STATUS OF PROBLEMS FROM COMPUTABILITY AND RANDOMNESS BY NIES

ANDRE NIES

Abstract. This is the second update on the status of open problems in Computability and Randomness [17]. The first appeared in March 2011.

Contents

1. Chapter 3

3.4.15. *Determine whether the sequence of bits of Ω in even positions can be superlow for some optimal machine.*

Solved negatively by Frank Stephan and Noam Greenberg, Nov 2010. Thus, we have a natural example of a low, but not superlow set.

(1) Recall that $J(e) = \varphi_e(e)$ whenever this is defined; otherwise $J(e)$ is undefined.

(2) Let $B = \Omega(0)\Omega(2)\Omega(4)\ldots$. Define a function f such that

$$f(e) \in B' \text{ iff } J(e) \text{ is defined and } B(J(e)) = 1;$$

note that $B(J(e)) = \Omega(2J(e))$.

(3) Now assume that B' is wtt-reducible to Ω wit'h bound $g(n)$. There is a recursive sequence of numbers $a_0, a_1, \ldots$ such that for each n,

$$a_n + g(f(a_n)) < J(a_n) < a_{n+1}.$$

Now one could use the wtt-reduction to compute $\Omega(2J(a_n))$ from $\Omega(0)\Omega(1)...\Omega(2J(a_n)-1)$ and hence build a partial computable martingale which succeeds on Ω, contradiction.

It works in fact for any ML-random set Z wtt above Halting problem in place of Ω.

3.6.9. *To what extend does van Lambalgen's (3.4.6) hold for weak 2-randomness?* In [4, Corollary 2.1] it is shown that there is a weakly 2-random of the form

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$A \oplus B$ such that A is not weakly 2-random relative to B (and B is not weakly 2-random relative to A). Thus, one direction of van Lambalgens theorem does not hold for weak 2-randomness.

A somewhat weaker, answer was later found in Kautz's PhD thesis [15].

However, the other direction holds: if A is weakly 2-random relative to B and B is weakly 2-random, then $A \oplus B$ is weakly 2-random.

3.6.23. *Is there a characterization of weak 2-randomness via the growth of the initial segment complexity?*

A partial affirmative answer is given in [14].

2. Chapter 5

$G(b)$ is the number of sets that are K-trivial with constant b.

5.2.16. *It is not hard to verify that $G \leq_{\mathrm{T}} \emptyset^{(3)}$. Determine whether $G \leq_{\mathrm{T}} \emptyset^{(2)}$. (This may depend on the choice of an optimal machine.)*

Barmpalias and Sterkenburg [6] have shown that $G \leq_{\mathrm{T}} \emptyset^{(2)}$ for any machine. This problem also appears in [11, Section 10.1.4].

Let C be an index set for a class of c.e. sets, namely $e \in C \wedge W_e = W_i \rightarrow i \in C$. We say that C is *uniformly* Σ^0_3 if there is a Π^0_2 relation P such that $e \in C \leftrightarrow \exists b\, P(e,b)$ and there is an effective sequence $(e_n, b_n)_{n\in\mathbb{N}}$ such that $P(e_n, b_n)$ and $\forall e \in C\, \exists n\, W_e = W_{e_n}$. In other words, C is the closure, under having the same index, of a projection of a c.e. relation contained in P. For instance, let $P(e,b)$ be $\forall n\, \forall s\, \exists t > s\, [K_t(W_{e,t}\upharpoonright_n) \leq K_s(n) + b]$.

5.3.33. *Is every Σ^0_3 index set of a class of c.e. sets uniformly Σ^0_3?*

Answered negatively by Frank Stephan, Nov 2010. See the 2010 Logic blog.

5.5.8. *Are there c.e. Turing degrees $\mathbf{a}, \mathbf{b}$ such that the K-trivial degrees coincide with $[\mathbf{0}, \mathbf{a}] \cap [\mathbf{0}, \mathbf{b}]$?*

A negative answer was given in [3] based on results from [2].

5.5.19. *Characterize lowness for Demuth randomness.*

Downey and Ng [12] have shown that lowness for Demuth randomness implies being computably dominated. This is a partial answer. Using this, [7] have given a full characterization via a concept called BLR-traceability, which implies jump traceability defined in Section 8.4. They build a perfect Π^0_1 class of sets that are low for Demuth randomness.

3. Chapter 6

6.3.17. *Decide whether $\emptyset' \leq_{LR} C \Leftrightarrow \exists B \leq_{\mathrm{T}} C\, [\emptyset' \in \mathcal{K}(B)]$.*

6.3.17. *Is there a minimal pair of c.e. sets that are uniformly a.e. dominated?*

Downey and Greenberg are working on some results under the stronger hypothesis that $\emptyset'$ is sjt by each set.

4. Chapter 7

7.5.13. *Decide whether for each high (c.e.) set C there is a (left-c.e.) set $Z \equiv_T C$ such that Z is partial computably random.*

The answer is NO in the general case. A high set C can be jump traceable by Ex. 8.6.2, but a partial computably random Z cannot be jump traceable by Theorem 7.6.7. The c.e. case it is still open.

7.6.11. *Decide whether the weaker hypothesis suffices in Theorem 7.6.7 that $\forall r \in S\, [C(Z\upharpoonright_r) \ \leq b \log r]$ for some infinite computable set S and $b \in \mathbb{N}$.*

5. Chapter 8

8.1.13. *If $\{X\colon\ X \leq_{LR} B\}$ is countable, is B low for Ω?*

Barmpalias and Lewis [5] have given an affirmative answer.

8.2.14. *Decide whether each weakly 2-random set is array computable.*

Barmpalias, Downey and Ng in [4] have shown that not all weakly 2-random sets are array computable. In fact, for each function g, there is a weakly 2-random Z and a function $f \leq_T Z$ such that f is not dominated by g.

8.3.16. *Characterize the class* $\text{Low}(\mathsf{W2R}, \mathsf{SR})$. *In particular, determine whether it coincides with being c.e. traceable.*

This class was shown to coincide with $\text{Low}(\mathsf{MLR}, \mathsf{SR})$, that is, c.e. traceability, by Bienvenu and Miller (May 2010).

8.4.9. *Characterize the sets that are low for Ω and jump traceable (each K-trivial set is). Characterize the sets that are computably traceable and jump traceable.*

8.4.22. *Decide whether $A \leq_{CT} B \Leftrightarrow \mathsf{SR}^B \subseteq \mathsf{SR}^A$ for each A, B.*

Miyabe [16, Thm. 4] has given an affirmative answer.

8.4.28. *Is each strongly jump traceable set strongly superlow?*

Diamondstone et al. [10] have given an affirmative answer. Also see the upcoming BSL survey of Greenberg and Turetsky on strong jump traceability.

8.5.26. *Is the class Shigh of superhigh sets Σ^0_3? Is each strongly jump traceable c.e. set in $\mathsf{Shigh}^\Diamond$?*

The second question has been answered in the affirmative by Hirschfeldt, Greenberg and Nies [13]. In fact $\mathsf{Shigh}^\Diamond$ coincides with sjt on the c.e. sets.

8.6.4. *We ask whether no further implications hold in Fig. 8.2 on page 362.*
(i) Can a c.e. traceable set be LR-hard?
(ii) Can a set that bounds only GL_1 *sets be LR-hard?*
(iii) Can a set that is low for Ω be superhigh?

(i) has been answered in the negative by Barmpalias [1]. In fact, he shows that if $\emptyset'$ is c.e. traceable by A, then A is not array computable, and in particular, not c.e. traceable. This yields a new implication in the diagram Fig 8.1 from array computable to not $\geq_{LR} \emptyset'$.

8.6.5. *Decide whether the sets that are computably dominated and in* GL_1 *are closed downward under* $\leq_{\mathrm{T}}$.

6. Chapter 9

A *generalized* Π^1_1*-ML-test* is a sequence $(G_m)_{m\in\mathbb{N}}$ of uniformly Π^1_1 open sets such that $\bigcap_m G_m$ is a null class. Z is Π^1_1*-weakly 2-random* if Z passes each generalized Π^1_1-ML-test.

9.2.17. *Is* Π^1_1*-weak 2-randomness stronger than* Π^1_1*-ML-randomness?*

An affirmative answer has been given by Chong and Yu [9]. A somewhat simpler proof was subsequently provided by Bienvenu, Greenberg and Monin [8, Section 5.2]. Each proof shows that in fact higher Ω is not higher weakly 2-random.

9.4.11. *Is each set that is low for* Π^1_1*-randomness in* Δ^1_1*?* Greenberg and Monin (Higher randomness and genericity, Forum of Mathematics: Sigma, to appear) have answered the question in the affirmative. This relied on Monin's earlier work on calculating the Borel rank of the set of Π^1_1 random sequences.

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