Automatic groups and Thompson's group F

Josh Bax
Supervisors: Bakhadyr Khoussainov and Andre Nies

July 22, 2011

1 Abstract

We show how some recent results relate to the currently open question of whether Thompson's group F is automatic. We begin with a brief introduction to (synchronous) automatic groups including some basic results, namely their characterisation as regularly generated groups which have the fellow traveller property and that their Dehn function is quadratic. This is followed by a description of the group F and proofs of some of its basic properties leading on to a discussion of results related to the growth function of F; the fellow traveller property and Guba's proof that the Dehn function of F is quadratic.

2 Preliminaries

Here we recap key concepts and standard terminology used within this survey.
An *alphabet* is a finite set of symbols, usually denoted by A. Elements of A are called *letters*. A *string* (equivalently a word) over an alphabet A is a finite sequence of letters from A. This can also be viewed as a function $s : \{1, ..., n\} \to A$. This representation is only conceptual, in what follows $s(k)$ will always refer to the length k prefix of s. A *language* over A is a set of strings over A. The largest possible language over A is the free monoid with respect to concatenation A^*, this contains every possible finite string over A.
A *finite state automaton* is a 5-tuple $F_A = (Q, A, \delta, q_0, F)$ where

- Q is the set of states,
- A is the alphabet of the automata,
- A *deterministic automaton* has transition function $\delta : Q \times A \to Q$,
- A *nondeterministic automaton* has transition function $\delta : Q \times A \to \mathcal{P}(Q)$,

- $q_0 \in Q$ is the start state,
- $F \subseteq Q$ is the set of final states.

An automaton F_A performs a computation on a string $s \in A^*$ by application of the transition function on the letters of s and the current state of the automata, where the first state is q_0. The computation is accepting if after reading all of the letters of s the automaton is in a state in F. Thus we can define the *language of an automaton* $L(F_A)$ as the subset of strings in A^* accepted by the automaton F_A.
A *regular expression* is a string representation of a language. Suppose L_1 and L_2 are automata recognisable languages over A. A regular expression over A is defined inductively as:

- $L(\epsilon) = \{\epsilon\}$, $L(()) = \emptyset$ and for any $s \in A^*$, $L(s) = \{s\}$.

Let r_1 and r_2 be regular expressions;

- $L(r_1|r_2) = L(r_1) \cup L(r_2)$.
- $L(r_1r_2) = \{st : s \in L(r_1) \text{ and } t \in L(r_2)\}$.
- $L(r_1^*) = L(r_1)^*$, i.e. the free monoid generated by $L(r_1)$.

Parentheses are used to group subexpressions.
A key result in the theory of automata states that the languages recognised by finite automata are equivalent to those defined by regular expressions. The set of all languages recognisable by either of these methods is called the *regular languages*. We will normally give a regular expression when referring explicitly to the language of an automaton.

A group may be written in a compact form as a set of *generators* and a set of *relators*. A group G is generated by the set X if G is the smallest group containing X. In an abstract sense, let A be some alphabet such that there is a surjective map $\pi : A \to X$. Define the *free group on* A, $F(A)$ as the subset of the language $(A|A^{-1})^*$ which does not contain any words with substrings aa^{-1} or $a^{-1}a$ for any $a \in A$. Note that $F(A)$ and $F(B)$ are isomorphic if $|A| = |B|$, so if $|A| = n$, $F(A)$ is normally referred to as the *free group of rank* n. The relators are words over A that act as 'simplification rules' to define a sublanguage of $F(A)$ in which no word has any relator as a substring and this sublanguage is representative of G in $F(A)$. What we have really done is define a surjective group homomorphism $\beta : F(A) \to G$ for which all of the relators are in the kernel. Since these are sufficient to define the group, the kernel of β is just the normal closure of the set of relators. Groups can then be specified as $G =< X|R >$ where X is the generating set and R is the set of relators. This is a *presentation* for G.

3 Automatic groups

Automatic groups formalise, in a sense, the notion of a "well-behaved" group from a computational perspective. Central to this is the group presentation. In general, groups have many different presentations and it is rarely clear which, if any, is the best to use. A major part of this difficulty is the fact that the homomorphism from the free group $F(A)$ to G is not guaranteed to be one-to-one. This means that any element of G may have many representatives in $F(A)$ and many more in A^*. The problem of recognising when two arbitrary words in A^* represent the same element of G (or, equivalently, a word represents the identity in G) is called the *word problem.*

Automatic groups have elements and operations recognisable by finite automata over A^*. More precisely the automata recognise the graphs of right multiplication by elements of A over a certain subset of the free group on A. This implies a certain simplicity of computation in both recognising normal forms for elements of G and for computing multiplications and equality. In fact automatic groups have a word problem solvable in quadratic time as will be shown later. Much of the material developed here is covered in greater detail in the now classic work by Epstein [8].

Definition 3.1. *A finitely generated group G with generating set A is called an automatic group if there exist the following automata:*

1. *W the word automaton, an automaton over A with language $L(W)$, such that $L(W)$ maps surjectively on to G.*

2. *For each $x \in A$ an automaton M_x such that M accepts $(v, w) \in A^* \times A^*$ if $v, w \in L(W)$ and $v = wx$ in G. These are called multipliers.*

3. *An automaton E over A such that $L(E) = \{(w, w) \in A^* \times A^* |$ if $w =_G w\}$. This is the equality recogniser.*

Together these are referred to as an automatic structure for G over A.

A technical note on the definition of the multipliers; these are a type of automata known as *many-variable automata.* They are defined over the alphabet $A' \times A'$ where $A' = A \cup \{\$\}$, assuming \$ is not in A. Any string in $A^* \times A^*$ has an equivalent in $(A' \times A')^*$ which can be found by padding the shorter string in the pair with the letter \$ so that both have the same length, then splitting these strings into letters from $(A' \times A')^*$.

Some examples of simple automatic groups:

The free group on X, has word automaton recognising the language X^* and multiplier automata M_x that recognise $\{(y, y) : y \in X\}^*(\$, x)$ for all $x \in X$.

Any finite group $G =< X >$. $L(W)$ can simply be the set of all group elements, this is a finite language over G. Similarly multipliers correspond to the language of pairs of group elements that differ by a single generator, also finite. Another classic example of automatic groups are the word-hyperbolic groups. A non-example is the Baumslag-Solitar groups. These have presentation $B(m,n) =< a, b|ba^m b^{-1} = a^n >$ which, to return to the point about presentations, look "easy" but are quite difficult to work with. In particular, the group $B(1,n)$ has an exponential Dehn function for $|n| \geq 2$. This means that given any word of length n representing the identity in $B(1,2)$, $O(2^n)$ other words equivalent to w can be derived. It will be shown later that automatic groups have quadratic Dehn function.

Definition 3.2. *Let G be a group with a finite set of generators A. The Cayley graph $\Gamma(G, A)$, is the graph with G as its set of vertices and such that (x, y) is an edge of Γ precisely when $y = xg$, for $g \in A$. Each edge is labelled by its respective generator and if $|g| = 2$, so that $y = xg$ and $x = yg$, then the edge between x and y is undirected. The vertex corresponding to the identity in G is referred to as the basepoint of the graph.*

As a simple example, the cyclic group $\mathbb{Z}_n =< a >$ has Cayley graph $\Gamma(G, \{a\})$ isomorphic to the cyclic graph on n vertices. Each edge has the form (a^i, a^{i+1}) for $0 \leq i < n$ and is labelled by a. The basepoint is the identity.

An important point to make is that the geometry of the Cayley graph of G for different generating sets is not necessarily similar. Take for example S_3, which is be generated by both $A_1 = \{(1,2),(1,3)\}$ and $A_2 = \{(1,2),(1,2,3)\}$. Clearly $\Gamma(G, A_2)$ must contain a directed 3-cycle, whereas $\Gamma(G, A_1)$ contains no such cycle, as every edge is necessarily undirected. Hence the two graphs cannot be isomorphic.

Next we introduce some important combinatorial structures that occur within the Cayley graph of a group.

Definition 3.3. *Given a presentation $< A|R >$ for a finitely generated group G, any representative of the identity can be written in the form*

$$w = \Pi_{i=1}^{n} v_i^{-1} r_i^{\pm 1} v_i \tag{1}$$

where $v_i \in F(A)$, $r_i \in R$ and $n \geq 0$. This is referred to as the disc with boundary w and combinatorial area n. The combinatorial area of the word w is the smallest possible n for which w has a decomposition (1)*. This is often written as $area(w)$.*

Consider $D_n =< r, f|r^n, f^2, (rf)^2 >$, the dihedral group of order n. A representative of the identity in D_n is $w = r^k f r^{(n-k)} f$ where $0 < k < n$. One disc for w is $\Pi_{i=0}^{n-2} r^i (rf)^2 r^{(n-i)}$. Another disc for w is $r^{(n-1)} f (r^n) r^{(n-1)} f$. Hence $area(w) = 1$.

The intuitive idea of a disc is that if w represents the identity in G, it corresponds to a cycle starting at the basepoint in the Cayley graph. Then a (possibly longer) cycle can be obtained from w by taking cycles labelled by the relators r_i and translating them by the v_i's such that the outer cycle remains w. In this way the construction provides a concrete way of representing all such representatives of the identity derived from w. As well, the combinatorial area of a word represents the smallest number of applications of relators of the presentation of G needed to derive the word. Looking at discs as subgraphs of a Cayley graph provides an avenue for investigating representatives of the identity from a geometric perspective.

Definition 3.4. *The Dehn diagram (also called van Kampen diagram) for a disc is the path in $\Gamma(G, A)$ starting at the basepoint and labelled by the word on the right hand side of* (1). *The boundary of this subgraph has label w and it has n internal cycles labelled by the relators $r_1, ..., r_n$.*

Dehn diagrams were used extensively by Guba to obtain his results on the complexity of the Dehn function which we define below.

Definition 3.5. *The Dehn function (also called the isoperimetric function) of G with respect to a generating set A is defined as*

$$\phi(n) = max\{area(w) : w =_G e \text{ and } |w| \leq n\}$$

where $area(w)$ is just the combinatorial area of w.

Although the Dehn function depends on the specific presentation of the group, its complexity class is an invariant of the group. To see this, first fix some presentation for G with Dehn function ϕ. In any other presentation for G, the generators of this new presentation may be written in terms of the old generators such that they are represented by words whose length are bounded by some constant C_{gens}. Also the relators of the old presentation are discs in the new presentation whose are must be bounded by some constant C_{rel} as they are finite. Hence if ϕ' is the Dehn function of the new presentation, it is the case that $\phi(n) \leq C_{rel}\phi'(nC_{gens})$.
Note that then name "Dehn function" is normally reserved for the smallest isoperimetric function of a group. But since we are primarily concerned with the complexity class of such functions we will simply refer to any representative isoperimetric function as a Dehn function. The fact that isoperimetric functions for a group have the same complexity and the following theorem demonstrate the usefulness of the Dehn function when investigating groups.

Theorem 3.1. The group G has a solvable word problem if and only if its Dehn function is computable.

The proof of this consists of showing that if the Dehn function is bounded by a computable function, then the number of representatives for the identity is also bounded and is therefore computable. It requires the lemma that if $v \in F(A)$ participates in a disc over A then it's maximum length is linearly proportional to the length of the boundary word. This together with the the definition of isoperimetric functions gives the result. The complete proof is found in Epstein [8].

The Cayley graph has the usual graph distance metric, namely $d(x, y)$ being the length of the shortest path between x and y, or infinite if no such path exists. Note that $\Gamma(G, A)$ must be connected because A generates G. Hence $d(x, y)$ is finite for all vertices x and y. Hence we have a well behaved metric for G:

Definition 3.6. *The word metric on G induced by A, $d_A : G \to \mathbb{N}$, is the length of the shortest path between two elements of G in the Cayley graph $\Gamma(G, A)$. Equivalently, $d_A(x, y)$ is the shortest word over A representing $x^{-1}y$. We write $|x|$ for $d_A(x, e)$.*

This naturally extends to a metric for A^* simply by applying the surjective map from A to G. Another useful metric is the synchronous distance, which is essentially the word distance between equal length prefixes of two words.

Definition 3.7. *The synchronous distance between two words over A^* is $d_{syn}(u, v) = sup\{d_A(u(t), v(t)) : t \leq min(|u|, |v|)\}$.*

The property of being an automatic group, while guaranteeing well-behaved presentations, also implies a certain geometric reqularity in the Cayley graph of the underlying group.

Theorem 3.2. Assume G is an automatic group with generators A and word automaton W. Let M_x be a multiplier automaton for some $x \in A$. For any $w_1, w_2 \in L(M_x)$ the word distance between $w_1(n)$ and $w_2(n)$ is bounded by a fixed constant k_A. This is called the often called the fellow traveller property or Lipschitz property.

Proof. Assume that all the automata of the automatic structure are normalised (they have no unreachable states and dead states are consolidated). Take k to be the largest number of states in any automaton of the automatic structure. Suppose $w_1, w_2 \in L(W)$ and (w_1, w_2) is accepted by M_x. After processing the first n letters of (w_1, w_2) M_x must be in a state no more than k steps away from an accepting state, otherwise it is in the consolidated dead state which is not possible as (w_1, w_2) is an accepted string. Hence there is an accepting string $(w_1(n)v_1, w_2(n)v_2)$, implying that $w_1(n)v_1 =_G w_2(n)v_2x$. Then there is a path in $\Gamma(G, A)$ from $w_1(n)$ to $w_2(n)$ labelled by $v_1xv_2^{-1}$ and this has length $2k - 1$. Thus the distance between prefixes is bounded by $k_A = 2k - 1$. □

It turns out that this property is in fact characteristic of automatic groups, as demonstrated in the following theorem.

Theorem 3.3. Let G be a group with finite generating set A. If $W = (S, A, \delta, s_0, F)$ is a regular automaton over A such that $L(W)$ maps surjectively onto G and $\Gamma(G, A)$ has the fellow traveller property, then G is automatic.

Proof. Suppose k_A is the Lipschitz constant for G. Define N to be the set $\{g \in G : |g| \leq k\}$. Now we must construct the multipliers of the automatic structure. We know that these satisfy the fellow traveller property, so any accepting pair has prefixes which are at less than distance k from one another. Then we can immediately reject any pair for which this does not hold. For those which it does, the difference between the pair in G has a shortest length representative of length less than k, so it lies in N which is finite. Hence we can keep track of this difference with a finite automaton. So for $x \in A \cup \{\epsilon\}$ define M_x as follows:

1. Starting with an automaton W' that recognises $L(W)\$^*$, we take the set of states to be $S' \times S' \times N \cup \{s_{fail}\}$ where S' is the set of states of W' and $s_{fail} \notin S'$.
2. The alphabet $A' = (A \cup \{\$\}) \times (A \cup \{\$\})$.
3. The start state is (s_0, s_0, e).
4. Define the transition function δ_x such that for $(a, b) \in A'$

$$\delta_x((a, b), (s_i, s_j, g)) \to (\delta(s_i, a), \delta(s_j, b), agb^{-1})$$

   where the letter \$ maps to the identity in G and if $agb^{-1} \notin N$ the resulting state is s_{fail}. This is a failure state, so for any $a \in A'$, $\delta_x(x, s_{fail}) \to s_{fail}$.

5. The set of final states is $\{(s_i, s_j, x) \in S \times S \times N : s_i, s_j \in F\}$.

Clearly this automaton has the desired multiplier behaviour and these multipliers, together with W, give an automatic structure for G on A. □

Note that there is a generalisation of automatic groups which do not necessarily have this property, these are called asynchronous automatic groups. The kind discussed here are then referred to as synchronous automatic groups.

Theorem 3.4. The property of automaticity is invariant under change of generators.

Proof. Suppose we have two generating sets X and Y for the group G and an automatic structure for G on X specified by the word automaton W_X and the Lipschitz constant k_X. Clearly, every generator in X has a representation in terms of Y. It is a theorem that the image of a regular language under substitution is a regular language. Call the above substitution f. So we have that $f(L(W_X))$ is a regular language over Y and since $L(W_X)$ maps surjectively onto G and $x =_G f(x)$, then $f(L(W_X))$ maps surjectively onto G also. Concretely we find the automaton W_Y over Y by replacing the letters on each of the arrows of W_X with the representation of that letter in terms of Y. An automaton with arrows labelled by strings (or regular expressions in general) is known as a generalised finite state automaton. A Lipschitz constant for the new structure can be found by multiplying k_X by the length of the longest string over Y required to represent a generator in X. By theorem 3.3 this specifies an automatic structure for G over Y. □

Theorem 3.5. Let word automaton W define an automatic structure for $G =< A >$. Given any $s \in A^*$, a representative of s in $L(W)$ can be found in time quadratic in the length of s.

Proof. We show that given $s \in L(W)$ we can find a representative of sx where $x \in A$, in linear time, so that starting with a representative of the identity in $L(W)$ we can build up a representative of any $w \in A^*$ by appending the letters of w one by one. Hence this entire process may be accomplished in time $O(|w|^2)$.
Let $s = s_0 s_1 ... s_n$ be any element of $L(W)$. We want to find a normal form of sx that is accepted by W, which we will call $\bar{sx}$. To do this, we enumerate the set of states of $M_x = (Q, A, \delta, q_0, F)$ reachable by a computation on (s, t), where t is a yet to be determined element of A^*. In other words, define $S_0 = \{q_0\}$, then define inductively $S_{i+1} = \{q \in Q : \delta(q_i, (s_i, t)) = q$ for $q_i \in S_i$, $t \in A^*\}$. Since M_x is finite, one of these sets must eventually contain a final state. This corresponds to a path of edges from S_0 to the final state. The label of this path is $\bar{sx} \in L(W)$ by definition. Since the size of each set S_i is bounded by a constant we must now show that the number of sets (equivalently the length of $\bar{sx}$) grows no more than $O(|s|)$ to complete the proof. By definition $(s, \bar{sx})$ is accepted by M_x. Then, if the run of M_x on $(s, \bar{sx})$ has more than $|s| + |Q|$ transitions, some state of M_x is visited twice after all of s has been read. Since any final state of M_x is no more than $|Q|$ transitions away from any (non-rejecting) state, there is a shorter accepting string. However $\bar{sx}$ is defined above to be the shortest string such that $(s, \bar{sx})$ is accepting, hence $|\bar{sx}| \leq |s| + |Q|$. □

Theorem 3.6. Let G be an automatic group with presentation $< A|R >$, then any word $w \in A*$ that represents the identity has a disc (1) that has area $O(|w|^2)$. Equivalently, G has a quadratic Dehn function.

Proof. Let $w \in A^*$ and W be a word automaton for G over A. Define N to be a constant larger than the number of states in any of the multiplier automata in the automatic structure for G implied by W. First we prove any representative of w in $L(W)$ is has length $O(|w|)$. Fix c as the length of some word $w_0 \in L(W)$ that maps to the identity in G. Now $(w_0, w(1))$ is accepted by some multiplier automaton, hence $w(1)$ has a representative in $L(W)$ of length at most $c + N$. Assume $w(t)$ has a representative of length at most $c + tN$ for some $t \leq |w|$. Following the argument above, $w(t+1)$ has a representative in $L(W)$ of length at most $c + tN + N = c + (t+1)N$. Then w has a representative of length at most $c + |w|N$.
Take two prefixes of w, $w(t)$ and $w(t+1)$ where $w(t+1) = w(t)x$ for $x \in A$. The paths in $\Gamma(G, A)$ to the respective elements of G called w_t and w_{t+1} have length less than $c + |w|N$ and by the fellow traveller property, these have synchronous distance less than some constant k. Hence the loop $w_t x w_{t+1}^{-1}$ can be decomposed into $c + |w|N$ loops of length $2k + 2$. Since there are $|w|$ many such prefix pairs, there are $O(|w|^2)$ such loops in total. We can build an equivalent presentation by adding the boundary equations of all loops of size $2k + 2$ to R. If w represents the identity element in this new presentation, then by the above construction w can always be decomposed into a disc with area $O(|w|^2)$, hence the Dehn function is $O(n^2)$. □

4 Thompson's group F

Thompson's group owes it's name to Richard Thompson who first encountered it in 1965 in the context of his work on the λ-calculus. It was originally described by Thompson as the geometry group of the composition law but later found other incarnations as a homeomorphism group (see below) and as the group of order-preserving automorphisms of the free Cantor algebra on a singleton set.
F is interesting in that it is a finitely generated infinite order group which is 'almost ablelian' in that every homomorphic image of F is abelian. Over the years F has served as a useful source for counter-examples to various conjectures. A complete overview of the basic theorems on F (and the other infinite Thompson groups) from the homeomorphism point of view is found in Cannon, Floyd and Parry's survey [4]. Another very readable introduction with an emphasis on tree diagrams (introduced in the next section) is the first chapter of Belk's thesis [1].

Definition 4.1. *A number is a dyadic rational if its denominator is a power of* 2*. A dyadic interval is an interval of the form* $[k/2^n, (k+1)/2^n]$ *for some* $k, n \in \mathbb{N}$*. A dyadic rearrangement is a piecewise linear homeomorphism that maps a dyadic partition onto another dyadic partition of the same size.*

Thompson's group has one definition as the set of dyadic rearrangements of the unit interval under composition. This is precisely the group of piecewise-linear homeomorphisms of the unit interval which are differentiable everywhere except for at a finite set of points which have dyadic rational coordinates and whose derivatives (where they exist) are powers of 2.

First we define an infinite sequence of functions and claim that F is generated by these. Then we exhibit an equivalent finite presentation.

Consider the following two functions:

$$f_0(x) := \begin{cases} 2x, & x \in [0, 1/4) \\ x + 1/4, & x \in [1/4, 2/4) \\ (x+1)/2, & x \in [1/2, 1] \end{cases} \quad f_1(x) := \begin{cases} x, & x \in [0, 1/2) \\ 2x - 1/2, & x \in [4/8, 5/8) \\ x + 1/8, & x \in [5/8, 6/8) \\ (x+1)/2, & x \in [3/4, 1] \end{cases}$$

f_0 represents a rearrangement of the dyadic partition $[0, 1/2], [2/4, 3/4], [3/4, 1]$ to the partition $[0, 1/2], [2/4, 3/4], [3/4, 1]$. A simpler notation is to just list the breakpoints of the partitions, for example the domain partition is $0, 1/2, 3/4, 1$.
So f_1 rearranges the partition $0, 1/2, 5/8, 3/4, 1$ to $0, 1/2, 3/4, 7/8, 1$. Note that by giving the domain and range partitions we have completely specified the functions. Below is a graphical method of describing elements of F which helps visualise multiplication.

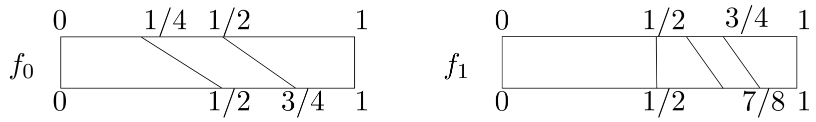

Fig. 1. Representation of f_0 and f_1 as rearrangements of dyadic partitions.

Note that conjugation of f_1 by f_0 essentially embeds the pattern of f_0 in the interval $[3/4, 1]$. This is a general pattern and with it in mind we define the sequence of functions $f_0, f_1, f_2 ...$ where f_0 and f_1 are defined as above and $f_i = f_0^{1-i} f_1 f_0^{i-1}$ for all $i \geq 2$. The following theorem states that giving two equal size standard dyadic partitions of $[0, 1]$ is enough to define an element of f.

Theorem 4.1. If $0 < x_1 < x_2 < ... < x_n = 1$ and $0 < y_1 < y_2 < ... < y_n = 1$ are sequences of dyadic rationals such that $[x_i, x_{(i+1)}]$ and $[y_i, y(i+1)]$ are standard dyadic intervals for $1 \leq i \leq n-1$, then there exists $f \in F$ such that $f(x_i) = y_i$. Also, if $x_i = y_i$ and $x_{(i+1)} = y_{(i+1)}$, then f is trivial on $[x_i, x_{(i+1)}]$.

Proof. Define $f : [0, 1] \to [0, 1]$ to be the function whose graph is the path in $\mathbb{R}^2$ from $(0, 0)$ to $(1, 1)$ comprised of the line segments with end points (x_i, y_i) and $(x_{(i+1)}, y_{(i+1)})$. Clearly this function is a piecewise linear homeomorphism with dyadic breakpoints. Since $[x_i, x_{(i+1)}]$ and $[y_i, y(i+1)]$ are standard dyadic intervals the line segment from (x_i, y_i) to $(x_{(i+1)}, y_{(i+1)})$ has slope that is a power of 2. Hence f is a dyadic rearrangement. If $x_i = y_i$ and $x_{(i+1)} = y_{(i+1)}$ then the line segment over the interval $[x_i, x_{(i+1)}]$ has slope one. Thus f is trivial on $[x_i, x_{(i+1)}]$. □

The sequence of functions $f_0, f_1, f_2, ...$ in F is in fact a generating set for F. The set is closed under composition and so forms a submonoid in F and every element of F can be written as a gf^{-1} where both g and f are in this monoid. This set gives rise to the following canonical infinite presentation for F.

Definition 4.2. *F is described by the presentation*

$$< x_0, x_1, x_2, ...| x_n x_k = x_k x_{n+1}, k < n > \tag{2}$$

The map setting $x_i = f_i$ for all $i \geq 0$ defines an isomorphism with F.

Given this infinite presentation we can show that F has a finite presentation with generators $\{x_0, x_1\}$.

Theorem 4.2. F has finite presentation

$$< x_0, x_1 | x_2^{x_1} = x_3, x_3^{x_1} = x_4 > \tag{3}$$

where $x_{n+1} = x_0^{-n} x_1 x_0^n$, for $n \geq 1$.

Proof. The definition of x_{n+1} is by induction on indices of elements. $x_2 = x_1^{x_0}$ and for arbitrary k, the inductive assumption $x_k = x_0^{1-k} x_1 x_0^{k-1}$ and the relator $x_{k+1} = x_k^{x_0}$ imply the result. Hence F is generated by x_0, x_1. It remains to show that the relators of (3) derive those of (2). We prove that $x_n^{x_k} = x_{n+1}$ for all $k < n$ again by induction on n. This is true by definition for $n = 2$ and 3. Assume the inductive hypothesis for arbitrary n. Now consider $x_n^{x_k}$ for some $k < n$. If $k = 0$ then the statement is true by definition. So $0 < k < n$. Expanding the above and simplifying:

$$\begin{aligned} x_n^{x_k} &= x_0^{1-k} x_1^{-1} x_0^{k-n} x_1 x_0^{n-k} x_1 x_0^{k-1} \\ &= x_0^{1-k} x_1^{-1} x_{n-k+1} x_1 x_0^{k-1} \\ &= x_0^{1-k} (x_{n-k+1})^{x_1} x_0^{k-1} \end{aligned}$$

So we all we must prove $(x_{n-k+1})^{x_1} = x_{n-k+2}$, as all other cases follow by conjugating both sides of this equation by x_0^{k-1}. This is true for $n - k + 1$ less than n, so we have $x_n^{x_k} = x_{n+1}$ for $1 < k < n$. The only remaining case is $k = 1$. This is done using the relations $x_{n-1} x_1 = x_n$, $x_n x_k = x_k x_{n+1}$ as above and $x_2x_1 = x_1x_3$. The derivation is as follows:

$$\begin{aligned} x_{n-1}x_2 &= x_2x_n \\ (x_{n-1}x_2)x_1 &= (x_2x_n)x_1 \\ x_{n-1}x_1x_3 &= x_2x_nx_1 \\ x_1x_nx_3 &= x_2x_nx_1 \\ x_1x_3x_{n+1} &= x_2x_nx_1 \\ x_2(x_1x_{n+1}) &= x_2(x_nx_1) \end{aligned}$$

Therefore in all cases $x_n^{x_k} = x_{n+1}$ and so the two relations of (3) suffice. Hence (3) is a presentation for F. □

Definition 4.3. *A positive element of $p \in F$ is a product of $\{x_0, x_1, ...\}$. A monotone positive element is any $p \in F$ such that p is positive and can be written as a product of generators with monotonically increasing indices., although we will not prove this here*

Theorem 4.3. Every $f \in F$ has a unique normal form $f = pq^{-1}$ where both p and q are monotone positive and if x_i occurs in both p and q, then x_{i+1} is in one of p or q. In addition, the indices of generators in p and q do not exceed the length of pq^{-1}, considered as a word over $\{x_0, x_1, ...\}$.

Proof. The proof consists of the observation that the relators of the infinite presentation yield rewriting rules for strings over $\{x_0, x_1\}$, where x_n is considered shorthand for $x_0^{1-i}x_1x_0^{i-1}$ and x_n^{-1} is $x_0^{1-i}x_1^{-1}x_0^{i-1}$, (for simplicity these shall be referred to as 'letters'). Here we present just enough detail to illustrate how to find these normal forms in practice.
Assuming an ordering on words such that $x_i < x_{i+1}$ and $x_i < x_i^{-1}$ for any $i \in \mathbb{N}$, it follows that systematic application of these rules yields a unique irreducible string of the required form.
Suppose $k < n$, then these rules follow from (2):

$$\begin{aligned} x_n^{-1}x_k &\to x_kx_{n+1}^{-1} \\ x_k^{-1}x_n &\to x_{n+1}x_k^{-1} \end{aligned}$$

These rules act to shift inverse elements right and so produce a string with a strictly lower lexicographic order. This string has the form ab where a is positive and b is not. Also we have:

$$\begin{aligned} x_nx_k &\to x_kx_{n+1} \\ x_k^{-1}x_n^{-1} &\to x_{n+1}^{-1}x_k^{-1} \end{aligned}$$

The immediate result of these rules is a string of lower lexicographic order. The eventual result is that every substring with all positive (resp. negative) letters is monotone.

Thus, once these rules can no longer be applied to the given string, it must have the form pq^{-1} where both p and q are monotone positive. If some letter x_i occurs in both p and q and x_{i+1} does not occur in either, then the reverse of the above rules can be applied to bring x_i and x_i^{-1} adjacent and cancel both (since the rule $x_i x_i^{-1} \to \epsilon$ is implicit). If x_{i+1} is in one of p or q, then nothing can be done. To prove that this system gives a unique normal form it is required to prove that applying the above rules in any order produces the same irreducible string. A full proof using tree diagrams is in [4] and an abridged version is in [11] uses the rewrite-rule approach above. □

As a demonstration of the above procedure consider normalising the word $x_5 x_3^{-2} x_1 x_3 x_5 x_3^{-1} x_2^{-1}$. First shift negative letters right:

$$x_5 x_1 x_4^{-2} x_3 x_5 x_3^{-1} x_2^{-1}$$
$$x_5 x_1 x_3 x_5^{-1} x_3^{-1} x_2^{-1}$$

Then reorder positive letters:

$$x_1 x_6 x_3 x_5^{-1} x_3^{-1} x_2^{-1}$$
$$x_1 x_3 x_7 x_5^{-1} x_3^{-1} x_2^{-1}$$

Finally delete the pair x_3, x_3^{-1} by shifting x_3^{-1} left.

$$x_1 x_3 x_7 x_3^{-1} x_4^{-1} x_2^{-1}$$
$$x_1 x_3 x_3^{-1} x_6 x_4^{-1} x_2^{-1}$$
$$x_1 x_6 x_4^{-1} x_2^{-1}$$

Note that the last step would not have been possible if either x_4 or x_4^{-1} had occurred in $x_1 x_3 x_7 x_5^{-1} x_3^{-1} x_2^{-1}$.

Now we will present some interesting properties of F.

Theorem 4.4. $F/[F,F] \simeq \mathbb{Z} \oplus \mathbb{Z}$.

Proof. A presentation for the abelianisation can be obtained by adding relators $x_n^{x_k} = x_n$ for $1 \leq$ and $0 \leq k$ to (2), the infinite presentation of F. Since this already has relators of the form $x_n^{x_k} = x_{n+1}$ the resulting presentation is

$$< x_0, x_1, ... | x_i = x_j \text{ for } i, j \geq 1 >$$

This is an abelian group generated by x_0 and x_1, both of which have infinite order. Thus, this is a presentation for $\mathbb{Z} \oplus \mathbb{Z}$. □

Theorem 4.5. $Z(F)$ is trivial.

Proof. Take $f \in Z(F)$. Suppose for contradiction that f is non-trivial. So we can choose some point $z \in [0,1]$ such that $z \neq f(z)$. Choose n large enough so that $z \in [k/2^n, (k+1)/2^n]$ and $f(z) \in [l/2^n, (l+1)/2^n]$ but $k \neq l$. Then by theorem 4.1, we can find some element $g \in F$ such that g is non-trivial on $[k/2^n, (k+1)/2^n]$ but trivial on $[l/2^n, (l+1)/2^n]$. It follows that $g(f(z)) \neq f(g(z))$ contradicting that $f \in Z(F)$. □

Theorem 4.6. For every proper non-trivial normal subgroup N, the group F/N is abelian.

Theorem 4.7. $[F, F]$ is simple.

Theorem 4.8. F contains a free abelian subgroup of every rank.

Proof. Define the set $X_n = \{x_0x_1^{-1}, x_2x_3^{-1}, ...x_{2n-2}x_22n-1^{-1}\}$. Let x_ax_{a+1} and $x_bx_{b+1}^{-1}$ for $0 \leq a < b$ be in X. Clearly $b - a > 2$. So using the rules from theorem 4.3 we have

$$(x_bx_{b+1}^{-1})^{x_ax_{a+1}^{-1}} =$$

$$(x_{b+1}x_{b+2}^{-1})^{x_{a+1}^{-1}}$$

$$(x_{a+1}x_{b+1}x_{a+1}^{-1})(x_{a+1}x_{b+2}^{-1}x_{a+1}^{-1})$$

$$(x_bx_{a+1})x_{a+1}^{-1}x_{a+1}(x_{a+1}^{-1}x_{b+1}^{-1})$$

$$= x_bx_{b+1}^{-1}$$

And since every element of F has infinite order it follows that $< X_n >= \mathbb{Z}^n$. □

Definition 4.4. *The growth function $g_L : \mathbb{N} \to \mathbb{N}$, of language L over alphabet A is defined as $g_L(n) = |L \cap A^n|$ or, in other words, simply the number of words in L having length n. In addition, the generating function for g_L is defined as the power series*

$$GF_L(x) = \Sigma_{n=0}^{\infty} g_L(n)x^n \tag{4}$$

Languages can be classified in terms of their growth function and generating function. Key categories are:

- exponential growth- when $g_L(n) = O(m^n)$.
- polynomial (sometimes termed subexponential) growth- when $g_L(n) = O(n^k)$ for some $k \in \mathbb{N}$.
- rational growth - this refers to the generating function GF_L being a rational function, that is it can be written as a ratio of polynomial functions.

As would be expected, this classification can be applied to finitely generated groups by forming the growth function of the language over a generating set. The property of having exponential growth is independent of the choice of generating set, as shown in [14].

Theorem 4.9. F has exponential growth.

Proof. We exhibit a subset of F which has an exponential growth function. This forms a lower bound for the growth function of F, thus it is exponential also. Consider the alphabet $\{x_0^{-1}, x_1\} \subset F$. Any word over this alphabet can be written in the form $x_1^{e_1} x_0^{-1} x_1^{e_2} ... x_0^{-1} x_1^{e_n}$, where each $e_i \geq 0$. By Theorem 4.3, (in particular, by repeated application of the rule $x_0^{-1} x_i \to x_{i+1} x_0^{-1}$ to the $n-1$ occurrences of x_0), we obtain the equivalent normal form in F, $x_1^{e_1} x_2^{e_2} ... x_n^{e_n} x_0^{1-n}$. This is unique and is fully specified by the sequence of exponents $e_1, e_2, e_3, ..., e_n$. Hence every element of $\{x_0^{-1}, x_1\}^*$ is distinct in F (in other words it is a free submonoid of F). For any n there are 2^n elements in $\{x_0^{-1}, x_1\}$. This is the lower bound for g_F. Thus F has exponential growth. □

Regular languages may have exponential or polynomial growth, but always have rational generating function. So, does the language given by (3) have rational generating function? Burillo proved in [3] that the monoid of positive words (i.e. $\{x_0, x_1\}^*$) has rational generating function and gave an explicit formula for the number of positive words of a given length. In the opposite direction, Elder, Fusy and Rechnitzer [7] in a 2010 paper calculate the first 1500 terms of the growth series and conjecture that the generating function "...contains square-root singularities, so is unlikely to be rational." A negative answer like this would demonstrate that no automaton has language precisely that of (3). At the present time this question remains open.

Another result pointing to F not being automatic is the fact that geodesic combings over the finite presentation of F do not have the fellow traveller property. This relies on the fact that F is not minimally almost as demonstrated by Belk and Bux [2] and separately by Cleary and Tabak[5].

Definition 4.5. *A group G is minimally almost-convex if, for every $g, h \in B^n(G)$ (where $B^n(G)$ is the n-ball centred on the basepoint in the Cayley graph of G) and such that the distance between g and h is exactly 2, then there is a path of length less than $2n$ between g and h within $B^n(G)$.*

Theorem 4.10. F is not minimally almost-convex.

Definition 4.6. *Given G generated by A, with Cayley graph $\Gamma(G, A)$, a combing of G is a map that associates with each $g \in G$ a path in $\Gamma(G, A)$ from g to the basepoint. A geodesic combing is a combing in which every path has minimal length in $\Gamma(G, A)$.*

Theorem 4.11. Any geodesic combing of F using the finite presentation does not have the fellow traveller property.

Proof. Suppose for contradiction that a geodesic combing of F has a Lipschitz contstant k. Choose $n > 2k$ and let a, b be words over $\{x_0, x_1\}$ representing geodesic paths of length less than n to two elements $\overline{a}, \overline{b} \in F$ such that $\overline{a}$ and $\overline{b}$ are at distance two from each other. Then there is another element $\overline{c} \in F$ at distance one from each of $\overline{a}$ and $\overline{b}$. So for $0 < i < n$ it is the case that the distance between $\overline{a(i)}$ and $\overline{c(i)}$ is less than k and similarly for $\overline{b(i)}$ and $\overline{c(i)}$. Hence the elements $\overline{a(i)}$ and $\overline{b(i)}$ are at distance at most $2k$ from each other. Since both are in $B^n(F)$ and by assumption $2k < 2n$, this contradicts that F is not minimally almost-convex. □

This immediately implies that there can be no automatic structure for F whose language $L(W)$ has only geodesics strings. In fact it's stronger than that. A theorem found in chapter 3 of Epstein [8] states that given the language $L(W)$ of an automatic structure, then, given some ordering on the alphabet A, the lexicographically shortest representatives within $L(W)$ for each element of G forms an automatic structure for G also. In particular no automatic structure for F can have a geodesic representative for every group element within its language. This was also shown by Cleary and Elder in 2006 [6].

Now we summarise S. B. Fordham's result[9], which was later extended[10] to the p-adic Thompson groups $F(p)$ (these are the groups of rearrangements of p-adic subintervals of $[0, 1]$). The result is developed using the tree representation for group elements.

Definition 4.7. *The tree of standard dyadic intervals is the rooted binary tree $\mathcal{T}$ with nodes labelled by intervals. It has root $[0, 1]$ and for every node the left child is the left half of the interval and the right child the right half.*

Clearly the edges of $\mathcal{T}$ represent the subset inclusion relation and any dyadic partition of $[0, 1]$ corresponds to a finite subtree with root $[0, 1]$.

Definition 4.8. *A tree diagram for $f \in F$ is a pair (T_1, T_2) of finite subtrees of $\mathcal{T}$ each with root $[0, 1]$ such that the leaves of T_1 are labelled by the intervals in the dyadic partition induced by the breakpoints of the domain of f and similarly T_2 has leaves labelled by the image of this partition under f.*

As an example consider the tree diagrams for the generators f_0 and f_1.

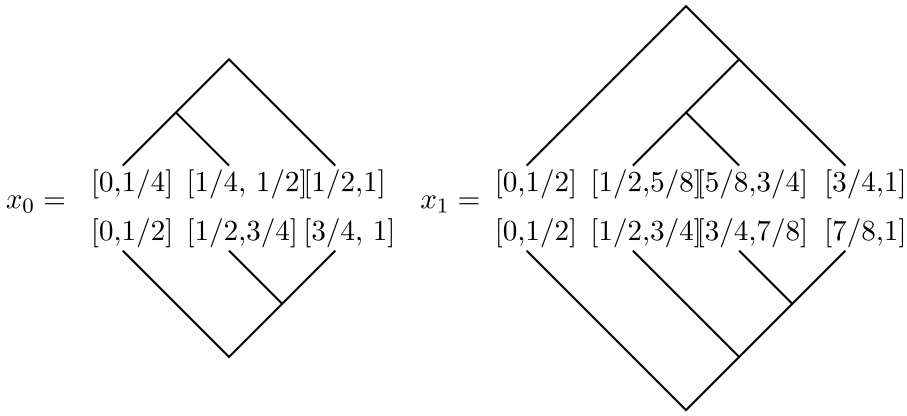

Fig. 2. Tree diagrams for generators x_0 and x_1.

Note that some trees may have redundant leaves. A *caret* is defined as a binary tree with just three nodes. If any carets of T_1 have leaves which are also leaves in T_1 and these leaves match the leaves of a caret in T_2, then the corresponding element of f will have the same gradient on both of the intervals that make up the carets leaves. Hence this is an unecessary subdivision and may be removed. Any tree diagram which has had all redundant carets removed is said to be reduced and this represents the equivalence class of all unreduced trees that also correspond to the given element of F. The product of tree diagrams (T_1, T_2) and (T_2, T_3) is (T_1, T_3). Multiplication in general can be carried out by reversing the reduction process; simply adding caret pairs to a tree diagram until the range and domain tree of the two elements to be multiplied match.

Guba has developed a generalisation of the method of representing group elements by pairs of diagrams in the above manner, these are called diagram groups and are introduced in [13]. They consist of matched pairs of diagrams where multiplication is accomplished in the above manner and subdiagrams which may be reduced are specified in a similar way to relators in a presentation. For example F has the single 'relator' $x^2 = x$ corresponding to caret reduction. In [13] he uses these methods to prove that the conjugacy problem for F is solvable (this problem is similar to the word problem only it identifies when two elements are conjugate).

Definition 4.9. *A minimal length function for a group $G = < X >$ is any function $\phi : G \to \mathbb{N}$ which satisfies the following defining properties:*
Let $g \in G$ and $x \in X$,

1. $\phi(1_G) = 0$.
2. *If* $\phi(g) = 0$ *then* $g = 1_G$.
3. $\phi(gx^{-1}) \geq \phi(g) - 1$.
4. *There always exists some* $y \in X$ *such that* $\phi(gy^{-1}) = \phi(g) - 1$.

In Fordham's paper, tree diagrams are modified by only keeping track of internal nodes (i.e. not leaves). A placeholder element is used instead to represent unreduced trees in a concise manner. He identifies just seven possible configurations for these internal nodes and thus creates a scheme to label each occurence of these with a multiset representing the sequence of generators required to produce them. The size of these labelling multisets can then be easily calculated. This is then shown to be a minimal length function for F. Since this method only requires a traversal of the tree and the configuration of the internal nodes are determined by their neighbours, this can be accomplished in linear time.

Theorem 4.12. Given a word $w \in \{x_0, x_1\}^*$ a minimal length representative for w in F can be found in $O(|w|)$ steps. In particular the word problem for F is solvable in linear time.

If F were shown not to be automatic, then the above theorem puts F in the interesting situation where it has a large number of properties of automatic groups yet is sufficiently complex that it doesn't have an automatic structure. The next section demonstrates another property that F has in common with automatic groups.

5 The Dehn function of F is quadratic

This section summarises a result by Guba [11]. Prior to this result Guba had proved that the Dehn function was bounded by a quintic polynomial [12]. As demonstrated above, the Dehn function of an automatic group is quadratic. So this result shifts the balance in favour of F being automatic. But even if it turns out that it is not, F would still be an illustrative boundary case. Here we develop a few of Guba's preliminary constructions with a view to illustrating the techniques used, but the proof of the main theorem is omitted.

Definition 5.1. *Define the group* $\mathcal{P}_r$ *as*

$$\mathcal{P}_r = < x_0, x_1, x_2, ... | x_j^{x_i} = x_{j+1} (0 < j - i \leq r) >$$

This group is equal to F *for* $r \geq 2$.

Using this presentation for F gives a nice intermediate form between the simplification power of (2) and the finiteness of (3). It gives rise to the following theorem relating Dehn diagrams of $\mathcal{P}_5$ and F.

Theorem 5.1. If Δ is a Dehn diagram over $\mathcal{P}_5$ with boundary word w and area N, then there is a corresponding Dehn diagram over the finite presentation of F with boundary w and area $\leq 13N$.

Proof. The proof amounts to showing that any inner cycle of Δ is divided into no more than 13 cycles in the Cayley graph of the finite presentation for F. Given an arbitrary diagram Δ of $\mathcal{P}_5$ we map this to a diagram of F by replacing every edge labelled by x_n with a path labelled by $x_0^{1-n}x_1x_0^{n-1}$ for $n \geq 2$. Given our chosen presentation of F, the smallest possible cycles have as boundary word either $x_2^{x_0}x_3^{-1}$ or $x_3^{x_0}x_4^{-1}$. Any inner cycle of Δ has boundary $x_j^{x_i}x_{j+1}^{-1}$ for $0 < j - i \leq 5$ as per definition 5.1. If $i = 0$, then since $x_j^{x_0} = x_{j+1}$ by definition, the image of the cycle with boundary $x_j^{x_0}x_{j+1}^{-1}$ is a point. If $i \geq 1$, then the cycle boundary can be written as $(x_0^{1-i}x_1^{-1}x_0^{i-1})x_j(x_0^{1-i}x_1x_0^{i-1})x_{j+1}^{-1}$, which is equivalent to $x_0^{1-i}x_{(j-i+1)}^{x_1}x_0^{i-1}x_{j+1}^{-1}$ by expanding the definition of x_j. The possible subscripts for the middle element are in the range $2 \leq j - i + 1 \leq 6$, since $0 < j - i \leq 5$ by definition 5.1. Note that $x_k^{x_1} = x_{k+1}$ is a relator in the infinite presentation of F. From this we infer that if $x_k^{x_1}$ occurs in the label of a cycle then there is a chord with label x_{k+1}^{-1}. Equivalently, any disc bounded by the cycle may be rewritten with this relator. As in the proof for theorem 4.2, these relators can be written as compositions of the 2 relators of (3). The largest of the possible elements, x_6, admits a decomposition into a disc over (3) with area at most 13. Hence the total area is no more that $13N$. □

Theorem 5.2. If every word w over $\{x_0^{\pm 1}, x_1^{\pm 1}\}$ has a disc in $\mathcal{P}_5$ with boundary $w(pq^{-1})^{-1}$, where pq^{-1} is the normal form of w and area in $O(|w|^2)$, then the Dehn function of F is quadratic.

Proof. If pq^{-1} is the normal form for w, then $w = pq^{-1}$ and so $w(pq^{-1})^{-1}$ is a disc. Also, if w is the boundary word of some Dehn diagram then $w = 1$ and $pq^{-1} = 1$. So pq^{-1} and w have the same area. The result in Theorem 5.1 shows that if the area of a Dehn diagram in $\mathcal{P}_5$ is quadratically bounded then so is the area of the corresponding diagram in F. The rest of the argument proceeds in the same manner as at the end of Theorem 3.6. □

These theorems reduce the problem to finding Dehn diagrams over $\mathcal{P}_5$ with a boundary in a standard form. The following constructions provide a means for doing so.

Definition 5.2. *The shift function on* (2) *is* $\psi(x_i) = x_{i+1}$ *for all* $i \geq 0$. *The analogous function on* (3) *is simply conjugation by* x_0.

Definition 5.3. *Let* p *and* q *be monotone positive words over* F. *The triangle diagram of* p *and* q *is the Dehn diagram over* (3) *with boundary* pqr^{-1}, *where* r *is the normal form of the word* pq.

This construct provides a geometric interpretation of the normalisation of two positive words. It was estimated in [7] that the normal form of an element and its geodesic representation (i.e. the label of the shortest path to it in the Cayley graph) differ by $O(1)$, so these diagrams indeed deserve their name. The proof that the area of these diagrams has the same complexity class as arbitrary Dehn diagrams proceeds by induction on the length of the word w. Essentially it shows that the Dehn diagram with boundary $w(pq^{-1})^{-1}$ can be 'cut up' by normalised words in such a way that these form the 'hypotenuse' to two monotone positive words and the diagram is divided into triangular diagrams. When doing this it is necessary to specify that neither the length of any 'hypotenuse' r nor the maximum subscript occurring as a letter of r is larger than the length of the word being divided.

Triangle diagrams have a regular internal structure which is used in a further simplification to "rectangular diagrams". To motivate this, it is useful to consider the process of generating the triangle diagram corresponding to the words p and q over (2). Suppose $p = x_{j1}x_{j2}...x_{jm}$ and $q = x_{k1}x_{k2}...x_{kn}$. Define the sequence of words $v_0, u_1, v_1, u_2, v_2, ...u_n, v_n$ such that:

- $v_0 = p$.
- u_i is the longest suffix of v_{i-1} such that it's first letter has a subscript higher than k_i.
- $v_i = \psi(u_i)$.

Starting with p we note that u_1 is the suffix $x_{ja}x_{ja+1}...x_{jm}$ of p such that $k_1 < j_a$. Since $v_1 = \psi(u_1)$ and $x_{k1} < x_{ja}$ it follows that $v_1 = u_1^{x_{k1}}$. We can picture this as "attaching" the word v_1 to the word u_1 in p by edges labelled with x_{k1}. In general the word v_i can be attached by x_{ki} to the suffix u_i of v_{i-1}. This continues until some u_j is empty. The result can be pictured as p lying horizontally with layers of rectangular cells stacked on top, flush against the right edge of p. These cells have borders labelled by u_i on the bottom x_{ki} on the vertical edges and v_i on top. So the rightmost vertical path of the diagram has label q
Now the prefix of p left by u_1 consists of letters with subscripts at most $k1$. Continuing by reading x_{k1}, then the prefix of v_1 left by u_2 and we see that letters are read in increasing order. We can follow this 'hypotenuse' path from the empty element to pq, this is the path r and clearly it represents the normal form of pq.

Definition 5.4. *Let $p = x_{i1}...x_{im}, q = x_{j1}...x_{jn}$ be monotone positive words over F such that $j_1 + k > i_{k+1}$ for $0 \leq k < m$. Then there is a Dehn diagram in $\mathcal{P}_5$ with boundary $q^p(\psi^m(q))^{-1}$ and with mn cells. This is called a rectangular diagram.*

Note that the boundary corresponds to the equality $q^p = \psi^m(q)$. This follows from the condition $j_1+k > i_{(k+1)}$, because $q^{x_{i1}} = x_{j1+1}x_{j2+1}...x_{jn+1} = \psi(q)$ and similarly $q^{x_{i1}...x_{ik}} = x_{j1+k}x_{j2+k}...x_{jn+k} = \psi^k(q)$ for k as above.

Due to their internal structure as described above, triangle diagrams can be covered by rectangular diagrams in such a way that the area of the combined covering diagrams is not more than a constant multiple of the area of the original triangle diagram. The following example will give an idea of how this may be done.

Define two monotone positive words, $p = x_0x_3x_4x_6x_8x_9x_{10}$ and $q = x_1^2x_7x_{10}x_{13}x_{14}x_{19}$. We can construct the triangle diagram for the pair and a graphical representation of this is shown below. Note that the numbers refer to the subscripts of the letters in the words and each of the horizontal lines corresponds to a v_i from the triangle diagram sequence. Now r can easily be read off from the diagram as $x_0x_1^2x_5x_6x_7x_9x_{10}x_{12}x_{13}^2x_{14}x_{16}x_{19}$.

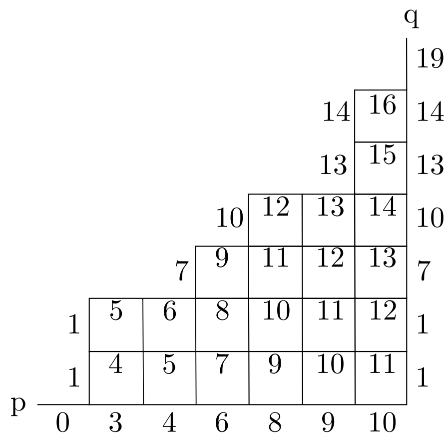

Fig. 3. Triangle diagram for p and q.

Next we want to show how this is covered by rectangular diagrams. Note that each of the suggestively drawn cells of the triangle diagram does in fact form a rectangular diagram, as the letter on the vertical edges always has a lower subscript than those on the bottom and the letter on the upper horizontal edge is just the shift of that on the bottom edge. For example, the first cell on the lower left is the rectangular diagram formed by x_3 and x_1 and its border is labelled by $x_3^{x_1}(\psi(x_3))^{-1}$. Guba shows that such a diagram can be covered by choosing subdiagrams recursively such that the fragments of r that are not on the edges of of the diagram have lengths at most $|r|/2$ and also that the difference between largest and least subscripts in such a diagram is not more than the range of subscripts in r.

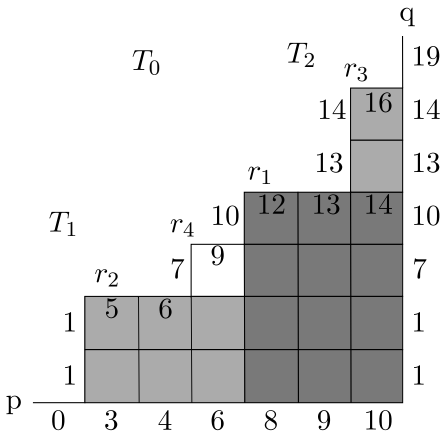

Fig. 4. Triangle diagram shaded to show rectangular subdiagrams.

To do this in our example we can first choose the rectangular subdiagram shaded dark grey. This has border label $(x_8x_9x_{10})^{x_1^2x_7x_{10}}(x_{12}x_{13}x_{14})^{-1}$ and its intersection wsectionith r is the word $r_1 = x_{10}x_{12}x_{13}$. Notice that the remaining parts of the original triangle diagram T_0 are also triangle diagrams which we will call T_1 and T_2. The hypotenuse of T_1 is $x_0x_1^2x_5x_6x_7x_9$ and that of T_2 is $x_{13}x_{14}x_{16}x_{19}$ both of the required length. These two triangle diagrams can be covered as well, T_1 by the light grey and white rectangle diagrams shown and T_2 by the single rectangle diagram corresponding to the subword r_3 of r. All of these diagrams have subscript ranges less than r. Hence if these can be shown to have quadratically bounded area then T_0 will also.

The remainder of the proof consists in demonstrating that rectangular diagrams have quadratically bounded area. This proceeds by breaking the problem into yet smaller pieces and considering Horizontal and Vertical subdiagrams of rectangular diagrams where p and q in the formulation of rectangle diagrams are assumed to consist of one letter in each respective case. The proofs in both cases are quite technical and aim to show that respective bounding cases exist within each diagram type.

All automatic groups have quadratic Dehn function but the converse is not known to be true. So this theorem neither proves nor disproves the automaticity of F. Its direct implication is that F has solvable word problem by theorem 3.1, however this was already known before this result and indeed the result from theorem 4.12 shows that the word problem for the finite presentation of F is linear.

6 Summary

We have seen that the class of automatic groups represents groups that are 'well-behaved' in a computational sense. In particular they have a regular language which maps onto the group, the fellow-traveller property and a quadratic word problem and quadratic Dehn function. Thompson's group F, in particular its finite presentation, has been shown to have some of these properties, yet shows signs of violating others. F has a word problem solvable in linear time and quadratic Dehn function which suggest that the equivalence class of representatives for a group element is not complicated. But it has an exponential growth function showing that the number of elements of any size grows extremely fast. Also the facts that the growth function appears to not be rational and that there can be no regular language containing all geodesics of F hints that there may not be a regular language for the finite presentation. If F eventually is proved to not be automatic it will nevertheless be a very interesting illustration of just where the boundary lies between automatic and non-automatic groups.

References

[1] J. M. Belk. Thompson's Group F. PhD thesis, Cornell University, 2004.

[2] J. M. Belk and K. Bux. Thompson's Group F is not Minimally Almost Convex. *eprint arXiv:math/0301141*, Publication Date: 01/2003.

[3] J. Burillo. Growth of Positive Words in Thompson's Group F. *Communications in Algebra*, vol. 32 no. 8, pp. 3087-3094, 2004.

[4] J. W. Cannon, W. J. Floyd and W. R. Parry. Introductory notes on Richard Thompson's groups. *www.math.binghampton.edu/matt/thompson/cfp.pdf* (ret. 05/2011), 2005.

[5] S. Cleary and J. Taback. Thompsons Group F is not Almost Convex. *Journal of Algebra* vol. 270, no.1, pp. 133149, 2003.

[6] S. Cleary, M. Elder and J. Taback. Cone types and geodesic languages for lamplighter groups and Thompson's group F, *Journal of Algebra* vol. 303 no. 2, pp. 476-500, 2006.

[7] M. Elder É. Fusy and A. Rechnitzer. Counting elements and geodesics in Thompson's group F. *Journal of Algebra*, vol. 324 no. 1, pp. 102-121, 2010.

[8] D. B. A. Epstein with J.W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Patterson, W.P. Thurston. *Word Processing in Groups*, Jones and Bartlett Publishers, 1992.

[9] S. B. Fordham. Minimal Length Elements of Thompsons Group F. *Geometriae Dedicata* vol. 99, no. 1, pp. 179220, 2003.

[10] S. B. Fordham and S. Cleary. Minimal length elements of Thompsons groups. *Geometriae Dedicata* vol. 141, no. 1, pp. 163-180, 2009.

[11] V. S. Guba. The Dehn function of Richard Thompson's group F is quadratic. *Inventiones Mathematicae*, vol. 163, no. 2, pp. 313-342, 2006.

[12] V. S. Guba. Polynomial upper bounds for the Dehn function of R. Thompson's group F. *J. Group Theory* 1, pp. 203-211, 1998.

[13] V. S. Guba and M. Sapir. Diagram Groups. *Memoirs of the American Math. Soc.*, vol. 130, no. 620, pp. 1-117, 1997.

[14] Pierre de la Harpe. *Topics in Geometric Group Theory.* University of Chicago Press, Chicago, 2000.