A hierarchy of the countably computable functions

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(Joint work with Kojiro Higuchi)

Motivations

1. Continuous = Computable Relative to an Oracle
2. We want to obtain deep understanding of the behavior of discontinuous (nonuniform), but degree-preserving maps.

Question

How can we study the effective content of discontinuous functions?

Our Answer

Effecivize Luzin's idea concerning countable-decomposability!

- (N. Luzin) A function $f : X \to Y$ is countably continuous or σ-continuous if f is decomposable into countably many continuous functions, that is, there is a countable cover $\{X_i\}_{i\in\omega}$ of X such that $f|_{X_i}$ is continuous.
- A function $f : X \to Y$ is countably computable if f is decomposable into countably many computable functions.

Proposition

A function $f : \omega^\omega \to \omega^\omega$ is countably computable
if and only if $f(x) \leq_T x$ for every $x \in \omega^\omega$.

Definition

$f : X \to Y$ is Π^0_1-finitely computable if there exists a finite increasing sequence $\{X_i\}_{i \le k}$ of Π^0_1 sets with $X_k = X$ such that $f|_{X_{i+1} \setminus X_i}$ is computable for every i.

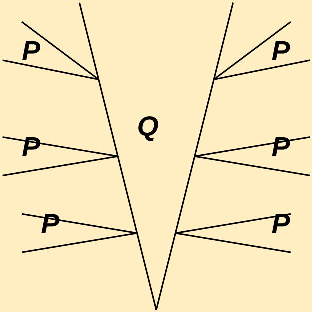
A set $P \subseteq 2^\omega$ is special if it is nonempty and it contains no computable element.

Anti-Cupping Theorem

Let $P \subseteq 2^\omega$ be a special Π^0_1 set. Then, there exists a Π^0_1 set $Q \subseteq 2^\omega$ such that

- there exists a computable function $f : P \to Q$,
- there exists a Π^0_1-finitely computable function $g : Q \to P$,
- for every $R \neq \emptyset$, there exists NO computable function $h : Q \times R \to P$.

Proof Idea

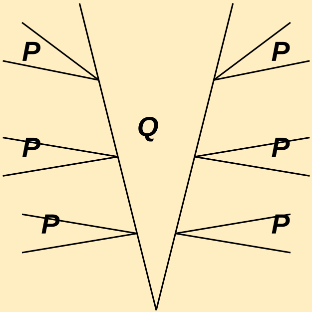


Definition (Concatenation)

For trees $P, Q \subseteq 2^{<\omega}$,

$$Q^\frown P = \{\sigma^\frown\langle 2\rangle^\frown\tau : \sigma \in Q,\ \tau \in P\}.$$

Proof Idea



$Q^\frown P$ represents the mass problem:

- "First we try to solve Q",
- "If we failed to solve Q, then next we try to solve P".

i.e., Solve "P or Q" *with a mind-change*!

Proof Idea

Definition (Concatenation)

For trees $P, Q \subseteq 2^{<\omega}$,

$$Q^\frown P = \{\sigma^\frown\langle 2\rangle^\frown\tau : \sigma \in Q,\ \tau \in P\}.$$

T: a computable binary tree with $[T]$ special.

Anti-Cupping Theorem

- $\exists$ a computable $f : [T] \to [T^\frown T]$,
- $\exists$ a Π^0_1-finitely computable $g : [T^\frown T] \to [T]$,
- $\forall R \neq \emptyset\ \neg\exists$ computable $h : [T^\frown T] \times R \to [T]$.

Definition

$f : X \to Y$ is <u>Γ-finitely computable</u> if there exists a finite increasing sequence $\{X_i\}_{i\leq k}$ of Γ sets with $X_k = X$ such that $f|_{X_{i+1}\setminus X_i}$ is computable for every i.

Proposition (Upper Bound)

The following are equivalent for any Π^0_1 classes $P, Q \subseteq 2^\omega$:

- There exists a finitely computable function $f : P \to Q$.
- There exists a Π^0_2-finitely computable function $g : P \to Q$.

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$f : X \to Y$ is Γ-finitely computable if there exists a finite increasing sequence $\{X_i\}_{i \le k}$ of Γ sets with $X_k = X$ such that $f|_{X_{i+1} \setminus X_i}$ is computable for every i.

Collapsing Theorem

There exists a special Π^0_1 class $Q \subseteq 2^\omega$ such that for every Π^0_1 class $P \subseteq 2^\omega$, the following are equivalent:

- there exists a finitely computable function $f : P \to Q$.
- there exists a Π^0_1-finitely computable function $g : P \to Q$.

Separation Theorem I

There are Π^0_1 classes $P, Q \subseteq 2^\omega$ such that:

- There exists a computable function $f : Q \to P$.
- There exists a Δ^0_2-finitely computable function $g : P \to Q$.
- There exists NO Π^0_1-finitely computable function $h : P \to Q$.

Separation Theorem II

There are Π^0_1 classes $P, Q \subseteq 2^\omega$ such that:

- There exists a computable function $f : Q \to P$.
- There exists a Π^0_2-finitely computable function $g : P \to Q$.
- There exists NO Δ^0_2-finitely computable function $h : P \to Q$.

Proof Idea

We introduce the notion of "disjunction" of trees.

$\sigma = (0, 7, 1, 11, 0, 2, 0, 1, 0, 5, 1, 4, 1, 22, 0, 7, \dots)$.

- $\mathtt{pr}_0(\sigma) = (7, 2, 1, 5, 7, \dots)$.
- $\mathtt{pr}_1(\sigma) = (11, 4, 22, \dots)$.

The disjunction of trees S and T is defined as follows:

$$S \triangledown_\infty T = \{\sigma : \mathtt{pr}_0(\sigma) \in S \text{ and } \mathtt{pr}_1(\sigma) \in T\}.$$

Proof Idea

Definition (Disjunction)

For $\sigma \in (2 \times \omega)^{<\omega}$, and $i < 2$,

- z^i_n = the n-th least element of $\{m : (\exists k)\ \sigma(m) = \langle i, k \rangle\}$.
- $\mathtt{pr}_i(\sigma)(n) = k$ if $z^i_n \downarrow$ and $\sigma(z^i_n) = \langle i, k \rangle$.
- $\mathtt{mc}(\sigma) = \#\{n : (\exists i, k, l)\ \sigma(n) = \langle i, k \rangle \& \sigma(n+1) = \langle 1 - i, l \rangle\}$

$\sigma = (0, 7, 1, 11, 0, 2, 0, 1, 0, 5, 1, 4, 1, 22, 0, 7, \dots)$.

- $\mathtt{pr}_0(\sigma) = (7, 2, 1, 5, 7, \dots)$.
- $\mathtt{pr}_1(\sigma) = (11, 4, 22, \dots)$.
- $\mathtt{mc}(\sigma) \geq 4$ (the number of times of mind-changes).

Proof Idea

Definition (Disjunction)

The disjunction of trees S_0 and S_1 is defined as follows:

$$S_0 \nabla_\infty S_1 = \{\sigma : (\forall i < 2)\ \mathtt{pr}_i(\sigma) \in S_i\}.$$
$$S_0 \nabla_n S_1 = \{\sigma : (\forall i < 2)\ \mathtt{pr}_i(\sigma) \in S_i \text{ and } \mathtt{mc}(\sigma) < n\}.$$

- $S_0 \nabla_1 S_1 \approx S_0 \oplus S_1$.
- $S_0 \nabla_2 S_1 \approx S_0 {}^\frown S_1$.

Proof Idea

Separation Theorem I

There is a computable trees $T \subseteq 2^{<\omega}$ such that:

- $\exists$ a computable function $f : [T] \to [T \nabla_\omega T]$.
- $\exists$ a Δ^0_2-finitely computable function $g : [T \nabla_\omega T] \to [T]$.
- $\neg\exists$ a Π^0_1-finitely computable function $h : [T \nabla_\omega T] \to [T]$.

Separation Theorem II

There is a computable trees $T \subseteq 2^{<\omega}$ such that:

- $\exists$ a computable function $f : [T] \to [T \nabla_\infty T]$.
- $\exists$ a Π^0_2-finitely computable function $g : [T \nabla_\infty T] \to [T]$.
- $\neg\exists$ a Δ^0_2-finitely computable function $h : [T \nabla_\infty T] \to [T]$.

Definition

$f : X \to Y$ is Γ-countably computable if there exists a uniform Γ covering $\{X_i\}_{i\in\omega}$ of X such that $f|_{X_i}$ is computable uniformly in i.

Π^0_α-countably computable iff $\Sigma^0_{\alpha+1}$-countably computable.

Proposition (Upper Bound)

The following are equivalent for any Π^0_1 classes $P, Q \subseteq 2^\omega$:

- There exists a countably computable function $f : P \to Q$.
- There exists a Π^0_2-countably computable function $g : P \to Q$.

Definition

$f : X \to Y$ is Γ-countably computable if there exists a uniform Γ covering $\{X_i\}_{i\in\omega}$ of X such that $f|_{X_i}$ is computable uniformly in i.

Separation Theorem III

Let $P \subseteq 2^\omega$ be a special Π^0_1 set. Then, there exists a Π^0_1 set $Q \subseteq 2^\omega$ such that

- there exists a computable function $f : P \to Q$,
- there exists a Π^0_1-countably computable function $g : Q \to P$,
- there exists NO finitely computable function $h : Q \to P$.

Definition

$f : X \to Y$ is Σ^0_2-excluded-middle computable if there exists a Σ^0_2 formula $\exists m \forall n \varphi(m, n, \alpha)$ and a computable sequence of computable functions $\{g_m\}_{m \in \omega}$ and h such that for every $\alpha \in X$,

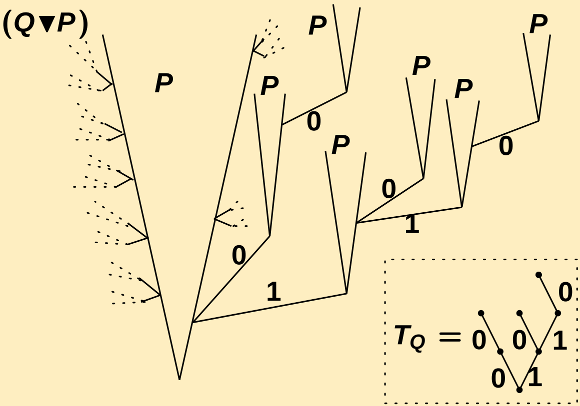
- $f(\alpha) = g_m(\alpha)$ if $\forall n \varphi(m, n, \alpha)$ is true.
- $f(\alpha) = h(\alpha)$ if $\exists m \forall n \varphi(m, n, \alpha)$ is false.

Anti-Cupping Theorem II

Let $P \subseteq 2^\omega$ be a special Π^0_1 set. Then, there exists a Π^0_1 set $Q \subseteq 2^\omega$ such that

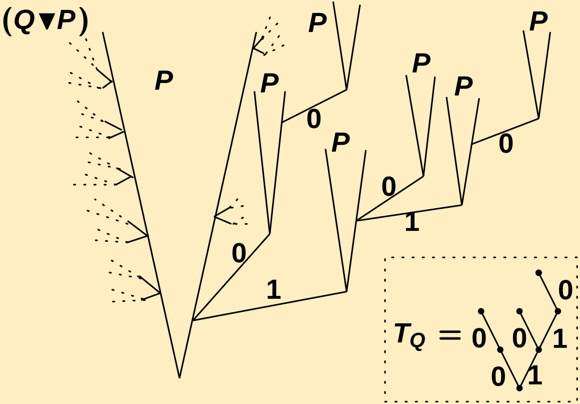
- $\exists$ a computable $f : P \to Q$,
- $\exists$ a Σ^0_2-excluded-middle computable $g : Q \to P$,
- $\forall R \neq \emptyset$, $\neg\exists$ Σ^0_2-countably computable $h : Q \times R \to P$.

- One can iterate the concatenation ⌢ such as $T^\frown T^\frown T^\frown T$.
- Indeed, one can iterate ⌢ along any well-founded tree.
- The hyperconcatenation $Q \blacktriangledown P$ is obtained by iterating ⌢ along an ill-founded tree T_Q.



Definition (Hyperconcatenation)

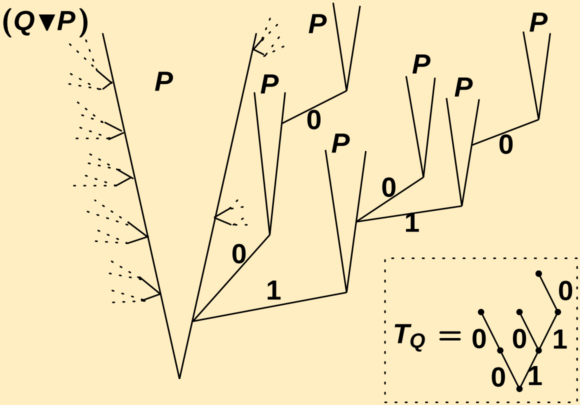
$$Q \blacktriangledown P = \{\sigma_0{}^\frown 2^\frown m_0{}^\frown \sigma_1{}^\frown 2^\frown m_1 \dots \sigma_i : (\forall j \leq i)\sigma_j \in P \text{ and } \langle m_0 \dots m_{i-1}\rangle \in Q\}.$$



The proof structure of Jockusch's Theorem (1987) is essentially the hyperconcatenation ▼ in the following sense.

Theorem

There exists a (uniform) computable function $f : \mathrm{DNR}_{k \cdot k} \to \mathrm{DNR}_k \blacktriangledown \mathrm{DNR}_k$.



Proof of Jockusch's Theorem

- $\exists$comp. $(a, b) \mapsto a \circ b$ s.t. $\Phi_{a \circ b}(a \circ b) = \langle \Phi_a(a), \Phi_b(b) \rangle$.
- Fix $g = \lambda n.\langle g_0(n), g_1(n) \rangle \in \mathbf{DNR}_{k \cdot k}$, where $g_0, g_1 \in k^\omega$.
- Then $(\forall a, b)\ g_0(a \circ b) \neq \Phi_a(a)$ or $g_1(a \circ b) \neq \Phi_b(b)$.

Proof of Jockusch's Theorem

- $\exists$comp. $(a, b) \mapsto a \circ b$ s.t. $\Phi_{a \circ b}(a \circ b) = \langle \Phi_a(a), \Phi_b(b) \rangle$.
- Fix $g = \lambda n.\langle g_0(n), g_1(n) \rangle \in \mathbf{DNR}_{k \cdot k}$, where $g_0, g_1 \in k^\omega$.
- Then $(\forall a, b)\ g_0(a \circ b) \neq \Phi_a(a)$ or $g_1(a \circ b) \neq \Phi_b(b)$.
- Consider the following Σ^0_2 sentence ψ:
  $\psi \equiv (\exists a)(\forall b)\ \Phi_b(b)\downarrow \to\ g_1(a \circ b) \neq \Phi_b(b)$.
  1. $\psi \Rightarrow$ we eventually construct $g_a = \lambda b.g_1(a \circ b) \in \mathbf{DNR}_k$.
  2. $\neg\psi \Rightarrow$ we eventually construct $h = \lambda a.g_0(a \circ b_a) \in \mathbf{DNR}_k$, where $b_a = \min\{b : g_1(a \circ b) = \Phi_b(b)\downarrow\}$.

Proof of Jockusch's Theorem

- $\exists$comp. $(a,b) \mapsto a \circ b$ s.t. $\Phi_{a\circ b}(a \circ b) = \langle \Phi_a(a), \Phi_b(b) \rangle$.
- Fix $g = \lambda n.\langle g_0(n), g_1(n) \rangle \in \mathrm{DNR}_{k\cdot k}$, where $g_0, g_1 \in k^\omega$.
- Then $(\forall a, b)\ g_0(a \circ b) \neq \Phi_a(a)$ or $g_1(a \circ b) \neq \Phi_b(b)$.
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  1. $\psi \Rightarrow$ we eventually construct $g_a = \lambda b.g_1(a \circ b) \in \mathrm{DNR}_k$.
  2. $\neg\psi \Rightarrow$ we eventually construct $h = \lambda a.g_0(a \circ b_a) \in \mathrm{DNR}_k$, where $b_a = \min\{b : g_1(a \circ b) = \Phi_b(b)\downarrow\}$.
- In other words, ($\exists$computable Φ)($\forall g \in \mathrm{DNR}_{k\cdot k}$)
  $\Phi(g) = \tau_0 * h(0) * \tau_1 * h(1) * \dots * \tau_{n+1} * h(n+1) * \dots$;
  1. $\psi \ \Rightarrow\ \Phi(g) = \tau_0 * h(0) * \tau_1 * h(1) * \dots * \tau_{a-1} * h(a-1) * g_a$;
  2. $\neg\psi \ \Rightarrow\ \Phi(g) =$
     $\tau_0 * h(0) * \tau_1 * h(1) * \dots * \tau_{a-1} * h(a-1) * \tau_a * h(a) * \dots$

Definition

$f : X \to Y$ is Σ^0_2-excluded-middle computable if there exists a Σ^0_2 formula $\exists m \forall n \varphi(m, n, \alpha)$ and a computable sequence of computable functions $\{g_m\}_{m \in \omega}$ and h such that for every $\alpha \in X$,

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T: a computable binary tree with $[T]$ special.

Anti-Cupping Theorem II

- $\exists$ a computable $f : [T] \to [T \blacktriangledown T]$,
- $\exists$ a Σ^0_2-excluded-middle computable $g : [T \blacktriangledown T] \to T$,
- $\forall R \neq \emptyset$, $\neg\exists$ Σ^0_2-countably computable $h : [T \blacktriangledown T] \times R \to T$.

We say that f is $\mathbf{\Sigma}^0_2$-EM* computable if it is of the form

$$f = g_1 \circ g_2 \circ \cdots \circ g_k,$$

where each g_i is $\mathbf{\Sigma}^0_2$-excluded-middle computable.

Separation Theorem IV

Let $P \subseteq 2^{\mathbb{N}}$ be a special $\mathbf{\Pi}^0_1$ set. Then there exists a $\mathbf{\Pi}^0_1$ set $Q \subseteq 2^{\mathbb{N}}$ such that:

- There exists a computable function $f : P \to Q$.
- There exists a $\mathbf{\Pi}^0_2$-countably computable function $g : Q \to P$.
- There exists NO $\mathbf{\Sigma}^0_2$-EM* computable function $h : Q \to P$.

- The notion of countable-decomposability has been an important notion in Descriptive Set Theory.
- The notion of countable-decomposability is also related to some notion of Algorithmic Learning.
- Perhaps, it is also related to the hierarchy of Excluded-Middle.

Further Work

Borel/hyperarithmetic version of countable-decomposability, etc.

Theorem (K., Gregoriades-K.)

The following are equivalent:

- f is $\underset{\sim}{\mathbf{\Pi}}^0_\alpha$ countably continuous.
- f is $\underset{\sim}{\mathbf{\Sigma}}^0_{\alpha+1}$ countably continuous.
- If $A \in \underset{\sim}{\mathbf{\Sigma}}^0_{\alpha+1}$ then $f^{-1}[A] \in \underset{\sim}{\mathbf{\Sigma}}^0_{\alpha+1}$.
- $A \in \underset{\sim}{\mathbf{\Sigma}}^0_{\alpha+1} \mapsto f^{-1}[A] \in \underset{\sim}{\mathbf{\Sigma}}^0_{\alpha+1}$ is continuous.

Theorem (K.)

The following are equivalent:

1. f is Π^0_α countably computable.
2. f is $\Sigma^0_{\alpha+1}$ countably computable.
3. $A \in \underset{\sim}{\mathbf{\Sigma}}^0_{\alpha+1} \mapsto f^{-1}[A] \in \underset{\sim}{\mathbf{\Sigma}}^0_{\alpha+1}$ is computable.

Learnability Characterizations

Theorem (de Brecht-Yamamoto, Higuchi-K.)

- f is Π^0_1 countably computable.
- f is Σ^0_2 countably computable.
- f is the discrete limit of a sequence of computable functions.
- f is identifiable in the limit.

Theorem (de Brecht-Yamamoto, Higuchi-K.)

- f is Π^0_1 finitely computable if and only if it is identifiable in the limit with bounded mind changes.
- f is Δ^0_2 finitely computable if and only if it is identifiable in the limit with bounded errors.