Infinite computations with random oracles

Philipp Schlicht, Universität Bonn
joint with Merlin Carl, Universität Konstanz

ARA 2014 Pretoria, 10 February 2014

Turing computations from many oracles

Theorem (Sacks)

Suppose that $x \in {}^{\omega}2$ is computable from all $y \in A$, where $A \subseteq {}^{\omega}2$ has positive measure. Then x is computable.

We consider the analogous problem for other models of computation.

The classical proof does not work for infinite time machines, since infinite time machines can read all bits of the input in a single computation.

A further problem is that except in the constructible universe L, we cannot enumerate all inputs $x \in {}^{\omega}2$.

Ordinal Turing machines (OTMs)

Koepke (2006): Consider a Turing program running on

- a tape of length *Ord* instead of the set ω of natural numbers
- in ordinal time.

The machine works as follows.

- The state is the *liminf* of the states at previous times.
- The head moves to the *liminf* of its previous positions.
- The contents of each cell is the *liminf* of the contents at previous times.

Example (Ordinal addition and multiplication)

1. $\alpha + \beta$ is defined by
   - $\alpha + (\beta + 1) = (\alpha + \beta) + 1$
   - $\alpha + \lambda = \sup_{\beta < \lambda} \alpha + \beta$ for limits λ
2. $\alpha \cdot \beta$ is defined by
   - $\alpha \cdot (\beta + 1) = (\alpha \cdot \beta) + \alpha$
   - $\alpha \cdot \lambda = \sup_{\beta < \lambda} \alpha \cdot \beta$ for limits λ

OTM computations

tape →

time ↓

	state	head	0	1	2	3	4	5	6	$\cdots$	ω	$\cdots$
0	0	0	0	0	0	0	0	0	0	$\cdots$	0	$\cdots$
1	1	1	1	0	0	0	0	0	0	$\cdots$	0	$\cdots$
2	0	2	1	0	0	0	0	0	0	$\cdots$	0	$\cdots$
3	1	3	1	0	1	0	0	0	0	$\cdots$	0	$\cdots$
4	0	4	1	0	1	0	0	0	0	$\cdots$	0	$\cdots$
5	1	5	1	0	1	0	0	0	0	$\cdots$	0	$\cdots$
6	0	6	1	0	1	0	1	0	0	$\cdots$	0	$\cdots$
7	1	7	1	0	1	0	1	0	0	$\cdots$	0	$\cdots$
8	0	8	1	0	1	0	1	0	1	$\cdots$	0	$\cdots$
9	1	9	1	0	1	0	1	0	1	$\cdots$	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
ω	0	ω	1	0	1	0	1	0	1	$\cdots$	0	$\cdots$
$\omega+1$	1	$\omega+1$	1	0	1	0	1	0	1	$\cdots$	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Infinite time Turing machines (ITTMs)

Hamkins-Kidder (2000): Consider the hardware of a Turing machine, but the time for a computation is *Ord*.

The machine works like an OTM, but if the *liminf* of the head position is ω, then the head moves to 0.

As for Turing machines, for OTMs and ITTMs, a typical program instruction $(a, s, a', s', d) \in \{0,1\} \times \omega \times \{0,1\} \times \omega \times \{-1,1\}$ is interpreted as follows:

"If the symbol currently read by the machine's read-write head is a and the machine is currently in state s, then overwrite a with the symbol a', change the machine state to s', and move the head according to d either to the left or to the right".

Computable sets

Definition

1. A real $x \in {}^{\omega}2$ is *OTM-computable* from a real $y \in {}^{\omega}2$ if there is an *OTM* P such that on input y, P halts with output x, i.e. $P^y = x$.
2. A set $A \subseteq {}^{\omega}2$ is *OTM-computable* from a real y if there is an *OTM* P such that for all $x \in {}^{\omega}2$, $x \in A$ if and only if P halts on input $x \oplus y$, i.e. $P^{x \oplus y} \downarrow$.

For infinite time Turing machines, there is a difference between writable (the program halts) and eventually writable (the program does not halt) sets of natural numbers.

The strength of infinite time machines

Definition

A set of inputs of natural numbers or of infinite sequences (reals) is recognizable if there is a program which halts for exactly these inputs.

ITTMs can recognize Π^1_1 and Σ^1_1 sets of reals (i.e. infinit 0-1-sequences) (Hamkins-Lewis).

OTMs can recognize exactly the Σ^1_2 sets of reals (Koepke-Seyfferth).

With OTMs with ordinal parameters, the constructible universe L can be computed (Koepke).

The constructible universe

The computable reals for various machines can be described by the levels of the constructible universe.

Definition

Let $L_0 = \emptyset$.

Let $L_{\alpha+1} = Def(L_\alpha, \in) = \{X \subseteq L_\alpha \mid X = \{x \in L_\alpha \mid (L_\alpha, \in) \models \varphi(x, a)\}$ for some $a \in L_\alpha$ and first order formula $\varphi\}$.

Let $L = \bigcup_{\alpha \in Ord} L_\alpha$.

Halting times

Definition

Let η^x denote the supremum of halting times of *OTMs* with oracle x.

Lemma

η is the least ordinal α such that some Σ_1 sentence is first true in L_α.

Lemma

The following conditions are equivalent for reals x, y.

1. *x is Δ^1_2 in y.*
2. *x is OTM-computable in the oracle y.*
3. *$x \in L_{\eta^y}[y]$.*

OTM computations in L

Theorem

Suppose that $V = L$. There is a real x and a co-countable set $A \subseteq {}^{\omega}2$ (in particular, comeager and of full measure) such that

1. *x is OTM-computable without ordinal parameters from every $y \in A$, but*
2. *x is not OTM-computable without parameters.*

Corollary

Assume that $V = L$.

1. *Let h be a real coding the halting problem for parameter-free OTMs. Then h is OTM-computable from every non-OTM-computable real x.*
2. *For all reals x and y, x is OTM-computable from y or y is OTM-computable from x.*

Cohen and random reals

Definition

Suppose that $M = L_\alpha$ and $x \in {}^\omega 2$.

1. x is Cohen over M if $x \in B$ for every comeager Borel set B with a Borel code in M.
2. x is random over M if $x \in B$ for every measure 1 Borel set B with a Borel code in M.

These notions are closely related to randomness in computability and to forcing in set theory.

OTM computations from many oracles

Theorem

1. *Suppose that for every $x \in {}^{\omega}2$, the set of random reals over $L[x]$ has measure 1. If A has positive Lebesgue measure and $x \in {}^{\omega}2$ is OTM-computable from every $y \in A$, then x is OTM-computable.*
2. *Suppose that for every $x \in {}^{\omega}2$, the set of Cohen reals over $L[x]$ is comeager. If A is nonmeager with the property of Baire and $x \in {}^{\omega}2$ is OTM-computable from every $y \in A$, then x is OTM-computable.*

Lemma

1. *The statement that for every real x, the set of random reals over $L[x]$ has measure 1 is equivalent to the statement that every Σ^1_2 set is Lebesgue measurable. This follows from $\omega_1^{L[x]} < \omega_1$ for all reals x.*
2. *The statement that for every real x, the set of Cohen reals over $L[x]$ is comeager is equivalent to the statement that every Σ^1_2 set hast the property of Baire.*

OTM computations with ordinal parameters

Lemma

A real x is OTM-computable from y with ordinal parameters if and only if $x \in L[y]$.

In L and in any model in which ${}^{\omega}2$ is not Lebesgue measurable, the problem is trivial.

The question is more interesting when the set of constructible reals has measure zero.

Theorem (Judah-Shelah)

There is a forcing $\mathbb{P}$ in L such that in any $\mathbb{P}$-generic extension of L, there is a measure one set A such that every $x \in A$ can be constructed from every $y \in A$, but A contains no constructible real.

Blass-Shelah forcing works, we constructed a much simpler forcing using an idea of Martin Goldstern.

Theorem

1. *Suppose that for every real x, there is a random real over $L[x]$. If A has positive measure and $x \in {}^{\omega}2$ is constructible from each $y \in A$, then $x \in L$.*
2. *Suppose that for every real x, there is a Cohen real over $L[x]$. If A is a nonmeager Borel set and $x \in {}^{\omega}2$ is constructible from each $y \in A$, then $x \in L$.*

Question

Is it consistent that there is a nonconstructible real x and a Borel set A of measure 1 such that x is OTM-computable without parameters from every $y \in A$?

All mentioned results also hold for nonmeager Borel sets of oracles instead of sets of oracles of positive measure.

ITTM halting times

Definition

Suppose that $x \in {}^{\omega}2$.

1. Let λ^x denote the supremum of ITTM-writable ordinals (the program halts) with oracle x.
2. Let ζ^x denote the supremum of eventually writable ordinals (the program does not necessarily halt)with oracle x.
3. Let Σ^x denote the supremum of ITTM-accidentally writable ordinals (The real appears at a random time)with oracle x.

Theorem (Welch)

For every real x, the reals writable (eventually writable, accidentally writable) in the oracle x are exactly those in $L_{\lambda^x}[x]$ ($L_{\zeta^x}[x]$, $L_{\Sigma^x}[x]$).

Theorem (Welch)

Suppose that x is a real. Then (ζ^x, Σ^x) is the lexically minimal pair of ordinals such that $L_{\zeta^x}[x] \prec_{\Sigma_2} L_{\Sigma^x}[x]$ and λ^x is minimal with $L_{\lambda^x}[x] \prec_{\Sigma_1} L_{\zeta^x}[x]$.

ITTM computations from many oracles

Lemma

Suppose that x is Cohen generic over $L_{\Sigma+1}$.

1. $L_\lambda[x] \prec_{\Sigma_1} L_\zeta[x] \prec_{\Sigma_2} L_\Sigma[x]$.
2. $\lambda^x = \lambda$, $\zeta^x = \zeta$ *and* $\Sigma^x = \Sigma$.

Proposition

Suppose that x is a real and that A is a comeager set of reals such that x is writable (eventually writable, accidentally writable) in every oracle $y \in A$. Then x is writable (eventually writable, accidentally writable).

Theorem

Suppose that x is a real and that A is a nonmeager Borel set of reals such that x is writable (eventually writable, accidentally writable) in every oracle $y \in A$. Then x is writable (eventually writable, accidentally writable).

Infinite time register machines (ITRMs)

An infinite time register machine works on finitely many register in ordinal time.

Theorem

Suppose that x is a real and A is a set of positive measure such that x is ITRM-computable from all $y \in A$. Then x is ITRM-computable.

α-Turing machines

The running time of an α-Turing machine is bounded by α.

Definition

Let $\bar{\alpha} = \omega_{\bar{\iota}}$ denote the least admissible ordinal γ such that L_{γ^+} does not contain a real coding γ, where γ^+ denotes the least admissible ordinal $> \gamma$.

Theorem

Suppose that $\alpha = \omega_\iota^{CK} < \bar{\alpha}$ is admissible, x is a real, A is a set of positive measure, and P is an α-Turing program such that $P^y = x$ for all $y \in A$. Then x is α-computable.

Theorem

There unboundedly many countable admissible ordinals α such that every real x which is α-computable from all elements of a set A of positive measure is α-computable.

Question

Which questions and results in algorithmic randomness should be studied for infinite time machines?