Point degree spectra of represented spaces[1]

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[1] Joint work with Takayuki Kihara.

Outline

Two questions

Introducing point degree spectra

The spectra

Applications

Names of real numbers

- There is no admissible representation of $\mathbb{R}$ with unique names
- But any real number has a well-defined Turing degree (equal to its decimal expansion)
- **Proof:** Make a case distinction $x \in \mathbb{Q}$ vs $x \in \mathbb{R} \setminus \mathbb{Q}$
- **Question** (Pour-El & Richards): Does every point in a computable metric space have a Turing degree[2]?

[2] I.e. a name computable from all its other names

An answer and a new question

Theorem (Miller 2004)

Some elements of $[0,1]^{\mathbb{N}}$ *do not have Turing degrees.*

Question (Brattka & Miller)

For which computable Polish spaces do all points have Turing degrees?

Countable isomorphisms and the second question

Definition

Let $\cong$ denote computable isomorphic. Say $\mathbf{X} \cong_{\sigma d} \mathbf{Y}$, iff $\exists (\mathbf{X}_i)_{i\in\mathbb{N}}, (\mathbf{Y}_i)_{i\in\mathbb{N}}$ s.t. $\mathbf{X} = \bigcup_{i\in\mathbb{N}} \mathbf{X}_i$ and $\mathbf{Y} = \bigcup_{i\in\mathbb{N}} \mathbf{Y}_i$ and $\forall i \in \mathbb{N}\ \mathbf{X}_i \cong \mathbf{Y}_i$.

By ^{c} the relativized version is distinguished.

Question (Motto-Ros, Schlicht & Selivanov)

Are there more $\cong^c_{\sigma d}$-equivalence classes of Polish spaces than $\mathbb{N}, \{0,1\}^{\mathbb{N}}, [0,1]^{\mathbb{N}}$?

Talking about computability

Definition

A *represented space* **X** is a pair (X, δ_X) where X is a set and $\delta_X :\subseteq \mathbb{N}^\mathbb{N} \to X$ a surjective partial function.

Definition

$f :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is a realizer of $F :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$, iff $(\delta_X(p), \delta_Y(f(p))) \in F$ for all $p \in \delta_X^{-1}(\mathrm{dom}(F))$.

$$\begin{array}{ccc} \mathbb{N}^\mathbb{N} & \xrightarrow{f} & \mathbb{N}^\mathbb{N} \\ \downarrow{\scriptstyle \delta_A} & & \downarrow{\scriptstyle \delta_B} \\ \mathbf{X} & \xrightarrow{F} & \mathbf{Y} \end{array}$$

Definition

$F :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ is called computable (continuous), iff it has a computable (continuous) realizer.

Turing and Medvedev degrees

Definition

Given $p, q \in \mathbb{N}^{\mathbb{N}}$, say $p \leq_T q$ iff $\exists F$ computable s.t. $F(q) = p$. Let $\mathfrak{T}$ be the partially ordered set of $\leq_T$ equivalence classes.

Definition

Given $A, B \subseteq \mathbb{N}^{\mathbb{N}}$, say $A \leq_M B$ iff $\exists F : B \to A$ computable. Let $\mathfrak{M}$ be the partially ordered set of $\leq_M$ equivalence classes.

We understand $\mathfrak{T} \subset \mathfrak{M}$.

The definition of point degree spectra

Definition

Given a represented space $\mathbf{X} = (X, \delta_{\mathbf{X}})$, define:

$$\mathrm{Spec}(\mathbf{X}) := \{\delta_{\mathbf{X}}^{-1}(\{x\})/\equiv_M \mid x \in X\} \subseteq \mathfrak{M}$$

Theorem

$\mathbf{X} \cong_{\sigma d} \mathbf{Y}$ *iff* $Spec(\mathbf{X}) = Spec(\mathbf{Y})$.

Dimension theory enters the fray

Definition

Let $\dim \emptyset = -1$ and

$$\dim(\mathbf{X}) = \inf\{\alpha \mid \forall U \in \mathcal{T}\ \forall x \in U\, \exists V \in \mathcal{T}\ x \in V \subseteq U \wedge \dim(\delta V) < \alpha\}$$

We set $\inf \emptyset = \infty$, and understand $\alpha < \infty$ for any ordinal α.

Theorem (e.g. Hurewicz & Wallmann)

For a Polish space $\mathbf{X}$ *the following are equivalent:*

1. $\dim(\mathbf{X}) < \infty$
2. $\mathbf{X} \cong^{c}_{\sigma d} A$ *for some* $A \subseteq \mathbb{N}^{\mathbb{N}}$.

Corollary

For a Polish space $\mathbf{X}$ *the following are equivalent:*

1. $\dim(\mathbf{X}) < \infty$
2. $\exists p \in \mathfrak{T}\ p \times \mathit{Spec}(\mathbf{X}) \subseteq \mathfrak{T}$

The continuous degrees

Definition (Miller 2004)

Define $\mathfrak{C} := \mathrm{Spec}([0,1]^{\mathbb{N}})$.

For a closed set $A \subseteq \{0,1\}^{\mathbb{N}}$, let $T(A) \subseteq \{0,1\}^{\mathbb{N}}$ be the set of codes of trees for A.

Theorem

$A \in \mathfrak{C}$ iff $\exists B \in \mathcal{A}(\{0,1\}^{\mathbb{N}})$ such that $A \equiv_M B \equiv_M T(B)$.

The enumeration degrees

Recall that $\delta_{\mathcal{O}} : \mathbb{N}^{\mathbb{N}} \to \mathcal{O}(\mathbb{N})$ defined via $n \in \delta_{\mathcal{O}}(p) \Leftrightarrow \exists i\ p(i) = n + 1$ is an admissible representation.

Definition

$\mathfrak{E} := \mathrm{Spec}(\mathcal{O}(\mathbb{N}))$

Theorem (Miller 2004)

$\mathfrak{C} \subsetneq \mathfrak{E}$

The fourth Polish space

Theorem

There is a Polish space $\mathbf{P}$ *with* $\mathfrak{T} \subsetneq Spec(\mathbf{P}) \subsetneq \mathfrak{C}$*.*

This answers the question by Motto-Ros, Schlicht & Selivanov in the affirmative.

A degree structure $\mathfrak{A}$?

Question

Define $\mathfrak{A} := \bigcup_{\mathbf{X}\text{admissible}} Spec(\mathbf{X})$. *Is* $\mathfrak{E} \subsetneq \mathfrak{A} \subsetneq \mathfrak{M}$*?*

Question

Is there some admissible $\mathbf{X}$ *with* $Spec(\mathbf{X}) = \mathfrak{A}$*.*

Probabilistic computability

Definition (Brattka, Gherardi & Hölzl 2013)

We call $f : \mathbf{X} \to \mathbf{Y}$ probabilistically computable, iff there is a computable $F :\subseteq \mathbf{X} \times \{0,1\}^{\mathbb{N}} \to \mathbf{Y}$ s.t.
$\forall x \in \mathbf{X}\ \lambda(\{p \in \{0,1\}^{\mathbb{N}} \mid F(x,p) = f(x)\}) > 0$.

Proposition (Brattka, Gherardi & Hölzl 2013)

Let $f : \mathbf{X} \to \mathbf{Y}$ be probabilistically computable and $\mathcal{S}pec(\mathbf{Y}) \subseteq \mathfrak{T}$. Then f is non-uniformly computable.

Proof.

use Theorem of Sacks □

Improving the result

Proposition

Let $f : \mathbf{X} \to \mathbf{Y}$ be probabilistically computable and $\mathit{Spec}(\mathbf{Y}) \subseteq \mathfrak{E}$. Then f is non-uniformly computable.

Proof.

use Theorem of Leeuw-Moore-Shannon-Shapiro $\square$

Shore Slaman Join Theorem

Let $J : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be the Turing jump. Then $\mathfrak{L}_{k,n}$ is defined as $(J^{-1})^{\circ k} \circ \lim^{\circ k+n}$.

Theorem

Let $\mathcal{S}pec(\mathbf{Y}) \subseteq \mathfrak{T}$ and $f : \mathbf{X} \to \mathbf{Y}$ be single-valued. Then if $(\mathrm{id} \times f) \leq_{sW} \mathfrak{L}_{k,n}$, then $\underline{f} : \mathbf{X} \to \mathbf{Y}^{(n)}$ is non-uniformly computable.