The Continuity Problem, Once Again

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Let $\mathcal{T} = (T, \tau)$ be a topological space with countable basis

$$B_0, B_1, \ldots, B_n, \ldots$$

Definition

$F\colon T \to T'$ is effectively continuous, if there is some computable $g\colon \mathbb{N}^2 \to \mathbb{N}$ such that for all n,

$$F^{-1}(B'_n) = \bigcup \{\, B_{g(n,a)} \mid a \geq 0 \,\}.$$

Effective continuity corresponds to computability.

How can F be computed? Via oracle machines!

Oracle questions: $z \in B_n$?

Input: y

0. $n := 0$;
1. $i := 0$;
2. $y \in B_{g(n,i)}$?
   - If YES: Output B'_n; $n := n + 1$; Goto 1;
   - If NO: $i := i + 1$; Goto 2;

By this way a sequence of open sets B'_n is generated. If it is a base of the neighbourhood filter $\mathcal{N}(z)$ of some $z \in T'$, set

$$F(y) = z.$$

Otherwise, $F(y)$ is undefined.

Definition

y is computable if there is a computable $h\colon \mathbb{N} \to \mathbb{N}$ such that $\{\, B_{h(a)} \mid a \geq 0 \,\}$ is a base of $\mathcal{N}(y)$.

Set $T_c = \{\, y \in T \mid y \text{ computable} \,\}$.

Let i be the code of a program that computes h. Then i is an index of y.

This defines a (partial) indexing $x\colon \mathbb{N} \rightharpoonup T_c$ (onto).

Definition

$F\colon T_c \to T'_c$ is effective if there is some computable $f\colon \mathbb{N} \rightharpoonup \mathbb{N}$ such that

$$F(x_i) = x'_{f(i)}.$$

f transforms the program computing approximations of the input.

Example

Functional programming (LISP, SCHEME): The data are itself programs that are manipulated by a program.

Abstract setting.

$\mathcal{T}$: Countable T_0 space T with indexing x,
Topology τ with countable basis $\mathcal{B}$ and total indexing B of $\mathcal{B}$.

$\mathcal{T}'$: Similar

Continuity Problem Given $F\colon T \to T'$ effective.
Is F effectively continuous?

Special cases.

- $\mathcal{P} = \{ f : \mathbb{N} \rightharpoonup \mathbb{N} \mid f \text{ computable} \}$.

Topology generated by

$$B_g = \{ f \in \mathcal{P} \mid \operatorname{graph} g \subseteq \operatorname{graph} f \},$$

where

$g \in \mathrm{FINFCT} =$ set of partial functions with finite domain.

Note that functions in FINFCT can be coded. Moreover, they form an enumerable dense subset of $\mathcal{P}$.

Indexing of $\mathcal{P}$:

$f = \varphi_i$, iff i is the code of a Turing machine computing f.

Myhill/Shepherdson (1955) Effective operators on the partial computable functions are effectively continuous.

Structure of $\mathcal{P}$

1. $\mathcal{P}$ is partially ordered by

$$f \sqsubseteq h \iff \mathrm{graph}(f) \subseteq \mathrm{graph}(h)$$

2. Topology determines the order:

$$f \sqsubseteq h \iff (\forall g \in \mathrm{FINFCT})[f \in B_g \Rightarrow h \in B_g]$$

3. $\{\, g \in \mathrm{FINFCT} \mid g \sqsubseteq f \,\}$ is directed and enumerable such that

$$\mathrm{graph}(f) = \bigcup \{\, \mathrm{graph}(g) \subseteq \mathrm{graph}(f) \mid g \in \mathrm{FINFCT} \,\}$$

4. Each enumerable directed set of finite functions has a least upper bound in $\mathcal{P}$.

Properties (3) and (4) say that $(\mathcal{P}, \sqsubseteq)$ is a domain.

Example

- CE = set of all computably enumerable subsets of $\mathbb{N}$.
- CE is partially ordered by $\subseteq$
- $A = \bigcup \{ E \subseteq A \mid E \text{ finite} \}$, for all $A \in \mathrm{CE}$.

Generalizations of the Myhill/Shepherdson Theorem to domains:

- Egli/Constable 1976
- Sciore/Tang 1978
- Weihrauch 1980

Pros Domains are closed under the function space construction: lift the order argumentwise.

Cons So far no way to measure the speed of approximations, no complexity measure.

Note. Domains are in general *not* Hausdorff!

- $\mathcal{R} = \{ f : \mathbb{N} \to \mathbb{N} \mid f \text{ total, computable} \}$
- $\mathcal{R}$ is a metric space:

$$\delta(f, g) = \begin{cases} 0 & \text{if } f = g, \\ 2^{-\min\{ n \mid f(n) \neq g(n) \}} & \text{otherwise.} \end{cases}$$

- The topology is generated by

$$B(g, m) = \{ f \in \mathcal{R} \mid \delta(g, f) < 2^{-m} \},$$

  where g is 0 almost everywhere.

Functions like g can be coded: code the finite initial segment where, for some n, $g(n) \neq 0$. Moreover, they form an enumerable dense subset of $\mathcal{R}$.

Indexing of $\mathcal{R}$:

$f = \varphi_i$, iff i is the code of a Turing machine computing f.

Kreisel/Lacombe/Shoenfield (1959) Effective operators on the total computable functions are effectively continuous.

Generalizations to recursive metric spaces.
Recursive metric spaces come with

- an enumerable dense subset S,
- $\delta \colon S \times S \to \mathbb{R}_c$ is effective.

Here, $\mathbb{R}_c$ is the set of computable real numbers.

Ceĭtin (1962), Moschovakis (1964) Effective operators between recursive metric spaces are effectively continuous.

Pros Speed of approximation can be measured. Complexity can be studied.

Cons Not closed under higher function spaces: $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ can never have a countably based topology (Hyland (1979)).

Situation so far:

- Positive solutions for topologically very different types of spaces
  - Domains: T_0, non-Hausdorff
  - Metric spaces: Hausdorff
- Negative result:

  Friedberg (1958) Effective operators are not continuous in general.

Friedberg constructed a map $G\colon \mathcal{R} \to \mathbb{N}_\perp$ that is effective, but not continuous.
Here $\mathbb{N}_\perp$ is the flat domain of the natural numbers, i.e., the set $\mathbb{N} \cup \{\perp\}$ with the order

$$u \sqsubseteq v \iff u = \perp \vee u = v.$$

Thus. In order to derive a general theorem, a further condition is needed that is satisfied in the positive solution cases.

Spreen/Young (1983) Every effective operator between effectively given T_0 spaces that has a *witness for noninclusion* is effectively continuous.

Spreen (1998)

- Every effectively continuous operator is effective.
- Every effectively continuous operator has witness for noninclusion, if the topology is semi-regular, i.e., if $B_n = \mathrm{int}(\mathrm{cl}(B_n))$.

Definition

$F\colon T \to T'$ has a witness for noninclusion, if the following holds:

$$x_i \in F^{-1}(B'_n), \quad B_m \not\subseteq F^{-1}(B'_n) \qquad \Longrightarrow$$

a witness $z \in B_m \setminus F^{-1}(B'_n)$ can effectively be found, uniformly in i, m, n.

Special cases

- $F\colon \mathcal{P} \to \mathcal{P}$

1. Effective maps are monotone with respect to $\sqsubseteq$.
2. $F(B_g) \not\subseteq B'_f \implies g \in B_g \setminus F^{-1}(B'_f)$. g is the witness!

Generalization. $F\colon$ Domain $\to$ T_0 space

- $F\colon \mathcal{R} \to \mathcal{R}$

1. $\mathrm{ext}(B'_n)$ is enumerable, uniformly in n.
2. Search for $z \in B_m$ with $F(z) \in \mathrm{ext}(B'_n)$. z is the witness!

Generalization. $F\colon$ T_0 space $\to$ recursive metric space.

Note. Only the case of Friedberg's operator is not covered.

Problem The witness for noninclusion condition seems to have no topological meaning. So, what is its role in the result?

Aim To point out that the condition appears naturally when in a classical context the Axiom of Choice is used.

Definition

$F\colon T \to T'$ is

- effectively pointwise continuous, if there is some computable $h\colon \mathbb{N} \to \mathbb{N}$ such that

$$F(x_i) \in B'_n \Longrightarrow x_i \in B_{h(i,n)} \subseteq F^{-1}(B'_n).$$

- effectively sequentially continuous, if there is some computable $k\colon \mathbb{N}^2 \to \mathbb{N}$ such that, if m is an index of a computable sequence $(y_a)_a$ in T converging to $y \in T$ and $F(y) \in B'_n$, then

$$(\forall a \geq k(m,n))F(y_a) \in B'_n.$$

Lemma (Spreen 1998)

1. *If $F\colon T \to T'$ is effectively continuous, then it is also effectively pointwise continuous.*
2. *If T has an enumerable dense subset and $F\colon T \to T'$ is effective, then if F is effectively pointwise continuous it is also effectively continuous.*

Remaining steps.

1. F effectively sequentially continuous $\Rightarrow$ Feffectively pointwise continuous.
2. F effective $\Rightarrow$ F effectively sequentially continuous.

Step 1.

Let us have a look to the classical proof: Given $y \in T$,

- Construct a sequence of basic open sets

$$U_0 \supseteq U_1 \supseteq \cdots \ni y.$$

- Assume that F is *not* continuous. Then

$$(\exists V \in \mathcal{N}(F(y)))(\forall a) F(U_a) \not\subseteq V$$

- **Choose** $y_a \in U_a \setminus F^{-1}(V)$, for all a.
- It follows that $y_a \to y$ and hence that $F(y_a) \to F(y)$, as F is sequentially continuous.
- Thus, $(\exists N)(\forall a \geq N) F(y_a) \in V$.
- Contradiction!

In an effective setting we cannot *choose* y_a. Here, we use the witness for noninclusion condition!

Step 2. F effective $\Longrightarrow$ F effectively sequentially continuous.

General assumption:

- Given the index of a computable convergent sequence in T we can compute an index of its limit.
- B'_n is enumerable, uniformly in n.

Assume.

- F is effective.
- (y_a) computable and convergent with $F(\lim_a y_a) \in B'_n$.

We use the Kleene's recursion theorem and/or Rogers' fixed point theorem to construct a computable sequence (z_a). Note that the recursion theorem allows impredicative constructions, i.e., in the construction process we can use (z_a) as if it were already constructed. In particular, using our general assumption, we can compute its limit $\lim_a z_a$ (better: an index of it). And since F is effective, we can apply F to $\lim_a z_a$ (better: apply the index function coming with F to the computed index of $\lim_a z_a$), and search whether we find $F(\lim_a z_a)$ in B'_n.

Construction of (z_a)

1. Follow the sequence (y_a) as long as

$$F(\lim_a z_a)$$

   has **not** been found in B'_n, or (z_a) does not converge.

2. If (z_a) converges and $F(\lim_a z_a)$ has been found in B'_n, say in step N_0, repeat y_{N_0} as long as

$$F(y_{N_0})$$

   has **not** been found in B'_n.

3. If, in step N_1, $F(y_{N_0})$ has been found in B'_n, repeat y_{N_0+1} as long as

$$F(y_{N_0+1})$$

   has **not** been found in B'_n.

4. ...

Suppose 1. (z_a) does not converge, or $F(\lim_a z_a)$ will never be found in B'_n.

Then

$$z_a = y_a \quad (a \geq 0).$$

Hence,

$$z_a \to \lim_a y_a.$$

But

$$F(\lim_a y_a) \in B'_n.$$

Contradiction!

Thus N_0 exists and depends computably on n and the index m of (y_a).

Suppose 2. $F(y_{N_0})$ will never be found in B'_n.
Then

$$\lim_a z_a = y_{N_0}.$$

As we have just seen,

$F(\lim_a z_a)$ will be found in B'_n,

i.e. $F(y_{N_0})$ will be found in B'_n, contradiction!

Thus, N_1 exists and depends computably on m, n.

By induction it follows that for all $c \geq N_0$,

$$F(y_c) \in B'_n.$$

Thus, $(F(y_a))_a$ converges to $F(\lim_a y_a)$.

Note. We have to use the Markov Principle

$$\frac{\neg\neg(\exists n)P(n)}{(\exists n)P(n)} \quad (MP)$$

in our proofs.

As shown by Beeson (1975, 1976) and Beeson/Ščedrov (1984),

> The continuity of effective operators cannot be derived in intuitionistic systems without MP.