INTUNDLA LODGE RETREAT: OPEN PROBLEMS

1. De Beer

Background: This question concerns the speed of convergence of Birkhoff's theorem.

Let $\overline{X} := (X, \Sigma, \mu)$ be a standard Borel space and let T be a measure-preserving automorphism of Σ such that the dynamical system $(\overline{X}, T)$ is ergodic (i.e. there are no non-trivial invariant Borel subsets). Birkhoff tells us that if $f \in L^1(X, \mu)$,

$$|\frac{1}{n}\sum_{i=1}^{n} f(T^i x) - \int f \; d\mu| \to 0 \tag{1}$$

for μ-a.e. $x \in X$. Now as shown in the paper by del Junco & Rosenblatt (as one example out of several), this convergence can happen arbitrarily slowly.

I am interested in the positive direction. The main reasons for this are that there are direct applications in number theory, as well as applications to the problem of finding new invariants to distinguish between non-isomorphic dynamical systems.

There are two distinct variants of the question. Let $(\overline{X}, T)$ be a fixed dynamical system and (a_n) some monotone decreasing sequence of positive reals with limit 0.

Question 1.1. Find a dense subspace $R \subset L^1(X, \mu)$ for which (**??**) converges to 0 faster than (a_n) for μ-a.e. $x \in X$ and $f \in R$.

Question 1.2.

Given some $f \in L^1(X, \mu)$, consider the sets $A \subseteq X$ for which (**??**) converges to 0 faster than (a_n) for $x \in A$. Is it possible to describe some such A in terms of f?

2. Davie

Question 2.1. Suppose in the setting of Birkhoff's theorem, we have a computable operator T and a computable function. What is the speed of convergence when starting from a Martin-Löf- random point?

3. Freer

Background: Recall that a homogeneous structure is *highly homogeneous* when its age has one structure of each finite size, up to isomorphism. For example, $(\mathbf{Q}, <)$ is highly homogeneous, because although there are $n!$-many linear orderings on $[n]$, they are all isomorphic.

Question 3.1. Is there a proof that high homogeneity of a structure implies that it has trivial definable closure (equivalently, that its age has the Strong Amalgamation Property) without using Peter Cameron's 1976 characterization of the 5 reducts of $(\mathbf{Q}, <)$?

The following proof idea of Arno is a good start, and illustrates the key difficulty:

Suppose $\overline{a}b$ witnesses the nontrivial definable closure of a structure $\mathcal{M}$, i.e., every automorphism of $\mathcal{M}$ fixing the tuple $\overline{a}$ pointwise also fixes an element $b \notin \overline{a}$. Let $c \notin \overline{a}b$ be any other element of $\mathcal{M}$. Because no automorphism that fixes $\overline{a}$ pointwise moves b to c, we have $\mathrm{tp}(c/\overline{a}) \neq \mathrm{tp}(b/\overline{a})$. In particular, $\overline{a}b$ and $\overline{a}c$ are nonisomorphic tuples of the same size.

The problem is as follows: While $\overline{a}b$ and $\overline{a}c$ are nonisomorphic *as tuples in that order*, there may be some isomorphism as unordered tuples, e.g., a_0a_1b and a_0ca_1 may be isomorphic ordered tuples. In fact, even this can't happen, but the only proof we know is either much more complicated combinatorially or else makes use of Cameron and Ackerman–Freer–Patel. The goal is to find a simple combinatorial argument that no permutation of $\overline{a}c$ can be isomorphic to $\overline{a}b$.

Question 3.2. Suppose the isomorphism class in Str_L of a countable structure in a countable language L admits continuum-many ergodic probability measures that are S_∞-invariant (with respect to the logic action). How many such ergodic invariant measures come from fundamentally different Borel L-structures, in the sense that they could not be obtained as random substructures of the same Borel structure with points chosen m-i.i.d. for different nondegenerate continuous probability measures m (on the domain of the Borel structure)?

(Note that not all ergodic invariant measures must arise in this way, by sampling substructures of a Borel L-structure, though when there are continuum-many ergodic invariant measures there are continuum-many of this form.)

4. Mukeru

Background: Fractional Brownian motion is an interesting stochastic process with many applications in engineering, telecommunications, financial mathematics,... Compared to the classical Brownian motion process, very little is known about its properties.

Question 4.1. Assume that $X = \{X(t) : t \geq 0\}$ is a fractional Brownian motion of Hurst index $0 < H < 1$. It is well known that the maximum function $M(t) = \sup\{X(s) : s \leq t\}$ has the same distribution as $|X(t)|$ for $H = 1/2$ (that is, when X is the classical Brownian motion). Nothing is known for the general case $H \neq 1/2$. Even the mean $E[M(t)]$ is not known. Same problem for the local times of fractional Brownian motion. This is an interesting open problem with significant impact on various applications of fractional Brownian motion.

Question 4.2. It has been obtained by different authors that some almost surely properties of Brownian motion are reflected in *every* complex oscillations (algorithmically random Brownian motion). For many other properties, the problem remains open. For example, "does every complex oscillation admit local times?"

5. Nies

We say that $Z \in 2^{\mathbb{N}}$ is density random if Z is ML-random and every Π^0_1 class $\mathcal{P}$ with $\mathcal{P} \ni Z$ has Lebesgue density 1 at Z. This is the same as: every left-r.e. Martingale converges along Z.

z is a weak Birkhoff point of T if $\lim_N \sum_{i<N} f \circ T^i(z)$ exists.

Question 5.1. Let T be a computable measure-preserving transformation on Cantor space. Let f be lower semicomputable. Is every density random a weak Birkhoff point of f?

If $f = 1_U$ for an effectively open U then ML-randomness of z suffices.

A metric measure space is a Polish metric space (M, d) with a Borel probability measure μ.

Question 5.2. Understand Gromov's obvious Lemma (2006 Book, 3 1/2 .6): given a m.m. space M. Let (x_i) be a sequence in M. Then either (x_i) has a convergent subsequence, or there is $\delta > 0$ such that $\mu B_\delta(x_i) \to 0$.

Idea: The second alternative should say that there is some $\delta > 0$ with $\liminf_i \mu B_\delta(x_i) = 0$. For a counterexample, suppose that $M = \bigsqcup_n X_n$ where each X_n is noncompact with (e.g.) diameter 2^{-n}, $d(x, y) = 2$ for all $x \in X_m$, $y \in X_n$ with $m \neq n$, and (e.g.) $\mu(X_n) = 2^{-n}$. Suppose that $h\colon \omega \times \omega \to \omega$ is a bijection and choose (x_i) such that $(x_{h(n,j)})_j$ is a discrete sequence in X_n for all n.

The proof of the weaker claim is straightforward. Assuming that the second alternative fails. We first claim that for every infinite $I \subseteq \omega$ and every $\delta > 0$, there is an infinite $I' \subseteq I$ of diameter $\leq \delta$. To see this, suppose that $\delta > 0$ and let $f(i, j) = 0$ for $i \neq j$ in I if $B_\delta(x_i) \cap B_\delta(x_j) = \emptyset$ and $f(i, j) = 1$ otherwise. By Ramsey's theorem, there is an infinite f-homogeneous $I' \subseteq I$. If f is constant 0, then $\mu(M)$ is infinite, contradicting the assumption. So f is constant 1 on I'.

Using this, we could try to inductively construct a sequence $I_0 \supseteq I_1 \supseteq$ of infinite sets with diameters $\leq \frac{1}{n}$ and $i_n := \min I_n < \min I_{n+1}$ for all n. Then $(x_{i_n})_n$ is a Cauchy subsequence of (x_i). Detail needed.

Gromov/Vershik have shown that measure preserving isometry of m.m. spaces is smooth, i.e., Borel reducible to identity on $\mathbb{R}$.

Question 5.3. Suppose we are given computable metric measure spaces M_0, M_1 that are measure preserving isometric. Show that there is a low measure preserving isometry.

Note: Melnikov and Nies have an example of such M_0, M_1 where every measure -preserving isometry is PA complete.

6. Schlicht

Background: The Gromov-Hausdorff distance $d_{GH}(X, Y)$ of two Polish spaces X, Y is defined as the infimum over the Hausdorff distances of the images of X, Y under isometric embeddings into arbitrary metric spaces Z. Two Polish spaces X, Y are E_{GH}-equivalent if $d_{GH}(X, Y) = 0$.

Question 6.1. Does E_1 (equality up to finite error on sequences of reals) Borel reduce to E_{GH} on Polish spaces?

Question 6.2. Is there an E_{GH}-equivalence class of Polish spaces which contains exactly n equivalence classes for some $n \geq 2$?

Question 6.3. In the constructible universe L, is there a Σ^1_2 or even Π^1_1 counterexample to Frostman's Lemma?

Answer: A Π^1_1 counterexample can be constructed in L using the Spector-Gandy theorem.

Question 6.4. Assuming projective determinacy, does Frostman's Lemma hold for all projective sets?

Question 6.5. Suppose that F is the Fraisse limit of a Ramsey class of not necessarily rigid finite structures. Is $Aut(F)$ amenable?

Note: It might follow from results mentioned in Cameron Freer's talk that $Aut(F)$ does not have to be amenable.

7. Pauly

Question 7.1. Given some computable $\mathrm{Aut}(\mathbf{Q})$-flow on a computably compact computable metric space, how hard is it to find a fixed point? Can we ensure PA complete?

Note that there is always a low one.

Question 7.2. Consider the maps $\dim_{\mathcal{H}} : \mathcal{A}([0,1]) \to [0,1]$ and $\mathrm{FractionalDensity} : 2^{\mathbb{N}} \to [0,1]$. One can quite easily show $\lim \leq_W \dim_{\mathcal{H}} \leq_W \lim\star\lim$ and $\lim \leq_W \mathrm{FractionalDensity} \leq_W \lim\star\lim$. But what are the precise classifications? Paul Potgieter's (2013 paper on arithmetic progressions, involving nonstandard analysis) construction may show $\mathrm{FractionalDensity} \leq_W \dim_{\mathcal{H}}$.

Question 7.3. Fix some dynamical system. Consider the (partial, multi-valued) map R mapping a function f and a starting point to the rate of convergence in Birkhoff's theorem. How hard is R? There are examples where the rate of convergence may be arbitrarily bad, but R is continuous, and examples where $\mathrm{C}_{\mathbb{N}} \times d_{\mathrm{MLR}}$ is reducible to R.

8. Fouché

If G is a topological group we write $C_{ub}(G)$ for the C^*-algebra consisting of the uniformly continuous bounded complex-valued functions on G. We denote by Γ the Gelfand functor on C^*-algebras.

A topological group is non-archimedean if its unit element has a system of neighbourhoods consisting of open subgroups of the group.

If D is a locally compact space, we write $\beta(D)$ for its Stone-Cech compactification. Note that if D is discrete, then $\beta(D)$ can be thought of as the space of ultrafilters on D, or as $\Gamma(\ell^\infty(D))$, the space of characters on the C^*-algebra $\ell^\infty(D)$.

If H is a subgroup of G we write $H\backslash G$ for the space of right-cosets $H\sigma$. If H is open in G then the strongest topology rendering the natural map $G \to (H\backslash G)$ continuous, is discrete.

If G is non-archimedean, it can be shown that

$$\Gamma C_{ub}(G) \approx \varprojlim_{H \leq_o G} \beta(H\backslash G),$$

where the inverse limit is over all the open subgroups of G.

Question 8.1. Let $G = \mathrm{Aut}(\mathbf{Q})$. Prove, without invoking Ramsey's theorem (finite version), the existence of a G-invariant element on $\Gamma C_{ub}(G)$. More generally, give a topological proof of the classical Ramsey theorem.

Question 8.2. Find an explicit embedding of the universal minimal S_∞-flow $M(S_\infty)$ into $\Gamma C_{ub}(S_\infty)$. Note that one can identify $M(S_\infty)$ with the space LO of linear orders on $\mathbb{N}$ topologised via its embedding into $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$.

Question 8.3. Let $\mathcal{L}$ be the class of finite structures of the form $(L,<)$, where L is a finite lattice and $<$ is a linear extension of L. Write $\mathbb{L}$ for the Fraïssé-limit of $\mathcal{L}$. Is the automorphism group of $\mathbb{L}$ extremely amenable? Or equivalently, is $\mathcal{L}$ Ramsey? My conjecture is that the Ramsey degree of a finite lattice L is given by

$$\frac{e(L)}{|\mathrm{Aut}(L)|},$$

where $e(L)$ is the number of linear extensions of L.

Let $\mathcal{L}_0$ be the class of finite lattices and $\mathbb{L}_0$ its Fraïssé limit. Is $\mathrm{Aut}(\mathbb{L}_0)$ amenable?

Question 8.4. Calculate the Ramsey degrees of finite graphs, for example, in a purely dynamical-topological way by using the Kechris-Pestov-Todorcevic (KPT) dynamical interpretation of Ramsey degrees. (This problem is due to KPT, but deserves repeating.)

Question 8.5. Determine the Weihrauch complexity of identifying the Fourier dimension of a compact subset of the reals.

The definition of the Fourier dimension of a compact set E of reals denoted by $\dim_f(E)$ is as follows:

$$\dim_f(E) = \sup\{\alpha : \exists_{\mu \in M^+(E)}\ |\hat{\mu}|^2(u) \ll |u|^{-\alpha}\}$$

Also recall that Hausdorff dimension $\dim_h$ has the following Fourier interpretion:

$$\dim_h(E) = \sup\{\alpha : \exists_{\mu \in M^+(E)} \int_{\mathbb{R}} |\hat{\mu}(u)|^2 |u|^\alpha \frac{du}{|u|} < \infty\}.$$