \otimes H$$
$$\varepsilon : H \to \mathbb{C}$$

*satisfying coassociativity and cocommutativity of the counit.:*

$$(\Delta \otimes \mathrm{id}_H) \circ \Delta = (\mathrm{id}_H \otimes \Delta) \circ \Delta \tag{3}$$
$$(\varepsilon \otimes \mathrm{id}_H) \circ \Delta = (\mathrm{id}_H \otimes \varepsilon) \circ \Delta = \mathrm{id}_H \tag{4}$$

The coassosiativity of the coalgebra can be described with the following commuting diagram. Notice that one can obtain this diagram by reversing the arrows of the diagram describing associativity of algebras.

$$\begin{tikzcd}
 & H\otimes H\otimes H & \\
H\otimes H \arrow[ur, "\Delta\otimes\mathrm{id}_H"] & & H\otimes H \arrow[ul, "\mathrm{id}_H\otimes\Delta"'] \\
 & H \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

FIGURE 3. Coassociativity of Coalgebra Coproduct.

Similarly we can reverse the arrows in figure 2 to obtain the diagram describing the cocommutativity of the counit.

$$
\begin{tikzcd}
 & H \arrow[dl, "\Delta"'] \arrow[d, "\mathrm{id}_H"] \arrow[dr, "\Delta"] & \\
H \otimes H \arrow[dr, "\varepsilon \otimes \mathrm{id}_H"'] & & H \otimes H \arrow[dl, "\mathrm{id}_H \otimes \varepsilon"] \\
 & \mathbb{C} \otimes H \simeq H \simeq H \otimes \mathbb{C} &
\end{tikzcd}
$$

FIGURE 4. Cocommutativity of Coalgebra Counit.

An algebraic object with the structure of both an algebra and a coalgebra is known as a bialgebra, if one also ensures that the products and units from both structures interact nicely.

**Definition 4.3.** *A bialgebra $(H, \nabla, \eta, \Delta, \varepsilon)$ is an algebra $(H, \nabla, \eta)$ and coalgebra $(H, \Delta, \varepsilon)$ satisfying:*

$$\Delta \circ \nabla = (\nabla \otimes \nabla) \circ (\mathrm{id}_H \otimes \tau \otimes \mathrm{id}_H) \circ (\Delta \otimes \Delta) \tag{5}$$

$$\varepsilon \otimes \varepsilon = \varepsilon \circ \nabla \tag{6}$$

$$\eta \otimes \eta = \Delta \circ \eta \tag{7}$$

$$\varepsilon \circ \eta = \mathrm{id}_{\mathbb{C}} \tag{8}$$

*where $\tau : H \otimes H \to H \otimes H$ is the flip function defined by $\tau(x \otimes y) = y \otimes x$ for all $x, y \in H$.*

These properties are illustrated in the three following commuting diagrams. The first of which describes how the coproduct and product interact. Composing the product with the coproduct does not immediately give identity but instead a flip is introduced.

$$
\begin{tikzcd}
H \otimes H \arrow[r, "\nabla"] \arrow[d, "\Delta \otimes \Delta"'] & H \arrow[r, "\Delta"] & H \otimes H \\
H \otimes H \otimes H \otimes H \arrow[rr, "\mathrm{id}_H \otimes \tau \otimes \mathrm{id}_H"] & & H \otimes H \otimes H \otimes H \arrow[u, "\nabla \otimes \nabla"']
\end{tikzcd}
$$

FIGURE 5. Product and Coproduct Interaction in Bialgebras.

The second figure describes both properties (6) and (7) which relates how the unit and counit interact with the coproduct and product respectively.

$$
\begin{tikzcd}
H \arrow[r, "\varepsilon"] & \mathbb{C} \\
H \otimes H \arrow[u, "\nabla"] \arrow[ur, "\varepsilon \otimes \varepsilon"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{C} \arrow[r, "\eta"] \arrow[dr, "\eta \otimes \eta"'] & H \arrow[d, "\Delta"] \\
 & H \otimes H
\end{tikzcd}
$$

FIGURE 6. Unit and Product Interaction in Bialgebras.

And we also want the counit to be a left inverse of the unit.

Finally, along with a bialgebra structure, a Hopf algebra also has an antipode which serves as a generalised inverse. In Example 4.1 the antipode serves to recover the inverse when embedding a group into the structure of a Hopf algebra.

**Definition 4.4.** *A Hopf algebra $(H, \nabla, \eta, \Delta, \varepsilon, S)$ is a bialgebra with a linear map $S : H \to H$ called the antipode, satisfying:*

$$\sum S(c_{(1)})c_{(2)} = \sum c_{(1)}S(c_{(2)}) = \varepsilon(c)1_H \quad \forall c \in H, \tag{9}$$

*where $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$.*

Note that here the indices of the coproduct are implied, allowing us to trade superscripts and notational conciseness for readability. As before, the properties of the antipode can be described with the following commuting diagram:

$$\begin{tikzcd}
& H \otimes H \arrow[rr, "S\otimes\mathrm{id}_H"] & & H \otimes H \arrow[dr, "\nabla"] & \\
H \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rr, "\varepsilon"] & & \mathbb{C} \arrow[rr, "\eta"] & & H \\
& H \otimes H \arrow[rr, "\mathrm{id}_H\otimes S"] & & H \otimes H \arrow[ur, "\nabla"'] &
\end{tikzcd}$$

FIGURE 7. Properties of the Antipode.

The group algebra of a finite group canonically forms a Hopf algebra. In Hopf theory such structures are considered trivial.

*Example* 4.1. Let $G$ be a finite group. We construct a Hopf algebra $H$ with underlying set $\mathbb{C}G$ - the group algebra of $G$ over $\mathbb{C}$. We define the coproduct, counit, and antipode on $G \in \mathbb{C}G$ by

$$\begin{aligned} \Delta(g) &= g \otimes g, \\ \varepsilon(g) &= 1, \\ S(g) &= g^{-1}. \end{aligned}$$

and extend it to $\mathbb{C}G$ by linearity.

**Remark 4.1.** *If $G$ is a finite group then $H = \mathbb{C}G$ is a Hopf algebra.*

*Proof.* We first show that $H$ has a coalgebra structure. Suppose $g \in G$. Then

$$((\Delta \otimes \mathrm{id}_H) \circ \Delta)(g) = (g \otimes g) \otimes g = g \otimes (g \otimes g) = ((\mathrm{id}_H \otimes \Delta) \circ \Delta)(g)$$

and

$$((\varepsilon \otimes \mathrm{id}_H) \circ \Delta)(g) = \varepsilon(g) \otimes g = g \otimes \varepsilon(g) = ((\varepsilon \otimes \mathrm{id}_H) \circ \Delta)(g).$$

Therefore $H$ has a coalgebra structure.

To show that $H$ is a bialgebra, suppose $g, h \in G$ and $e \in G$ is the identity. Then the products agree with the units as

$$(\varepsilon \circ \nabla)(g \otimes h) = \varepsilon(gh) = 1 = \varepsilon(g)\varepsilon(h) = (\varepsilon \otimes \varepsilon)(g \otimes h)$$

and

$$(\Delta \circ \eta)(1) = \Delta(\eta(1)) = \Delta(e) = e \otimes e = (\eta \otimes \eta)(1).$$

Moreover the product and coproduct agree as

$$\begin{aligned}
&(\nabla \otimes \nabla) \circ (\mathrm{id}_H \otimes \tau \otimes \mathrm{id}_h) \circ (\Delta \otimes \Delta)(g \otimes h) \\
=&(\nabla \otimes \nabla) \circ (\mathrm{id}_H \otimes \tau \otimes \mathrm{id}_h)(g \otimes g \otimes h \otimes h) \\
=&(\nabla \otimes \nabla)(g \otimes h \otimes g \otimes h) \\
=&gh \otimes gh \\
=&\Delta(gh) \\
=&(\Delta \otimes \nabla)(g \otimes h)
\end{aligned}$$

and thus $H$ is a bialgebra. Finally, for $H$ to be a Hopf algebra, we have

$$\begin{aligned}
\nabla \circ (S \otimes \mathrm{id}_H) \circ \Delta(g) &= \nabla \circ (S \otimes \mathrm{id}_H)(g \otimes g) \\
&= \nabla(S(g) \otimes g) \\
&= \nabla(g^{-1} \otimes g) = e \\
\nabla \circ (\mathrm{id}_H \otimes S) \circ \Delta(g) &= \nabla \circ (\mathrm{id}_H \otimes S)(g \otimes g) \\
&= \nabla(g \otimes S(g)) \\
&= \nabla(g \otimes g^{-1}) = e
\end{aligned}$$

and

$$(\eta \circ \varepsilon)(g) = \eta(\varepsilon(g)) = \eta(1) = e.$$

□

By this construction, the group structure of $G$ is not lost and is encoded in the group-like elements.

**Definition 4.5.** *If $H$ is a Hopf algebra then $g \in H$ is called group-like if*

$$\Delta(g) = g \otimes g.$$

The inverses of the group-like elements can be recovered from the antipode. In fact for any Hopf algebra, the group-like elements form a group with the inverses given by the antipode.

**Lemma 4.1.** *Let $H$ be a Hopf algebra and $G = \{h \in H \mid \Delta(h) = h \otimes h\}$ be the set of group-like elements of $H$. Then $G$ forms a group under multiplication in $H$.*

*Proof.* First note that $1_H$ is clearly the identity element of $G$ and the associativity follows from the associativity of the product on $H$. Now suppose $g, h \in H$. Then $G$ is closed under multiplication as

$$\begin{aligned}
\Delta(gh) = (\Delta \circ \nabla)(g \otimes h) &= (\nabla \otimes \nabla) \circ (\mathrm{id}_H \otimes \tau \otimes \mathrm{id}_H) \circ (\Delta \otimes \Delta)(g \otimes h) \\
&= \nabla \otimes \nabla(g \otimes h \otimes g \otimes h) = gh \otimes gh
\end{aligned}$$

so $g \cdot h \in G$.

First note that $((\varepsilon \circ \mathrm{id}_H) \circ \Delta)(g) = \varepsilon(g)g = g$ and therefore $\varepsilon(g) = 1$. Now to show that $S(g)$ is an inverse for $g$ we have that

$$S(g) \cdot g = \nabla(S(g) \otimes g) = (\nabla \circ (S \otimes \mathrm{id}_H) \circ \Delta)(g) = (\eta \circ \varepsilon)(g) = \eta(1) = 1_H = e$$

the same result holds for $g \cdot S(g)$ therefore $S(g) = g^{-1}$. Hence $G$ forms a group. $\square$

We can also describe the notion of a dual Hopf algebra. In finite dimensions, the dual vector space of a Hopf algebra is naturally a Hopf algebra itself.

**Definition 4.6.** *Let $H = (H, \nabla, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra and $H^*$ be the dual vector space of $H$.*

*Firstly $H^*$ forms an algebra with unit map $\eta^* : \mathbb{C} \to H^*$ that takes $1_{\mathbb{C}}$ to $\mathrm{id}_H$ and product map $\nabla^* : H^* \otimes H^* \to H^*$ defined by*

$$\nabla^*(f \otimes g)(a) = \sum f(a_{(1)})g(a_{(2)})$$

*for all $f, g \in H^*$ and $a \in H$ where $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$.*

*Now $H^*$ forms a coalgebra with counit $\varepsilon^* : H^* \to \mathbb{C}$ defined by*

$$\varepsilon^*(f) = f(1_H)$$

*for all $f \in H^*$. Moreover $H^*$ has coproduct $\Delta^* : H^* \to H^* \otimes H^*$ defined by*

$$\Delta^*(f)(a \otimes b) = f(ab).$$

*for all $f \in H^*$ and $a, b \in H$.*

*Finally $H^*$ admits an antipode $S^*$ given by*

$$S^*(f)(a) = S(f(a))$$

*for all $f \in H^*$ and $a \in H$.*

4.2. **Hopf Algebras of Prime Dimension.** It is well known that the only finite groups of prime order are cyclic. Due to the large amount of structure present, a similar characterisation applies to Hopf algebras. We present the main theorem of the following paper [15] by Zhu. Here we omit some lemmata but replicate the reasoning of the main proof with some steps filled in.

**Theorem 4.1** (Zhu, 1994)**.** *Let $H$ be a Hopf algebra of prime dimension $p$ over $\mathbb{C}$. Then $H$ is isomorphic to the group algebra $\mathbb{C}\mathbb{Z}_p$ of the integers modulo $p$.*

The case for $p = 2$ is trivial. In fact, low-dimensional associative algebras over $\mathbb{C}$ have been classified [13] and there is only one unital associative algebra of dimension 2. Thus all Hopf algebras of dimension two are isomorphic to $\mathbb{C}\mathbb{Z}_2$.

Now let $H$ be a Hopf algebra of prime dimension $p$. For the rest of the proof we assume that $p \neq 2$. We wish to reduce the problem to the semisimple case. Here a Hopf algebra is semisimple if it is semisimple as an associative algebra i.e. it has trivial Jacobson radical. Thus we have the following lemma:

**Lemma 4.2.** *Every Hopf algebra of prime dimension is semisimple.*

The approach is to first look at elements of $H$ known as **integrals**. These are elements of $H$ that generate a one dimensional left $H$-module.

**Definition 4.7.** *An element $\Lambda \in H$ is a left integral if $h\Lambda = \varepsilon(h)\Lambda$ for all $h \in H$.*

Similarly we can define a notion of right integral. Every Hopf algebra has a left and right integral and they are unique up to scalar multiplication. The canonical example is present in a group algebra.

*Example* 4.2. Let $G$ be a finite group and $A = \mathbb{C}G$ the corresponding Hopf algebra. Then the element $\Lambda = \sum_{g\in G} g$ is a left and right integral.

*Proof.* This can be seen by the fact that left and right multiplication by $h \in G$ is a group endomorphism and thus the action of $h$ is simply to reorder the elements of $G$. Therefore

$$h\Lambda = h\sum_{g\in G} g = \sum_{g\in G} hg = \sum_{g\in G} g = \Lambda = \Lambda h.$$

□

When the left and right integrals of a Hopf algebra coincide, the Hopf algebra is called unimodular.

**Definition 4.8.** *A Hopf algebra is unimodular if the left integrals are the right integrals.*

In particular, by example 4.2 and the uniqueness of the integral, a group algebra is unimodular.

We wish to show that a prime dimensional Hopf algebra $H$ is unimodular. To do this, suppose for a contradiction that $H$ is not unimodular and let $\Lambda$ be the unique left integral of $H$. By the uniqueness of $\Lambda$, there is some element $f$ in the dual Hopf algebra $H^*$ such that $\Lambda h = f(h)\Lambda$ for all $h \in H$. Since $H$ is not unimodular, we know that $f \neq \varepsilon$ as otherwise $\Lambda$ would be a right integral. Thus $f$ is a non-trivial group-like element of $H^*$. Since $f$ generates a non-trivial subgroup of the group-like elements we have a sub Hopf algebra $\mathbb{C}\langle f\rangle$ with dimension $q$ dividing $\dim(H^*) = p$. Thus we have that $H^* = \mathbb{C}\langle f\rangle = \mathbb{C}\mathbb{Z}_p$ and therefore $H \simeq \mathbb{C}\mathbb{Z}_p$. Therefore $H$ is unimodular - a contradiction. Thus both $H$ and $H^*$ are unimodular.

We now show that $H$ is semisimple. Since both $H$ and $H^*$ are unimodular, it can be shown, as in Corollary 5.7 of [8], that the order of the antipode $S$ is 1,2, or 4. Now consider the eigenspaces $H_+$ and $H_-$ of $S^2$ with eigenvalues 1 and $-1$ respectively. That is

$$H_+ = \{h \in H \mid S^2(h) = h\},$$
$$H_- = \{h \in H \mid S^2(h) = -h\}.$$

Since $H_+ \cap H_- = \{0\}$ and $S^4 = 1$ we have that $H = H_+ \oplus H_-$. Also note that because of this decomposition there is a basis of $H$ in which the representation of $S^2$ in this basis is a diagonal matrix of $\dim(H_+)$ 1s and $\dim(H_-)$ $-1$s and therefore the trace of $S^2$ is given by.

$$\mathrm{Tr}(S^2) = \dim(H_+) - \dim(H_-),$$

Now suppose for a contradiction that $H$ is not semisimple. By Theorem 1 of [9] we obtain that $\mathrm{Tr}(S^2) = 0$. Therefore we obtain $\dim(H) = 2\dim(H_+)$. But since $\dim(H)$ is an odd prime we obtain a contradiction. Therefore $H$ is semisimple.

4.2.1. *Hopf Modules and Characters.* To proceed, we must first discuss the modules and then the characters of Hopf algebras. A Hopf algebra module is simply a module over the underlying ring structure. The character of a Hopf module is defined as follows:

**Definition 4.9.** *A character of $H$-module $V$ is a dual-vector $\chi_V \in H^*$ such that $\chi_V(a) = \mathrm{Tr}|_V(a)$ for all $a \in H$ where $\mathrm{Tr}|_V(a)$ is the trace of the linear operator defined by $v \mapsto a \cdot v$ for all $v \in V$.*

The simplest character we can construct is the trivial character which is defined via the counit.

*Example* 4.3. Let $\mathbb{C}$ be a one-dimensional module over $H$ with the action defined by $a \cdot \lambda = \varepsilon(a)\lambda$ for all $a \in H$. The corresponding character is $\varepsilon$.

Now note that characters are closed under addition and multiplication. If $\chi_V$ and $\chi_U$ are characters corresponding to $H$-modules $V$ and $U$, the maps $\chi_V\chi_U$ and $\chi_V + \chi_U$ are characters of the modules $V \otimes U$ and $V \oplus U$ respectively. With this notion we can define a unital subalgebra of $H^*$ that contains all characters.

**Definition 4.10.** *The character ring $C(H)$ of $H$ is the subalgebra of $H^*$ over $\mathbb{Q}$ spanned by the irreducible characters of $H$.*

Here the unit is given by the trivial character $\varepsilon$.

It can then be shown that $C(H) \otimes_{\mathbb{Q}} \mathbb{C} \subset H^*$ is semisimple. By the Artin-Wedderburn theorem of finite dimensional semisimple algebras, it follows that $C(H) \otimes_{\mathbb{Q}} \mathbb{C}$ can be decomposed into the finite product of matrix algebras over division algebras. Thus we can write

$$C(H) \otimes_{\mathbb{Q}} \mathbb{C} \simeq \prod_{\alpha=1}^{\alpha} M_{n_\alpha}(D_\alpha)$$

for some finite dimensional division algebras $D_\alpha$ over $\mathbb{C}$. We then choose a basis of matrix units $E_{\alpha ij}$ for $1 \leq \alpha \leq N$ and $1 \leq i, j \leq n_\alpha$. It can again be shown that the trace of each unit is a positive integer dividing the dimension of $H$.

Thus we obtain a decomposition for the identity.

$$\sum_{\alpha=1}^{N} \sum_{i=1}^{n_\alpha} E_{\alpha ii} = 1$$

4.2.2. *Proof of Theorem 4.1.* We note that $\varepsilon$ is the identity of $C(H)$ and take the trace of the decomposition to obtain

$$\sum_{\alpha=1}^{N} \sum_{i=1}^{n_\alpha} \mathrm{Tr}(E_{\alpha ii}) = \mathrm{Tr}(\varepsilon) = p.$$

Since each $E_{\alpha ii}$ divides $p$ we have two cases. In the first there is only one $E_{\alpha ii}$ with trace equal to $p$. Therefore $C(H) \otimes_{\mathbb{Q}} \mathbb{C}$ is one-dimensional. Hence $H$ has only trivial character $\varepsilon$. However this is impossible as only one dimensional algebras have only trivial irreducible character. In the second case we have that there are $p$ matrix units, each with trace $p$, and so $C(H) \otimes_{\mathbb{Q}} \mathbb{C}$ has dimension $p$. Thus $C(H) \otimes_{\mathbb{Q}} \mathbb{C}$ is a $p$-dimensional subalgebra of $H^*$ over $\mathbb{C}$ and therefore $C(H) \otimes_{\mathbb{Q}} \mathbb{C} = H^*$. Here $H$ is then a direct sum of $p$ copies of $\mathbb{C}$ and therefore $H \simeq \mathbb{C}\mathbb{Z}_p$.

4.3. **Quantum Groups as Deformations.** By definition, a quantum group is a Hopf algebra but in most contexts they are used as deformations of a usually commutative algebra. We first show how Lie algebras and groups can be described as Hopf algebras and then provide some deformed examples.

4.4. **Lie Groups as Hopf Algebras.** We follow the construction given in [2]. Let $G$ be a Lie group and $A = Fun(G)$ the set of complex-valued differentiable functions on $G$. By piece-wise multiplication and addition, $A$ forms an algebra. Now define coproduct $\Delta$, counit $\varepsilon$, and antipode $S$ by

$$\begin{aligned}\Delta(f)(g,h) &= f(gh)\\ \varepsilon(f) &= f(e)\\ S(f)(g) &= f(g^{-1}).\end{aligned}$$

for all $f, g, h \in A$.

Another interesting construction from [10] arises from complex semisimple Lie groups. If $G$ is a complex semisimple Lie group then it has a faithful representation in $GL_n(\mathbb{C})$ which by theorem 6.3 in [11] is algebraic so

$$G = \{x \in M_n \mid p(x) = 0\}$$

for some collection of polynomials $p$. Hence $G$ forms an algebraic variety with coordinate ring $\mathbb{C}[G]$ generated by $n^2$ variables $x^i_j$ for $0 \leq i, j \leq n$ modulo $\langle p \rangle$. The coproduct and counit are then derived from matrix multiplication

$$\begin{aligned}\Delta(x^i_j) &= \sum_k x^i_k \otimes x^k_j\\ \varepsilon(x^i_j) &= \delta^i_j\end{aligned}$$

That is, the coproduct is matrix multiplication and the counit is the trace. The antipode is given by defining the antipode of the matrix of generators as the adjugate matrix.

4.5. **Lie Algebras as Hopf Algebras.** We note that to construct a Hopf algebra from a Lie algebra we take the universal enveloping algebra and endow it with the appropriate structure. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over field $\mathbb{C}$ with generators $g_1, \ldots, g_n$. Its universal enveloping algebra is the free associative algebra $\mathbb{C}\langle 1, g_1, \ldots, g_n \rangle$ modulo the commutation relations $[g_i, g_j] = g_i g_j - g_j g_i$ for all $1 \leq i, j \leq n$. We take this algebra and define the coproduct, counit, and antipode by

$$\begin{aligned}\Delta(g_i) &= g_i \otimes I + I \otimes g_i\\ \varepsilon(g_i) &= 0\\ S(g_i) &= -g_i.\end{aligned}$$

4.6. SL(2) **and** $\mathrm{SL}_q(2)$**.** We now give an example of the Hopf algebra of a Lie group and the collection of Quantum groups generalising it. For $\mathrm{SL}(2,\mathbb{C})$ (the Lie subgroup of $\mathrm{GL}(2,\mathbb{C})$ which we abbreviate to SL(2)) we use the Lie group construction. Since $\mathrm{SL}(2) = \{x \in M_2 \mid \det(x) = 1\}$, the corresponding Hopf algebra becomes $\mathbb{C}[\mathrm{SL}(2)] = \mathbb{C}[a,b,c,d]/\langle ad - bc - 1\rangle$. The coproduct is given by matrix multiplication as in Section 4.4, the counit gives the diagonal elements and the antipode is given by

$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

To form $\mathrm{SL}_q(2)$ we generalise these relations. Pick $q \in \mathbb{C}^*$ and we start in the free associative algebra $\mathbb{C}\langle a,b,c,d\rangle$ and mod out by "quantised" commutation relations

$$ca = qac, \quad ba = qab, \quad db = qbd, \quad dc = qcd, \quad bc = cb, \quad da - ad = (q - q^{-1})bc$$

and a quantised determinant relation

$$ad - q^{-1}bc = 1.$$

The coproduct and counit are given as in the classical case and the antipode by

$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}.$$

We thus obtain a quantum group, i.e a class of Hopf algebras parameterised by $q \in \mathbb{C}^*$ where taking $q \to 1$ yields the classical case.

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