A Simple Linear-Time Algorithm for Finding Path-Decompositions of Small Width

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Outline

1. Introduction
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2. Preliminary Definitions
   - Boundaried graphs
   - Path-decompositions
   - Topological tree obstructions

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   - Proof of correctness
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**Pathwidth** is related to several VLSI layout problems:
- vertex separation
- gate matrix layout
- edge search number
- ...

Usefullness of bounded **treewidth** in:
- study of graph minors (Robertson and Seymour)
- input restrictions for many NP-complete problems
- (fixed-parameter complexity)
General problem(s) is NP-complete

*Input:* Graph $G$, integer $t$

*Question:* Is tree/path-width($G$) $\leq t$?

Algorithmic development (fixed $t$):

- $O(n^2)$ nonconstructive treewidth algorithm by Robertson and Seymour (1986)
- $O(n^{t+2})$ treewidth algorithm due to Arnborg, Corneil and Proskurowski (1987)
- $O(n \log n)$ treewidth algorithm due to Reed (1992)
- $O(2^{t^2} n)$ treewidth algorithm due to Bodlaender (1993)
- $O(n \log^2 n)$ pathwidth algorithm due to Ellis, Sudborough and Turner (1994)
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A distinguished set of vertices labeled 1, 2, . . . , k, is called the **boundary** of a (finite simple) graph.

A boundary size *k* **factorization** of a graph *G* is two *k*-boundaried graphs *A* and *B* such that *G* = *A* ⊕ *B*.

**Example**

The ⊕ operator on two 3-boundaried graphs *A* and *B* is illustrated below.

\[ A \quad 1 \quad 2 \quad 3 \quad 1 \quad 2 \quad 3 \quad A \oplus B \]

\[ A \quad B \]

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Linear-Time Path-Decomposition Algorithm
Definition

A *path-decomposition* of a graph $G = (V, E)$ is a sequence $X_1, X_2, \ldots, X_r$ of subsets of $V$ that satisfy the following:

1. $\bigcup_{1 \leq i \leq r} X_i = V,$
2. for every edge $(u, v) \in E$, there exists an $X_i$ such that $u \in X_i$ and $v \in X_i$, and
3. for $1 \leq i < j < k \leq r$, $X_i \cap X_k \subseteq X_j$.

Definition

The *pathwidth* of a path-decomposition $X_1, X_2, \ldots, X_r$ is $\max_{1 \leq i \leq r} |X_i| - 1$. The *pathwidth* of a graph $G$ is the minimum pathwidth over all path-decompositions of $G$. 
Path-decompositions

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Definition

An *(homeomorphic)* embedding of a graph $G_1 = (V_1, E_1)$ in a graph $G_2 = (V_2, E_2)$ is an injection from vertices $V_1$ to $V_2$ such that the edges $E_1$ are mapped to disjoint paths of $G_2$.

Example

![Graph Embedding Example](image-url)
Topological order

Definition
The set of homeomorphic embeddings between graphs gives a partial order, called the *topological order*.

Definition
A *lower ideal* $\mathcal{J}$ in a partial order $(\mathcal{U}, \succeq)$ is a subset of $\mathcal{U}$ such that if $X \in \mathcal{J}$ and $X \succeq Y$ then $Y \in \mathcal{J}$. The *obstruction set* for $\mathcal{J}$ is the set of minimal elements of $\mathcal{U} - \mathcal{J}$. 
Recursively generated tree obstructions

Some recursive rules for generating all topological tree obstructions of pathwidth $t$:

1. The tree $K_2$ is the only obstruction of pathwidth 0.
2. If $T_1$, $T_2$, and $T_3$ are any 3 tree obstructions for pathwidth $t$ then the tree $T$ consisting of a new degree 3 vertex attached to any vertex of $T_1$, $T_2$ and $T_3$ is a tree obstruction for pathwidth $t + 1$. 
Embedding tree obstructions in binary trees.

This shows that the complete binary tree of height $h(t) = 2t + \frac{1}{2}$ and order $f(t) = 2^{2t+1} - 1$ has pathwidth greater than $t$. 
Theorem

Let $H$ be an arbitrary undirected graph, and let $t$ be a positive integer. One of the following two statements must hold:

1. The pathwidth of $H$ is at most $f(t) - 1$.
2. $H$ can be factored: $H = A \oplus B$, where $A$ and $B$ are boundaried graphs with boundary size $f(t)$, the pathwidth of $A$ is greater than $t$ and less than $f(t)$. 
Proof idea of main result

- Assume we can embed a guest tree $B_{h(t)}$ in the host graph $H$ then we know that the pathwidth of $H$ is greater than $t$. (e.g. height $h(t) \geq \lg f(t)$)

- Refer to the vertices of $B_{h(t)}$ as tokens, and call tokens placed (or unplaced) if they are (not) mapped to vertices of $H$ in the current partial embedding.

- A vertex $v$ of $H$ is tokened if a token maps to $v$.

- Let $P[i]$ denote the set of vertices of $H$ that are tokened at time step $i$.

  The sequence $P[0], P[1], \ldots, P[s]$ will describe either a path-decomposition of $H$ or of a factor $A$. 
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- Let $P[i]$ denote the set of vertices of $H$ that are tokened at time step $i$. The sequence $P[0], P[1], \ldots, P[s]$ will describe either a path-decomposition of $H$ or of a factor $A$. 
Identifying tokens and tokened vertices

We recursively label the tokens of the guest binary tree by the following standard rules:

1. The root token of $B_{h(t)}$ is labeled by the empty string $\lambda$.
2. The left child token and right child token of a height $h$ parent token $P = b_1 b_2 \cdots b_h$ are labeled $P \cdot 1$ and $P \cdot 0$, respectively.

The token placement algorithm is described as follows.

1. Initially consider that every vertex of $H$ is colored blue.
2. A vertex of $H$ has its color changed to red when a token is placed on it, and stays red if the token is removed.
3. Only blue vertices can be tokened, and so a vertex can only receive a token once.
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Linear-time algorithm (grow part)

function GrowTokenTree
1   if root token $\lambda$ is not placed on $H$ then
   arbitrarily place $\lambda$ on a blue vertex of $H$
endif
2   while there is a vertex $u \in H$ with token $T$ and blue neighbor $v$, and token $T$ has an unplaced child $T \cdot b$ do
2.1   place token $T \cdot b$ on $v$
endwhile
3   return \{tokened vertices of $H$\}
program PathDecompositionOrSmallFatFactor
1 \( i \leftarrow 0 \)
2 \( P[i] \leftarrow \text{call} \ \text{GrowTokenTree} \)
3 \( \text{until } \left| P[i] \right| = f(t) \) or \( H \) has no blue vertices \( \text{repeat} \)
3.1 pick a token \( T \) with an unplaced child token
3.2 remove \( T \) from \( H \)
3.3 if \( T \) had one tokened child then
    replace all tokens \( T \cdot b \cdot S \) with \( T \cdot S \)
endif
3.4 \( i \leftarrow i + 1 \)
3.5 \( P[i] \leftarrow \text{call} \ \text{GrowTokenTree} \)
enduntil
done
Trying to embed a complete binary tree of height 3.
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Illustration of algorithm execution

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Some properties of the algorithm:

- The root token $\lambda$ of $B_{h(t)}$ is placed at most once for each component of $H$. But can move in steps 3.2-3.3.

- GrowTokenTree only returns when either $B_{h(t)}$ is completely embedded or there are no blue neighbors for the unplaced tokens.

- The algorithm terminates since each iteration of step 3.2 a tokened red vertex becomes untokened. (This can happen at most $n$ times.)
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Why the algorithm is correct (2 of 2)

Why $P[0], \ldots, P[s]$ is a path-decomposition of $H$ or $A$?

- Since each vertex $u$ is tokened at most once, the interpolation property holds.
- Let $(u, v)$ be an edge and assume vertex $u$ is tokened first. We only untoken a vertex when there is an unplaced child token step 3.2.
  Thus, vertex $v$ will be tokened as a child token of $u$. Therefore, there is some $P[i]$ containing both $u$ and $v$.

If all tokens of $B_{h(t)}$ are embedded into a subgraph of $H$ we claim that $A$ contains a subdivision of $B_{h(t)}$. Since GrowTokenTree only attaches pendant tokens to parent tokens we need only observe that the operation in step 3.3 subdivides the edge between $T$ and its parent.
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Why the algorithm runs in linear time

- If graph $H$ has more than $t \cdot n$ edges then the pathwidth is greater than $t$ (i.e. input has $O(n)$ edges).
- All operations on $B_{h(t)}$ are constant time.
- In GrowTokenTree step 2 if we find an edge $(u, v)$ where $v$ is a red vertex, we can delete it.
  Also it is safe to remove $(u, v)$ after step 2.1.
  Therefore, across all calls, each edge of $H$ needs to be considered at most once.
- The number of iterations in PathDecompositionOrSmallFatFactor is at most $n$. 
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- The number of iterations in PathDecompositionOrSmallFatFactor is at most $n$. 
Corollary

Any subtree of the binary tree $B_h(t)$ that has pathwidth greater than $t$ may be used in the pathwidth algorithm.

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Every graph with no minor isomorphic to forest $F$, where $F$ is a minor of a complete binary tree $B$, has pathwidth at most $c = |B| - 2$. 
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Corollary

Every graph with no minor isomorphic to forest $F$, where $F$ is a minor of a complete binary tree $B$, has pathwidth at most $c = |B| - 2$.

This is basically the main result of Bienstock, Robertson, Seymour and Thomas (1991) that for any forest $F$ there is a constant $c$, such that any graph not containing $F$ as a minor has pathwidth at most $c$. 
Illustration of algorithm (revised)

Trying to directly embed the Tree-1 obstruction.

After step 2,

\[ P[0] = \{a, b, c, d, e\} \]

After 3.3, \( T = 1 \)

\[ P[1] = \{a, b, d, e, f, h\} \]
Illustration of algorithm (revised)

Trying to directly embed the Tree-1 obstruction.

$T = 10$, after 3.5
$P[2] = \{a, b, d, e, h\}$

$T = 1$, after 3.5
$P[3] = \{a, b, e, h, i\}$

$T = 1$, after 3.5
$P[4] = \{a, b, g, h, i\}$
We have presented a simple linear-time algorithm (for each fixed constant $t$) that either establishes that the pathwidth of a graph is greater than $t$, or finds a path-decomposition of width at most $O(2^t)$.

- The width is equal to the number of tokens used. In practice this may be smaller than the complete binary tree.
- Can the width of the path-decomposition be bounded to the number of vertices in tree obstructions?
- There may be placement heuristics that can improve our performance on “typical” instances.
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Summary

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- The width is equal to the number of tokens used. In practice this may be smaller than the complete binary tree.
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- There may be placement heuristics that can improve our performance on “typical” instances.
Thank you!
**Definition**

A layout $L$ of a graph $G = (V, E)$ is a one to one mapping $L : V \rightarrow \{1, 2, \ldots, |V|\}$.

For a graph $G = (V, E)$ we conveniently write a layout $L$ as a permutation of the vertices $(v_1, v_2, \ldots, v_n)$.

For any layout $L = (v_1, v_2, \ldots, v_n)$ of $G$ let

$$V_i = \{v_j \mid j \leq i \text{ and } (v_j, v_k) \in E \text{ for some } k > i\}$$

for each $1 \leq i \leq n$.

**Definition**

The vertex separation of a graph $G$ with respect to a layout $L$ is $\text{vs}(L, G) = \max_{1 \leq i \leq |G|} \{|V_i|\}$.

The vertex separation of a graph $G$, denoted by $\text{vs}(G)$, is the minimum $\text{vs}(L, G)$ over all layouts $L$ of $G$. 