Coarse Reducibility and Randomness

Radboud University Nijmegen

Rutger Kuyper

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Joint work with D. Hirschfeldt, C. G. Jockusch, Jr. and P. Schupp.

Asymptotic density

Definition. Let $A \subseteq \omega$. Then we define:

$$\overline{\rho}(A) = \limsup_n \frac{|A \cap [0, n)|}{n}$$
$$\underline{\rho}(A) = \liminf \frac{|A \cap [0, n)|}{n}.$$

If $\overline{\rho}(A) = \underline{\rho}(A)$, then $\rho(A) = \overline{\rho}(A)$.

Coarse reducibility

Definition. A *coarse description* of A is a set $D \subseteq \omega$ such that $\rho(A \triangle D) = 0$.

Definition. We say that $A \subseteq \omega$ is *coarsely computable* if A has a computable coarse description.

Example. (Jockusch–Schupp) For every finitely generated group, the word problem is coarsely computable.

Definition. We say that $A \leq_c B$ if there is a Turing functional Φ such that for any coarse description D of B, $\Phi(D)$ is a coarse description of A (i.e. if the set of coarse descriptions of A Medvedev-reduces to the set of coarse descriptions of B).

Bases for randomness

One would naively expect for non-computable A: if X is A-random, then X does not compute A.

Definition. We call A a *base for 1-randomness* if there exists an $X \geq_T A$ which is A-random.

If A is K-trivial, then it is a base for 1-randomness by Kučera-Gács.

Theorem. (Hirschfeldt, Nies and Stephan) *A set A is a base for 1-randomness if and only if A is K-trivial.*

An embedding of the Turing degrees

We can embed the Turing degrees into the coarse degrees. First, let $I_n = [2^n, 2^{n+1})$.

Let $F(A)$ repeat A infinitely often, i.e. $F(A) = \{\langle n, i\rangle \mid n \in A \wedge i \in \omega\}$.

Finally, let $E(A) = \bigcup_{m \in F(A)} I_m$.

Then E is an embedding of the Turing degrees into the coarse degrees.

Coarse bases for randomness

One could ask once more: is it possible for non-computable A that X is A-random, yet $E(A) \leq_c X$?

Note that, if $E(A) \leq_c X$, then every coarse representation of X computes A non-uniformly, i.e. $\{A\}$ Muchnik-reduces to the set of coarse representations of X. Thus, we could also ask the slightly weaker question, if we let

$$X^c = \{A \subseteq \omega \mid \rho(X \triangle D) = 0 \to A \leq_T D\},$$

is it the case $X^c = \mathbf{0}$ for 1-random X?

Coarse bases for randomness

Theorem. (Hirschfeldt et al.) *If X is 1-random, then every element of X^c is K-trivial.*

Corollary. (Hirschfeldt et al.) *If X is weakly 2-random, then $X^c = \mathbf{0}$.*

Theorem. (Hirschfeldt et al.) *If $X \leq_T \emptyset'$ is 1-random, then $X^c \neq \mathbf{0}$.*

Coarse bases for randomness

Definition. (Hirschfeldt et al.) Let $n \in \omega$, let $X \subseteq \omega$ and let $0 \leq i < n$. Then we let $X_i^n(k) = X(nk + i)$. Furthermore, we let $X_{\neq i}^n = \oplus_{j \neq i} X_j^n$.

Lemma. (Hirschfeldt et al.) *Let $n \in \omega$, $X \subseteq \omega$ 1-random and A non-K-trivial. Then there is an $0 \leq i < n$ such that $X_{\neq i}^n \not\geq_T A$.*

Proof. Towards a contradiction, assume every $X_{\neq i}^n$ computes A. Then X certainly computes A. We will show that X is A-random, which is a contradiction. We have:

X_0^n 1-random in $X_{\neq 0}^n \Rightarrow X_0^n$ 1-random in A.
X_1^n 1-random in $X_{\neq 1}^n \Rightarrow X_1^n$ 1-random in $A \oplus X_0^n \Rightarrow X_0^n \oplus X_1^n$ 1-random in A. □

Coarse bases for randomness

Theorem. (Hirschfeldt et al.) *If X is 1-random, then every element of X^c is K-trivial.*

Proof. (Sketch) Let A be non-K-trivial and let X be 1-random. We will construct a coarse description D of X which does not compute A. In step e, we diagonalise against Φ_e. Let $0 \leq i \leq 2^{e+1}$ be such that $X_{\neq i}^{2^{e+1}} \not\geq_T A$, which exists by the lemma. Then there are two options: either Φ_e splits along $X_{\neq i}^{2^{e+1}}$, in which case there is a finite string σ such that D defined by

$$D(j) = \begin{cases} X(j) & \text{if } j \neq i \mod 2^{e+1} \\ \sigma(2^{-e-1}(j-i)) & \text{otherwise} \end{cases}$$

does not compute A. Otherwise, we can force divergence in a similar way.

□

Coarse bases for randomness

Theorem. (Hirschfeldt et al.) *If X is 1-random, then every element of X^c is K-trivial.*

Proof. (Sketch)
In both cases, $\rho(X \triangle D) \leq \frac{1}{2^{e+1}}$ and we have only defined D on a coinfinite set. We use the remaining space in the later steps. □

A question

Thus, we have seen: $X^c \subseteq \mathcal{K}$ for every 1-random X, while $X^c \neq \mathbf{0}$ for $X \leq_T \emptyset'$.

Question. (Hirschfeldt) Is every K-trivial A in X^c for some 1-random X?

I will sketch why this is false.

Oberwolfach randomness

Introduced by Bienvenu, Greenberg, Kučera, Nies and Turetsky to study the *covering problem*. It lies between difference randomness and balanced randomness.

Theorem. (Bienvenu et al.) *There is a K-trivial set A such that no set $X \geq_T A$ is Oberwolfach random. In fact, the upper cone of A is captured by a single Oberwolfach test.*

In particular, we have:

Corollary. *There is a K-trivial set A such that the upper cone of A is captured by a single balanced test.*

Computing from parts of a 1-random

Theorem. (Bienvenu et al.) *There is a K-trivial set A such that for every 1-random X, either A is not computable from the left half X_0 or it is not computable from the right half X_1.*

We generalise this result (using a different proof) as follows:

Theorem. *There is a K-trivial set A such that for every 1-random X and every $n \in \omega$ there exists an $0 \leq i < n$ such that $X^n_{\neq i}$ does not compute A.*

Assuming this fact, the answer to the question follows by a similar argument as the one which showed that $X^c \subseteq \mathcal{K}$.

Computing from all parts of a 1-random

Theorem. *There is a K-trivial set A such that for every $n \in \omega$ there exists an $0 \leq i < n$ such that $X^n_{\neq i}$ does not compute A.*

Proof. Let A be a K-trivial whose upper cone is captured by a balanced test, say $(G_{m,s})_{m \in \omega}$. We may assume G_m changes at most 2^m times. Fix $n \in \omega$. Let

$$H_{m,s} = \{X \mid \forall 0 \leq i < n(X^n_{\neq i} \in G_{m,s})\}.$$

Now, let $U_m = \bigcup_{s \in \omega} H_{m,s}$. We claim: $(U_m)_{m \in \omega}$ is a Solovay test.

Lemma. (Loomis–Whitney inequality) *Let $U \subseteq [0,1]^n$ be an open set. Then, if we let $\pi_i : [0,1]^n \to [0,1]^{n-1}$ be the projection*

$$\pi_i(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),$$

we have:

$$\lambda(U)^{n-1} \leq \lambda(\pi_1(U)) \ldots \lambda(\pi_n(U)).$$

Computing from all parts of a 1-random

Theorem. *There is a K-trivial set A such that for every $n \in \omega$ there exists an $0 \leq i < n$ such that $X^n_{\neq i}$ does not compute A.*

Proof. Let A be a K-trivial whose upper cone is captured by a balanced test, say $(G_{m,s})_{m\in\omega}$. We may assume G_m changes at most 2^m times. Fix $n \in \omega$. Let

$$H_{m,s} = \{X \mid \forall 0 \leq i < n(X^n_{\neq i} \in G_{m,s})\}.$$

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we have:

$$\lambda(U)^{n-1} \leq \lambda(\pi_1(U)) \dots \lambda(\pi_n(U)).$$

Computing from all parts of a 1-random

Theorem. *There is a K-trivial set A such that for every $n \in \omega$ there exists an $0 \leq i < n$ such that $X^n_{\neq i}$ does not compute A.*

Proof. Therefore, we have:

$$\lambda(H_{m,s}) \leq \lambda(G_{m,s})^{\frac{n}{n-1}} \leq 2^{\frac{-mn}{n-1}}$$

and thus

$$\lambda(U_m) \leq 2^m 2^{\frac{-mn}{n-1}} = (2^{\frac{1}{n-1}})^{-m}.$$

□

References

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