Kolmogorov complexity and harmonic analysis - ARA 2014

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It is important to add here that order is not to be identified with predictability. Predictability is a property of a special kind of order such that a few steps determine the whole order (... as in curves of low degree) but there can be complex and subtle orders which are not in essence related to predictability
(... a good painting is highly ordered, and yet this order does not permit one part to be predicted from another). (David Bohm)

In 1869 Heine proposed to Cantor the problem of determining whether or not every trigonometric series $\sum_{n\in\mathbb{Z}} c(n)e^{inx}$ that converges to 0 at all real numbers x will have all its coefficients $c(n)$ necessarily equal to 0; or equivalently, whether two trigonometric series which converge to the same limit for every real number x are (formally) equal in the sense that their coefficients are the same. In 1870, by making extensive use of Riemann's work in his Habilitationschrift (1854) on trigonometric series, Cantor proved that the answer to Heine's problem was in the affirmative.

Eventually Cantor showed that the same holds true if a trigonometric series converges everywhere with the possible exception of a closed countable set.

This is arguably the first example of a mathematical argument that is based on a transfinite induction.

These works of Riemann and Cantor pioneered a very highly developed theory of so-called sets of uniqueness in harmonic analysis with wide-ranging implications for number theory (Diophantine approximation) and descriptive set theory

(Lusin, Menshov, Salem, Zygmund, Meyer, Solovay, Kaufman, Kechris, Louveau ...)

A set $E \subset [0,1]$ is a *set of uniqueness* if every trigonometric series which converges to 0 for $x \in [0,1] \setminus E$ is identically 0, or equivalently, if any two trigonometric series that converge to the same limit for every real $x \in [0,1] \setminus E$ are identical. (Intuitively, this means that $[0,1] \setminus E$ sufficiently "large" to ensure that if the two series already agree pointwise on it, then they are the same on $[0,1]$ (they have the same coefficients).) A set which is not a set of uniqueness is called a *set of multiplicity*. More specifically, a subset $M \subset [0,1]$ is a set of multiplicity iff there exist two (formally) distinct trigonometric series that converge to the same limit outside M.

For the purpose of this talk, it is important to note that a compact subset E of the unit interval is a set of multiplicity if and only if there is a distribution T (in the sense of Schwartz) being supported by E such that its Fourier transform is such that $\hat{T}(u) \to 0$ as $|u| \to \infty$.

The following question was posed by Beurling and solved in the affirmative by Salem in 1950:

> *Given a number $\gamma \in (0,1)$, does there exist a closed set on the line whose Hausdorff dimension is γ and that carries a non-zero Radon measure μ whose Fourier transform $\hat{\mu}(u) = \int_{\mathbf{R}} e^{iux} d\mu(x)$ is dominated by a constant times $|u|^{-\gamma/2}$ as $|u| \to \infty$?*

Theorem 1 *(F, Davie, Mukeru 2013) Write $(0,1)_r$ for the set of real computable numbers in the unit interval and KC for the set of infinite binary numbers which are random in the sense of Kolmogorov-Chaitin-Levin-Martin-Löf. There is a Π^0_2 predicate over $\mathbf{R} \times (0,1)_r \times KC$ which for each $(\gamma, \omega) \in (0,1)_r \times KC$, defines a Salem set $S_\gamma(\omega)$ of Fourier dimension γ.*

We show that such sets can be constructed by looking at Cantor type ternary sets E with computable ratios ξ and then to consider the image of E under an algorithmically random Brownian motion. Along these lines we find, for every infinite binary string ω, which is algorithmically random, and every computable real γ in the unit interval, a $\Pi_2^0(\omega, \gamma)$ compact set $S_\gamma(\omega)$ which is a Salem set of Hausdorff dimension γ. We emphasise that these sets are uniformly definable in γ and ω.

Such sets are instances of what is now called *Salem sets*. It is well-known that given a compact subset E of $[0,1]$ with Hausdorff dimension $\gamma \in (0,1)$, the number $\gamma/2$ is critical for Beurling's question to have an affirmative answer since any Radon measure μ supported by E is such that the function $u \mapsto |u|^{\alpha}\hat{\mu}(u)$ is not bounded for any $\alpha > \gamma/2$. Note that a set meeting with Beurling's condition will necessarily be a set of multiplicity.

Salem proved that the answer to Beurling's question is in the affirmative by constructing for every γ in the unit interval, a *random* Radon measure μ (over a convenient probability space) whose support has Hausdorff dimension γ and which satisfies his requirement with probability one.

In fact we show that each $S_{\gamma}(\omega)$ carries a Frostman measure μ which is the pushout of a computable Frostman measure under a complex oscillation canonically associated with ω satisfying

$$|\hat{\mu}(u)|^2 \ll |u|^{-\gamma}$$

as $|u| \to \infty$.

Hence the Fourier dimension of $S_{\gamma}(\omega)$ is γ.

Theorem 2 *(F, Davie, Mukeru) Write* $(0,1)_r$ *for the set of real computable numbers in the unit interval and KC for the set of infinite binary numbers which are random in the sense of Kolmogorov-Chaitin-Levin-Martin-Löf. There is a* Π^0_2 *predicate over* $\mathbf{R} \times (0,1)_r \times KC$ *which for each* $(\gamma, \omega) \in (0,1)_r \times KC$*, defines a Salem set* $S_\gamma(\omega)$ *of Fourier dimension* γ*.*

Theorem 3 *(F 2012) If x is a complex oscillation and r is a real number then*

$$\forall_{\ell}\exists_{n}\exists_{t_1,\ldots,t_6\in[0,1]\cap\mathbb{Q}}\left[\ |n((t_1+t_2+t_3)-(t_4+t_5+t_6))-r|<\frac{1}{\ell}\ \right]\wedge$$
$$\forall_{1\leq i\leq 6}\ \ |x(t_i)|<\frac{1}{\ell}.$$

Almost sure version:

Theorem 4 *For a continuous version X of Brownian motion over the unit interval, we have, almost surely,*

$$\mathbb{R}=\bigcup_{n=1}^{\infty}n(Y_X-Y_X),$$

where

$$Y_X=Z_X+Z_X+Z_X,$$

and Z_X is the zero set of X. Moreover, almost surely, for any finite set A of real numbers, the set Y_X will contain an affine (rescaled and translated) copy of A.

Set

$$E = \left\{ \frac{1}{2} + \sum_{k=2}^{\infty} \epsilon_k \frac{1}{2^{k^2}} : \epsilon_k \in \{-1, 1\} \text{ for all } k \right\}. \tag{1}$$

Theorem 5 *If x is a complex oscillation then the elements of the image $x(E)$ of the set E under x will be linearly independent over the field of rational numbers.*

Is $x(E)$ a set of multiplicity???