1 Topoi

The origin of topoi in category theory came from the Grothendieck school of algebraic geometry. In the 1950s Grothendieck had introduced the idea of an abelian category to unify the categories of sheaves of abelian groups in his work on homological algebra. Turning to the categories of sheaves of sets in the 1960s, these were captured by the concept of a Grothendieck topos.

The topoi we are looking at are not all Grothendieck topoi but came out of Lawvere's work. Lawvere focused on the existence of a truth value object in each Grothendieck topos and their connection to a Grothendieck topology. Subsequently he developed a concept of a topos in the 1960s in terms of the existence of a subobject classifier which is like a truth value object. This was originally referred to as an elementary topos but is now just called a topos.

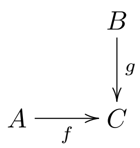
A motivating factor in Lawvere's work with topoi was the desire to give a categorical version of Cohen's proof of the independence of the continuum hypothesis from the axioms of set theory. This seemed plausible due to connections between Cohen's forcing technique and sheaf theory which is tied up with Grothendieck topoi.

The aim of this section is to define a topos, look at some constructions possible in a topos and consider the example of an effective topos. Before defining a topos we expand on the discussion of limits above and describe exponential objects and the subobject classifier. Once the notion of a topos is established we look at the ability to construct an integer and rational object from a natural number object. Lastly we briefly describe the effective topos - a universe where everything is computable.

1.1 Limits

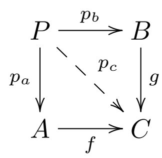
In general a limit is a cone over an arbitrary diagram such that all other cones have a unique morphism into the limit. So far we have only considered the limit over the empty diagram, a pair of objects and a parallel pair of morphisms which are the terminal object, the product and the equalizer respectively.

Another important example is the pullback which categorically expresses notions such as the inverse image and an equivalence relation. The pullback is a limit over the following diagram.



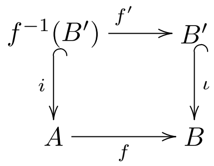
A pullback, P with morphisms p_x makes a commuting square as $f \circ p_a = p_c =$

$g \circ p_b$. We typically leave out the diagonal and state that $f \circ p_a = g \circ p_b$.



Example 1. *In* **Set** *a standard example of a pullback for the diagram above is* $P = \{(a, b) : f(a) = g(b)\}$ *with corresponding projections.*

Example 2. *Let* $f : A \to B$ *be a function in* **Set**. *The inverse image of* $B' \subseteq B$ *is found by the pullback of the two functions* f *and the inclusion* $\iota : B' \to B$,



where $f^{-1}(B') = \{x \in A : f(x) \in B'\}$ *and* f' *is the restriction of* f *to the domain* $f^{-1}(B')$ *and* $i : f^{-1}(B') \to A$ *is an inclusion.*

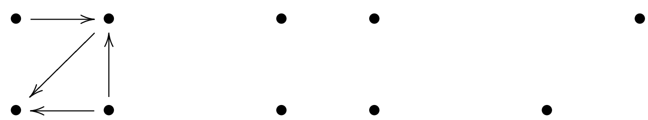
We do not need to establish the existence of each sort of limit individually. Once we know that a category has limits over any finite diagram, then we know that the category has a multitude of tools for expressing the interaction of its objects.

Theorem 1. *If the category* $\mathfrak{A}$ *has equalizers and all finite products then* $\mathfrak{A}$ *has limits over all finite diagrams.*

Proof. Let $\mathfrak{J}$ be a finite index category. A limit over the diagram $D : \mathfrak{J} \to \mathfrak{A}$ can be constructed as the equalizer over a suitable pair of morphisms between two products; the product of the objects in D and the product of all the objects which are codomains in D.

As $\mathfrak{J}$ is a finite category let $Ob\mathfrak{J}$ be the finite set of objects in $\mathfrak{J}$.

Let $Ob\mathfrak{J}$ be the index category consisting only of objects from $\mathfrak{J}$ with no morphisms. Construct a second index category $Co\mathfrak{J}$ that contains a subset of the objects of $Ob\mathfrak{J}$. In particular $Co\mathfrak{J} = \{u : \exists n \neq u \in Ob\mathfrak{J}, \mathbf{Hom}_{\mathfrak{J}}(n, u) \neq \emptyset\}$. $Co\mathfrak{J}$ is the index category with all of the codomain objects of non-identity morphisms from $\mathfrak{J}$ as illustrated by the following diagram.



$\mathfrak{J}$ $\qquad$ $Ob\mathfrak{J}$ $\qquad$ $Co\mathfrak{J}$

Objects and morphisms in the diagram D will be denoted Dn and $D(f)$ respectively where n is an object and f a morphism in $\mathfrak{J}$. Throughout this proof n is used for an object in $Ob\mathfrak{J}$ while u denotes an object in $Co\mathfrak{J}$.

Define two products; the first over all the objects of D which are indexed by $Ob\mathfrak{J}$ and the second over all the codomain objects of D indexed by $Co\mathfrak{J}$.

$$\prod Ob\mathfrak{J} = \prod_{n \in Ob\mathfrak{J}} Dn \quad \text{with morphisms } (O_n : \prod Ob\mathfrak{J} \to Dn)_{n \in Ob\mathfrak{J}}$$

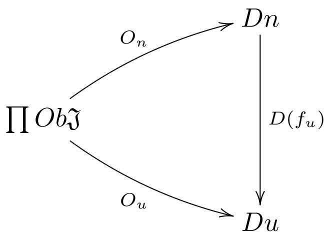
$$\prod Co\mathfrak{J} = \prod_{u \in Co\mathfrak{J}} Du \quad \text{with morphisms } (C_u : \prod Co\mathfrak{J} \to Du)_{u \in Co\mathfrak{J}}$$

Construct two morphisms, $r, s : \prod Ob\mathfrak{J} \to \prod Co\mathfrak{J}$ as follows.

By definition, for each $u \in Co\mathfrak{J}$ there exists some morphism $f_u : n \to u$ in $\mathfrak{J}$ where $n \neq u \in Ob\mathfrak{J}$. For each $u \in Co\mathfrak{J}$ let $r_u, s_u : \prod Ob\mathfrak{J} \to Du$ be defined as

(1) $r_u = D(f_u) \circ O_n$

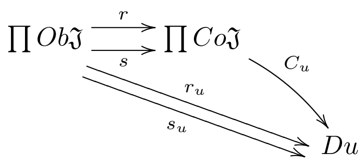
(2) $s_u = O_u$



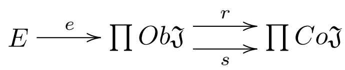
For each $u \in Co\mathfrak{J}$ there exists $r_u, s_u : \prod Ob\mathfrak{J} \to Du$ hence there exists $r =< .., r_u, .. >$, $s =< .., s_u, .. >$: $\prod Ob\mathfrak{J} \to \prod Co\mathfrak{J}$ which commute as follows.

(3) $C_u \circ r = r_u$

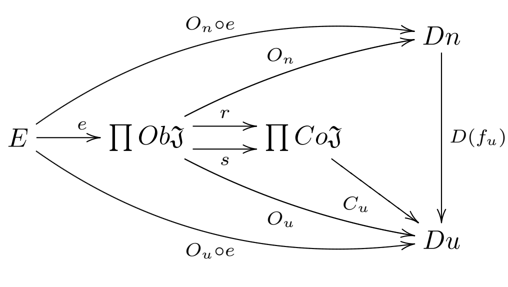
(4) $C_u \circ s = s_u$



Given the parallel pair of morphisms r, s there exists an equalizer, E with morphism $e : E \to \prod Ob\mathfrak{J}$.



What remains to be shown in that E is the limit over the diagram D with the family of morphisms $(O_n \circ e : E \to Dn)_{n \in Ob\mathfrak{J}}$.

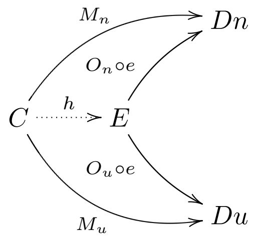


First we require that E is a cone over the diagram. This means the morphisms associated with E must commute with morphisms in the diagram.

Given $D(f_u) : Dn \to Du$ we require $D(f_u) \circ O_n \circ e = O_u \circ e$.

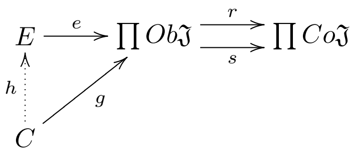
$$\begin{aligned} D(f_u) \circ O_n \circ e &= C_u \circ r \circ e && \text{by (1) and (3)} \\ &= C_u \circ s \circ e && \text{by definition of an equalizer} \\ &= O_u \circ e && \text{by (2) and (4)} \end{aligned}$$

Secondly, if E is a limit cone then any other cone over the diagram D must factor through E. Let C be an arbitrary cone over D with morphisms $(M_n : C \to Dn)_{n \in Ob\mathfrak{J}}$. We need to show that there exists a unique morphism $h : C \to E$ such that $O_n \circ e \circ h = M_n$ for all $n \in Ob\mathfrak{J}$.



Using the properties of E as an equalizer of r and s we can show that the unique h must exist.

If there is a morphism $g : C \to \prod Ob\mathfrak{J}$ such that $s \circ g = r \circ g$ then the unique $h : C \to E$ exists by the property of the equalizer E. It remains to be shown that such a g exists.



As C is a cone over the diagram indexed by $\mathfrak{J}$ it is also a cone over the diagram indexed by $Ob\mathfrak{J}$. By the property of the limit $\prod Ob\mathfrak{J}$ there exists a unique morphism $g : C \to \prod Ob\mathfrak{J}$ such that

$$(5) \quad O_n \circ g = M_n.$$

To verify that $s \circ g = r \circ g$ it is sufficient to check that $C_u \circ s \circ g = C_u \circ r \circ g$ for each $u \in Co\mathfrak{J}$.

$$\begin{aligned} C_u \circ s \circ g &= O_u \circ g && \text{by (4) and (2)} \\ &= M_u && \text{by (5)} \\ &= D(f_u) \circ M_n && \text{by property of the cone } C \\ &= D(f_u) \circ O_n \circ g && \text{by (5)} \\ &= C_u \circ r \circ g && \text{by (3) and (1)} \end{aligned}$$

Therefore the unique morphism $h : C \to E$ exists by the property of the equalizer such that $g = e \circ h$. This guarantees that $O_n \circ e \circ h = M_n$ so the cone C factors through E, and E is the limit over D. □

This is a useful theorem to guarantee the existence of finite limits. Equivalent results are possible in terms of different sorts of limits.

Lemma 2. *If a category* $\mathfrak{A}$ *has finite products and pullbacks, then* $\mathfrak{A}$ *has finite limits.*

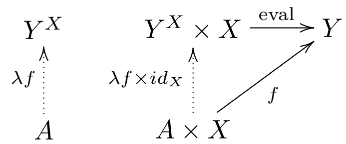
This result follows from above because an equalizer of $f, g : A \to B$ is also a pullback of f and g. We can just rephrase the previous result in terms of pullbacks instead of equalizers.

Once we have guaranteed the existence of finite limits in a category we can do many diagram chasing proofs like the one given above.

1.2 Exponential Object and Evaluation Morphism

The exponential object and evaluation morphism allow us to curry a function. Given some $f : A \times X \to Y$ the curried function will be $\lambda f : A \to (X \to Y)$. This allows us to change between a function of two variables to a single variable.

Definition 1. *Let* X *and* Y *be objects in the category* $\mathfrak{A}$. *An* exponential *of* Y *by* X *is an object* Y^X *along with a morphism* $\text{eval} : Y^X \times X \to Y$ *such that for any object* A *and morphism* $f : A \times X \to Y$ *there exists a unique morphism* $\lambda f : A \to Y^X$ *such that the following triangle commutes,*



An exponential is unique up to isomorphism.

Example 3. *Given* X *and* Y *in* **Set** *the functions from* X *to* Y *form a set which is the object* Y^X. *The morphism* eval *is defined by* $\text{eval}(f, x) = f(x)$ *where* $f : X \to Y$ *and* $x \in X$.

The same construction is valid in **FinSet**. There are only finitly many functions between two finite sets and the eval function is still defined.

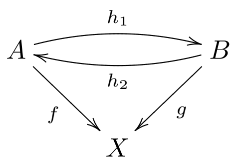
Example 4. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be two categories in the category of all small categories. The functor category* $\mathfrak{B}^{\mathfrak{A}}$ *is the exponential object with standard* eval *morphism.*

1.3 Subobject Classifier

The useful concept of a subset or subgroup is traditionally defined in terms of element membership. As category theory is based on morphisms rather than elements the definition of a subobject is based on the idea of an inclusion morphism.

Consider the category **Set** and objects X and S where $S \subseteq X$. The image of the inclusion function $\iota : S \to X$ is S. While ι is monic, there are many other monics $m : S' \to X$ whose image is S. All such monics define the subset S in a way equivalent to ι. We can define the subobjects of X by formalising the notion of equivalent monics.

Definition 2. *Let $f : A \to X$, $g : B \to X$ be two monomorphisms in $\mathfrak{A}$. Let $f \sim g$ be an equivalence relation defined by the existence of $h_1 : A \to B$ and $h_2 : B \to A$ such that $f = g \circ h_1$ and $g = f \circ h_2$.*



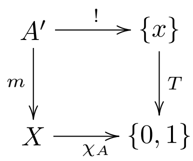
In the example in **Set** above, the equivalence class $[\iota]$ is the set of all injective functions into the set X whose image is the subset S. There are no other inclusion functions in $[\iota]$ and the subset S is uniquely determined by $[\iota]$.

Definition 3. *A* subobject *of an object A in $\mathfrak{A}$ is an equivalence class of monomorphisms under $\sim$.*

In set theory a subset is equivalent to a characteristic function. The analogous tool for categories is to characterise subobjects using a specific pullback.

Let $S \subseteq X$ then the characteristic function $\chi_S : X \to \{0, 1\}$ is defined as $\chi(x) = 1$ if $x \in S$ and 0 otherwise.

Conversely given a characteristic function $\chi_A : X \to \{0, 1\}$ we can determine the subset $A \subseteq X$ using a pullback. Define $T : \{x\} \to \{0, 1\}$ such that $T(x) = 1$.



As $\{x\}$ is a terminal object in **Set** then ! is the unique function from A' into $\{x\}$. Furthermore as T is monic, by a property of pullbacks m must also be monic.

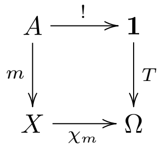
The pullback means that for any $a \in A'$,

$$T \circ !(a) = T(x) = 1 = \chi_A \circ m(a)$$

The elements in the image of m form the subset of X for which χ_A takes the value 1. As other monics $m' : A'' \to X$ satisfy the pullback above $[m]$ is the subobject of X.

The equivalence class of monics, $[m]$ is not the same as the subset A but from a categorical perspective they interact with χ_A in the same way.

Definition 4. *Let $\mathfrak{A}$ be a category with terminal object $\mathbf{1}$. A* subobject classifier *is an object Ω along with the monic $T : \mathbf{1} \to \Omega$ such that for any monic $m : A \to X$ there exists a unique morphism $\chi_m : X \to \Omega$ such that the following is* *a pullback.*



The equivalence class of monics $[m]$ *is the subobject of* X *corresponding to* χ_m.

In **Set**, **FinSet** and **Grp** the subobject classifier is simply {0,1} with the standard characteristic functions.

Lawvere refers to Ω as the truth value object and the morphism T as a way to single out the value true from Ω.

1.4 A Topos

Now we can present the axioms for when a category is a topos. We will give some common examples and counterexamples as well as methods for constructing new topoi from old ones.

Definition 5. *A category* $\mathfrak{A}$ *is a topos if it has;*

1. *Limits over all finite diagrams;*
2. *For all objects* A, B *in* $\mathfrak{A}$ *an exponential object* A^B *with evaluation morphisms* $\mathrm{eval} : A^B \times B \to A$*;*
3. *A subobject classifier with the object* Ω *and morphism* $T : \mathbf{1} \to \Omega$.

Example 5. *The categories* **Set** *and* **FinSet** *are topoi. The category of countable sets and functions between them,* **CountSet***, is not a topos because it lacks exponential objects.*

Given the sets $\{0, 1\}$ and ω in **CountSet**, assume the exponential object $\{0, 1\}^\omega$ exists. Then $Hom(\mathbf{1}, \{0, 1\}^\omega)$ would be a countable set. That is a contradiction because $Hom(\mathbf{1}, \{0, 1\}^\omega)$ is isomorphic to the uncountable set $Hom(\omega, \{0, 1\})$.

Example 6. *The category* **G-Set** *is a topos. Recall that the objects are sets,* X *under the group action of* G *such that* $h \cdot (g \cdot x) = (h \cdot g) \cdot x$ *where* $h, g \in G$ *and* $x \in X$.

Products and equalizers are the same in **G-Set** as in **Set** because they preserve G-actions. Let $m, n : X \to Y$ be two G-maps and consider the standard equalizer from **Set**, $E = \{x \in X : m(x) = n(x)\}$. If E is an object in **G-Set** then it must be closed under G-actions.

Let $x \in E$, then by definition of the G-maps and the equalizer $m(g \cdot x) = g \cdot m(x) = g \cdot n(x) = n(g \cdot x)$. Therefore $g \cdot x \in E$ and E is closed under G-actions. Similarly products are closed under G-actions so **G-Set** has finite limits.

Given the G-Sets X and Y, the exponential object Y^X is the set of all G-maps $f : X \to Y$. If Y^X is an object in **G-Set** then a G-action has to be defined on the maps. Let $f \in Y^X$ and $g \in G$ then define gf to be the map which acts on $x \in X$ by

$$x \longmapsto gf(g^{-1}x).$$

This is a G-action because for $g, h \in G$,

$$\begin{aligned} h(gf)(x) &= h(gf(h^{-1}x)) \\ &= hg(f(g^{-1}h^{-1}x)) \\ &= ((hg)f)(x). \end{aligned}$$

Lastly the subobject classifier is $\Omega = \{0, 1\}$ with the trivial group action defined on it and the morphism $T : \mathbf{1} \to \{0, 1\}$ which picks out the value 1.

The category **G-Set** is only one example of how to generate new topoi from old ones.

Lemma 3. *If $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are topoi then the cartesian product $\mathfrak{T}_1 \times \mathfrak{T}_2$ is a topos.*

Proof. Finite limits, exponentials and subobject classifer are defined component wise. In particular $(Y_1, Y_2)^{X_1, X_2} = (Y_1^{X_1}, Y_2^{X_2})$ and the subobject classifier is (Ω_1, Ω_2). □

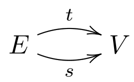
Theorem 4. *For a small category $\mathfrak{A}$, the functor category $\mathbf{Set}^{\mathfrak{A}}$ is a topos.*

We will give two examples to illustrate this result. Recall that in a functor category the objects are functors $F : \mathfrak{A} \to \mathbf{Set}$ and the morphisms are natural transformations between functors.

The category **G-Set** can be expressed as a functor category $\mathbf{Set}^{\mathbf{G\text{-}m}}$ where **G-m** is a monoid whose morphisms correspond to elements of the group G. A functor from **G-m** to **Set** is like a group action on a set.

The second illustration of the functor category construction is given by the category of directed multi-graphs.

Let **Gr** be the category with two objects, V, E and two non-identity morphisms.



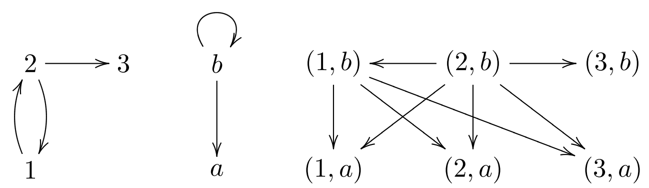
The category **Gr** is like a graph with vertice and edge sets and a way of assigning a source and target vertex to each edge.

The category of all directed multi-graphs is equivalent to the functor category $\mathbf{Set}^{\mathbf{Gr}}$. As **Gr** is a small category, $\mathbf{Set}^{\mathbf{Gr}}$ is a topos.

Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be graphs in $\mathbf{Set}^{\mathbf{Gr}}$.

The product $G \times H = (V_G \times V_H, E_G \times E_H)$ such that for (x, y) and (x', y') in $V_G \times V_H$ we have (x, y) is adjacent to (x', y') if $(x, x') \in E_G$ and $(y, y') \in E_H$.

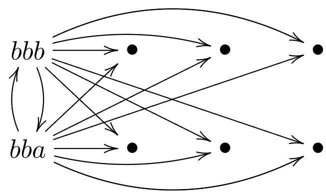
Consider the following example of two graphs G and H and their product.



G H G × H

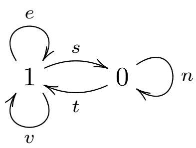
The exponential graph $H^G = (Hom(V_G, V_H), E)$ such that f and g in $Hom(V_G, V_H)$ are adjacent when $(x, x') \in E_G$ implies $(f(x), g(x')) \in E_H$.

Following on from the example above, let $f \in Hom(V_G, V_H)$ be represented as $f(1)f(2)f(3)$, then this is the graph of H^G.



The terminal object is the graph with one vertex and only the identity morphism.

The subobject classifier is the following graph with morphism $T : \mathbf{1} \to \Omega$ that picks out e.



The subobject classifier gives the different possibilities for the relationships between vertices and edges. For the subgraph $m : S \to G$ the classifing map $\chi_m : G \to \Omega$ acts as follows;

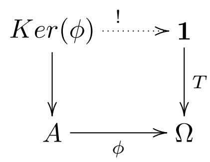
- If a vertex is in S it is mapped to 1, otherwise mapped to 0;
- An edge in S is mapped to e;
- An edge that is not in S can be mapped to four possibilities;
  - (v) If the source and target vertices are in the subgraph;
  - (s) If the source vertex is in the subgraph;
  - (t) If the target vertex is in the subgraph;
  - (n) If neither the edge nor vertices are in the subgraph.

Example 7. *A partially ordered set can be treated as a category. If it has no greatest element then there is no terminal object. In this case finite limits are not defined and it is not a topos.*

Example 8. *It is not always possible to put a topology on the set of continuous functions between two topologies. This means* **Top** *lacks exponential objects and is not a topos.*

Example 9. *The category* **Ab** *is not a topos because there does not exist a subobject classifier.*

The terminal object $\mathbf{1}$ in **Ab** is the zero group. The group homomorphism $T : \mathbf{1} \to \Omega$ must send the zero group to $0 \in \Omega$. Given any $\phi : A \to \Omega$ the pullback must give $Ker(\phi) = \phi^{-1}(0)$ as the subgroup of A.



Hence Ω must be an abelian group with a copy of every quotient $A/Ker(\phi)$ for every abelian group A which is impossible.

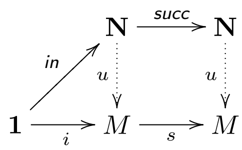
The categories which are topoi have the ability to express a lot of mathematical concepts and operations. For this reason topoi are sometimes presented as an alternative foundation for mathematics. We will now look at the possibility of expressing concepts about the natural numbers in topoi.

1.5 Natural Numbers, Integers and Rationals

In the category **Set**, $\mathbb{N}$ is an object but the natural number 3 is not. Categorically the singleton $\{3\}$ is indistinguishable from $\{\pi\}$. It is not possible to use the elements to give a categorical definition of $\mathbb{N}$. Instead we should characterise $\mathbb{N}$ in terms of morphisms.

In set theory the natural numbers can be generated by defining an initial element and a successor relation. A similar idea holds in category theory; a morphism from a terminal object is like selecting an initial element and a successor relation is just a particular non-identity morphism from an object to itself.

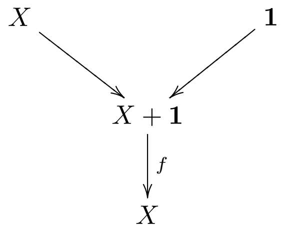
Definition 6. *A* natural number object *in $\mathfrak{T}$ is an object $\mathbf{N}$ along with two morphisms $\mathsf{in} : \mathbf{1} \to \mathbf{N}$ and $\mathsf{succ} : \mathbf{N} \to \mathbf{N}$ such that for any other object M with morphisms $i : \mathbf{1} \to M$ and $s : M \to M$ there exists a unique morphism $u : \mathbf{N} \to M$ making the following diagram commute.*



Not surprisingly the set of the natural numbers is the natural number object in the topos **Set**. In any category a natural number object is unique up to isomorphism.

To define the conditions for having a natural number object we need to specify which morphisms will accompany it. Thinking back to set theory, the property that the natural numbers form an infinite set can be captured by the existence of an isomorphism from $\mathbb{N} \cup \{x\}$ to $\mathbb{N}$. Freyd used the idea of an appropriate isomorphism to give the condition for a natural number object in a topos.

Theorem 5. *If there exists an object X in $\mathfrak{T}$ such that there is an isomorphism f from the coproduct of X and the terminal object $\mathbf{1}$ to X then X is the natural* *number object in $\mathfrak{T}$.*



The coproduct of X and $\mathbf{1}$ is like taking their disjoint union which suggests we have added something to X. Yet the isomorphism f attests to the fact that $X + \mathbf{1}$ is still no different to X. This expresses the notion of an infinite object categorically.

A category which is not a topos may have a natural number object. What is significant about having a natural number object in a topos is that we have the tools of finite limits at our disposal. This makes it possible to construct an integer object and a rational number object using similar ideas to the set theoretic constructions.

Example 10. *The set theoretic construction of the integers is given by*

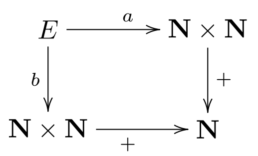
$$\mathbb{Z} = \{(n, m) : n, m \in \mathbb{N}\}/ \ \sim,$$

where $(n, m) \sim (n', m')$ *if and only if* $n + m' = n' + m$. *The equivalence class* $[(n, m)]$ *represents the integer* $n - m$.

In category theory a pullback is used to define an equivalence relation and a coequalizer is used to quotient.

Let $\mathbf{N}$ be the natural number object in a topos $\mathfrak{T}$. Let $+ : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ be the additive morphism that is defined recursively using the property of the natural number object.

Let E be the equivalence relation given by the following pullback.



In **Set** E would be $\{(n, m, n', m') : +(n, m') = +(n', m)\}$.

Define two morphisms $p, p' : E \to \mathbf{N} \times \mathbf{N}$. In **Set** p and p' would act on E by,

$$p : (n, m, n', m') \longmapsto (n, m),$$

$$p' : (n, m, n', m') \longmapsto (n', m').$$

We define the integer object $\mathbf{Z}$ as the coequalizer of the morphisms p and p'. This quotients by the relation $\sim$.

In a general category we do not know explicitly what p and p' are. However we can give their construction in terms of the morphisms from the pullback and product already defined. Then in any category p and p' behave in the same way, and their coequalizer is the integer object of that category.

First from the pullback we have the morphisms a and b which in **Set** were projections of a 4-tuple onto a pair. Secondly, as $\mathbf{N} \times \mathbf{N}$ is a product in the topos it comes with projections $\pi_1, \pi_2 : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$. In **Set** these give the first and second elements of an ordered pair.

Categorically p and p' can be constructed such that $p =< \pi_1 \circ a, \pi_2 \circ b >$ and $p' =< \pi_1 \circ b, \pi_2 \circ a >$.

Using pullbacks and coequalizers in a similar way a rational number object $\mathbf{Q}$ can be constructed from $\mathbf{N}$ and $\mathbf{Z}$.

Example 11. *In the set theoretic construction the rationals are given by*

$$\{(z, n) : z \in \mathbb{Z}, n \in \mathbb{N}\} / \sim,$$

where $(z, n) \sim (z', n')$ *if and only if* $z(n' + 1) = z'(n + 1)$. *The equivalence class* $[(z, n)]$ *represents the rational* $\frac{z}{n+1}$.

Again in the categorical situation a coequalizer will be used to quotient an equivalence relation constructed by a pullback.

In particular given that $\mathbf{N}$ and $\mathbf{Z}$ are objects in $\mathfrak{T}$ the product $\mathbf{Z} \times \mathbf{N}$ exists with morphisms $\pi_1 : \mathbf{Z} \times \mathbf{N} \to \mathbf{Z}$ and $\pi_2 : \mathbf{Z} \times \mathbf{N} \to \mathbf{N}$.

Let $m : \mathbf{Z} \times \mathbf{N} \to \mathbf{Z}$ be a morphism which acts like the multiplication of an integer with a natural number. Then $m \circ (Id_{\mathbf{Z}} \times \mathsf{succ})$ acts like multiplication of an integer with the successor of a natural number.

Define E as the object given by the following pullback,

$$\begin{array}{ccc}
E & \xrightarrow{\quad a \quad} & \mathbf{Z} \times \mathbf{N} \\
{\scriptstyle b}\downarrow & & \downarrow{\scriptstyle m \circ (Id_{\mathbf{Z}} \times \mathsf{succ})} \\
\mathbf{Z} \times \mathbf{N} & \xrightarrow[m \circ (Id_{\mathbf{Z}} \times \mathsf{succ})]{} & \mathbf{Z}
\end{array}$$

In **Set** E is $\{(z, n', z', n) : z(n' + 1) = z'(n + 1)\}$.

Define the rational number object $\mathbf{Q}$ as the coequalizer of the following diagram,

$$E \underset{< \pi_1 \circ b, \pi_2 \circ a >}{\overset{< \pi_1 \circ a, \pi_2 \circ b >}{\rightrightarrows}} \mathbf{Z} \times \mathbf{N} \longrightarrow \mathbf{Q}$$

In **Set** the construction of the integer and rational object is the categorical translation of the set theoretic construction. However the advantage of expressing it categorically is that the same result will hold in any other topos with a natural number object.

1.6 Effective Topos

The effective topos, **Eff** was first described by J. M. E. Hyland in 1982. It is based on the idea of Kleene's recursive realizability.

Eff is not a Grothendieck topos. This is despite the fact that Grothendieck topoi are based on the idea of sheaves, and Powell showed there is a parallel between sheaf models and realizability.

We will give a brief definition of **Eff** before turning to the subject of analysis. Hyland claims that analysis in **Eff** is essentially constructive real analysis similar to the Markov school. We will give one theorem which holds in **Eff** but not in classical analysis.

Objects in **Eff** are pairs $(X, \sim)$ consisting of a set X with a map $\sim: X \times X \to P(\mathbb{N})$.

For $(x, y) \in X$ the image under the map $\sim$ is denoted $(x \sim y)$. The natural numbers in $(x \sim y)$ index partial functions. These partial functions realize something about the similarity of x and y.

Let ϕ_e denote the partial function from $\mathbb{N}$ to $\mathbb{N}$ computed by the e^{th} Turing machine. Define $\phi_e(n) \downarrow$ to mean that ϕ_e halts on input n.

Let $\langle -, - \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a pairing function.

The map $\sim$ must satisfy conditions for symmetry and transitivity. Namely there exists s and t in $\mathbb{N}$ such that;

1. $n \in (x \sim y) \Rightarrow \phi_s(n) \downarrow \wedge \phi_s(n) \in (y \sim x)$
2. $n \in (x \sim y) \wedge m \in (y \sim z) \Rightarrow \phi_t(\langle n, m \rangle) \downarrow \wedge \phi_t(\langle n, m \rangle) \in (x \sim z)$.

Example 12. *The natural number object is* $\mathbf{N} = (\mathbb{N}, \sim)$*, where* $(x \sim y) = \{x\} \cap \{y\}$.

A morphism from $(X, \sim)$ to $(Y, \sim)$ is an equivalence class of strict, single valued, relational and total maps $M : X \times Y \to P(\mathbb{N})$. A full definition can be found in [Oos08].

The subobject classifier is larger than previous examples to account for the different possible subsets of $\mathbb{N}$ which contain the indexes for the significant partial functions.

In particular we have $\Omega = (P(\mathbb{N}), \sim)$ where

$$(X \sim Y) = \{\langle e_0, e_1 \rangle : \forall x \in X \phi_{e_0}(x) \in Y \wedge \forall y \in Y \phi_{e_1}(y) \in X\}.$$

1.7 Analysis in Eff

There are two set theoretic constructions of the real numbers using Cauchy sequences or Dedekind cuts. Classically the Dedekind reals are isomorphic to the Cauchy reals. A Dedekind real number object and a Cauchy real number object can be constructed in any topos with a natural number object. In a topos the Dedekind reals do not necessarily coincide with the Cauchy reals.

Lemma 6. *In* **Eff** *the object of the Dedekind reals is isomorphic to the Cauchy reals object.*

There is only one real number object up to isomorphism in **Eff**. Let **R** denote this real number object.

The following result in **Eff** contradicts the classical Bolzano-Weierstrass theorem.

Theorem 7. *There exists a bounded monotone sequence of rationals in* **Eff** *which has no limit in* **R**.

Proof. Let $f : \mathbb{N} \to \mathbb{N}$ be an injective function which enumerates the halting set $\{(e, x) : \phi_e(x) \downarrow\}$ without repetition. Using f we will construct a bounded monotone sequence $(r_n)_{n \in N}$ of rationals defined below.

$$r_n = \sum_{i=0}^{n} 2^{-f(i)}$$

Assume for a contradiction that $(r_n)_{n \in N}$ has a limit L.

$$\forall k, \exists N_k, \forall n > N_k \quad |r_n - L| < 2^{-k}$$

There exists a computable function $g(k) = M_k$ where $M_k > N_k$ by countable choice such that,

$$\forall k, \forall n > g(k) \quad |r_n - L| < 2^{-k}$$

Given $k \in \mathbb{N}$, if there exists an i such that $f(i) = k$ then for any $n < i$ we have $r_n + 2^{-k} \leq r_i < L$ hence $|r_n - L| > 2^{-k}$.

For all $k \in \mathbb{N}$ if $f(i) = k$ then $i \leq g(k)$. This contradicts the fact that the image of f is undecidable. □

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