Weakly represented families in reverse mathematics

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3rd, December, 2013

Outline

In this talk, a new notion of classes of sets, i.e. weakly representable family of sets, is introduced into principles in reverse mathematics and we then investigate the interplay among different principles. Specifically, in this talk, we will mainly cover the interplay between the following principles (defined later):

- COH
- COHW
- MEET
- HIM
- DOM
- AVOID

Base system

The base system we are working in is RCA_0 which consists of the following axioms:

(1) Basic Axioms:
- $n+1 \neq 0$
- $m+1 = n+1 \rightarrow m = n$
- $m+0 = m$
- $m+(n+1) = (m+n)+1$
- $m \cdot 0 = 0$
- $m \cdot (n+1) = (m \cdot n) + m$
- $\neg m < 0$
- $m < n+1 \rightarrow (m < n \vee m = n)$

(2) Induction Axiom: $(\varphi(0) \wedge \forall n(\varphi(n) \rightarrow \varphi(n+1))) \rightarrow \forall n(\varphi(n))$, here $\varphi \in \Sigma^0_1$

(3) Comprehension Axioms: $\exists X \forall n(n \in X \leftrightarrow \varphi(n))$ where $\varphi(n) \in \Delta^0_1$

Some second order principles and definitions

Definition (Cohesive set)

Given a family of sets $\mathscr{F}$, a set G is said to be $\mathscr{F}$-cohesive if for any $A \in \mathscr{F}$, either $G \subseteq^* A$ or $G \subseteq^* \overline{A}$.

Statement (COH)

For every sequence $\vec{R} =< R_i : i \in \omega >\in S$, there exists a $\vec{R}$-cohesive set.

The variant to *COH* is COHW. Before we introduce COHW, we need some new notions.

Definition (Weakly represented partial function)

A (partial) function f is said to be weakly represented by a set A if for every x,y, $\exists z \langle x, y, z\rangle \in A$ *if and only if* $x \in dom(f) \wedge f(x) = y$ ***[representation]*** *and* $\forall x, y, y', z, z' [(\langle x, y, z\rangle \in A \wedge \langle x, y', z'\rangle \in A) \rightarrow y = y'] \wedge [(\langle x, y, z\rangle \in A \wedge z < z') \rightarrow \langle x, y, z'\rangle \in A]$ ***[consistency and monotonicity]*** *and* $\exists z \langle x, y, z\rangle \in A \rightarrow \forall t < x \exists y', z' \langle t, y', z'\rangle \in A$ ***[downward closure]****.*

Definition (Weakly representable family of total functions)

A class of functions $\mathscr{F}$ is said to be weakly representable if $\mathscr{F} \subseteq S$ and there exists a uniform family $\langle A_e : e \in M \rangle \in S$ and a class of partial functions (namely, there exists $A \in S$ such that $\langle A_e : e \in M \rangle \leq_T A$) such that

- *f_e is weakly represented by A_e*
- *A total function $f \in \mathscr{F}$ if and only if $f = f_e$ for some $e \in M$*

Definition (Weakly representable family of sets)

A family of sets $\mathscr{S}$ is said to be weakly representable if the family of their corresponding characteristic functions is weakly representable.

Statement (COHW)

For every weakly represented family of sets $\mathscr{F}$, there exists a $\mathscr{F}$-cohesive set.

COH and COHW are not equivalent

It is easy to observe that COHW implies COH since every uniform family is weakly represented by some recursive set, which exists in the model of RCA_0.

Theorem

COH does not imply COHW over RCA_0.

Proof sketch.

There exists a ω-model of COH which is not a model of COHW. Indeed, we could consider the sequence $\langle A_i : i \in \omega \rangle$ such that A_{i+1} is 1-generic relative to A_i and A'_{i+1} is PA-complete and hyperimmune-free relative to A'_i. The second order part of the model would then be $\{B \in 2^\omega : B \leq_T A_0 \oplus \cdots \oplus A_n \text{ for some } n \in \omega\}$. This model consists of non-high 1-generic sets. By the (relativised) PA-completeness of the jump of the sets in the model, it satisfies COH. However, it does not contain any DNR function. But the existence of DNR function is implied by the existence of any non-high cohesive set (Jockusch-Stephan 1993). Therefore, COHW is not true in this model. □

Is the failure of DNR crucial in separating COH and COHW? We know, in ω-models, if A is of PA-degree, then there exists a uniform sequence of sets recursive in A such that it includes every recursive sets (Jockusch). In this case $WKL_0 + COH$ implies COHW in the standard models. Actually we could produce a uniform family containing the class of all recursive sets using a $\{0,1\}$-DNR function over RCA_0, whose existence is implied by WKL_0. It is natural to ask:

Question

Is COH and COHW equivalent over RCA_0+DNR?

To attempt to solve the problem, notice each Martin-Löf random set is of DNR degree. The construction above could be modified by inverting the jump to get Martin-Löf random sets whose jump is PA-complete (relativised) in the model. If we could answer the following question negatively, we get a negative answer to the question above.

Question

Given any non-high Martin-Löf random set X such that X' is PA-complete relative to $\emptyset'$, does there exists a cohesive set G such that $G \leq_T X$?

There are other evidences that show the differences between COH and COHW.

Statement (AVOID)

Given any weakly represented family of total functions $\mathscr{F}$, *there exits a function* g *such that for each* $f \in \mathscr{F}$ $\{n \in M : f(n) = g(n)\}$ *is bounded.*

Theorem

COH does not imply AVOID.

Theorem

COHW implies AVOID in ω *models. More precisely, given any r-cohesive set G, there is a recursive procedure to produce a total function f such that* $\{n \in \omega : f(n) = \varphi_e(n)\}$ *is finite for any total recursive function* φ_e.

Two more principles

Statement (HIM)

Given any weakly represented family of total functions $\mathscr{F}$, there exits a function g such that for each $f \in \mathscr{F}$ $g(x) > f(x)$ for some $x > b$ for each $b \in M$.

Statement (MEET)

Given any weakly represented family of total functions $\mathscr{F}$, there exits a function g such that for each $f \in \mathscr{F}$ $\{n \in M : f(n) = g(n)\}$ is unbounded.

Theorem

MEET and HIM are equivalent in the context of standard models.

Theorem

AVOID and MEET/HIM are independent of each other.

Domination Principle

Statement

Given any weakly represented family of total functions $\mathscr{F}$, there exits a function g such that for each $f \in \mathscr{F}$ $g(x) > f(x)$ for all $x > b$ for some $b \in M$.

Theorem

DOM implies COH (even COHW) over $RCA_0 + I\Sigma_2^0$.

Question

Could it be done in $B\Sigma_2^0$ or even weaker systems that DOM implies COH or COHW?

First order consequences

In Cholak, Jockusch, Slaman's 2001 paper [On the strength of ramsey's theorem for pairs], COH is shown to be Π^1_1-conservative over $\mathrm{I}\Sigma^0_1$. In fact, same argument could be applied to COHW, since the whole construction is done outside the model, making use of the fact that the model is countable (then reorder the requirement in the order type of ω). Do we have the same property for DOM?

The answer is affirmative. Before we go to some details about the theorem, we could look at some supporting evidence.

Lemma (Cholak, Jockusch, Slaman)

$SRT^2_2 + RCA_0 \vdash B\Sigma^0_2$.

Theorem

DOM does not imply SRT^2_2 over RCA_0.

We apply Hechler Forcing to prove the Π^1_1-conservation of DOM.

Theorem

Given a countable model M of RCA_0, there exists a function $g : M \mapsto M$ such that g dominates all recursive functions in M and $M[g] \models RCA_0$.

Definition (poset)

The conditions would be of the form $(s, f) \in M^{<M} \times M^M$, while the pre-conditions would be $(s, f) \in M^{<M} \times M^{<M}$. The extensional relationship between conditions is as follows.

- $(t, f) \leq (s, g)$ *if* $t \supseteq s$ *and* $f(x) \geq g(x) \forall x \in dom(f) \cap dom(g)$ *and* $t(y) \geq g(y)$ *for* $|s| \leq y < |t|$
- *If a set X is fixed, then we write (s, i) short for (s, φ_i^X) where φ_i^X is a total function.*

Definition

(Forcing for Δ^0_0 formulas)

- *Given (s,f) a condition or pre-condition, then* $(s,f) \Vdash g(x) = z$ *iff* $x < |s|$ *and* $s(x) = z$; $(s,f) \Vdash g(x) \neq z$ *iff* $x < |s|$ *and* $s(x) \neq z$
- *For other cases of* $\theta(g)$ *where* θ *is* Δ^0_0 *it follows by the normal inductive relation decided by its truth in the ground model M.*

Definition

(Forcing for Π^0_1 *and* Σ^0_1 *formulas)*
Let $\varphi(g)$ *be* $\forall x \theta(x,g)$ *with* $\theta \in \Delta^0_0$.

- $(s,h) \Vdash \varphi(g)$ *iff for any pre-condition* $(t,f) \leq (s,h)$ *and any* $w \in M$ $(t,f) \Vdash \theta(w,g)$ *holds.*
- $(s,h) \Vdash \neg\varphi(g)$ *iff there exist* $w \in M$ *and pre-condition* (t,f) *such that* $(s,h) \leq (t,f)$, $(t,f) \Vdash \neg\theta(w,g)$.

Sketch.

First of all, fix a listing of total recursive functions $\{\phi_e : e \in M\}$ and Π^0_1 formulas with x, g being the only free variables (other cases could be easily reduced to this case) $\{\varphi_e(x,g) : e \in M\}$. The requirements we wish to meet are ($e \in M$ in the following):

- R_{3e} : g is defined at e.
- R_{3e+1} : g satisfies (s,e) for some finite function s, i.e. $s \subseteq g$ and g dominates ϕ_e. Note that here the indices e is chosen such that ϕ_e is total.
- R_{3e+2} : For e-th Π^0_1 formula $\varphi(x,g)$, either $M[g] \models \forall x \varphi(x,g)$ or there exists a least $b \in M$ such that $M[g] \models \neg\varphi(b,g) \wedge \forall a < b \varphi(a,g)$

By the countability of the model, rearrange the requirements in the order type of ω. It can be checked that at each stage, there is a extension of condition that forces the current active requirement. □

Question

Is it possible to add a dominating function whose definability could be controlled?

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