A Simple Linear-Time Algorithm for Finding Path-Decompostions of Small Width

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Outline

1. Introduction
   - Motivation
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   - Boundaried graphs
   - Path-decompositions
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   - Proof of correctness
   - Other results

Motivation

- Pathwidth is related to several VLSI layout problems:
  - vertex separation link
  - gate matrix layout
  - edge search number
  - . . .
- Usefullness of bounded treewidth in:
  - study of graph minors (Robertson and Seymour)
  - input restrictions for many NP-complete problems
  - (fixed-parameter complexity)

History

- General problem(s) is NP-complete
  Input: Graph G, integer t
  Question: Is tree/path-width(G) $\leq t$?
- Algorithmic development (fixed t):
  - $O(n^2)$ nonconstructive treewidth algorithm by Robertson and Seymour (1986)
  - $O(n^{t+2})$ treewidth algorithm due to Arnberg, Corneil and Proskurowski (1987)
  - $O(n \log n)$ treewidth algorithm due to Reed (1992)
  - $O(2^{t^2} n)$ treewidth algorithm due to Bodlaender (1993)
  - $O(n \log^2 n)$ pathwidth algorithm due to Ellis, Sudborough and Turner (1994)

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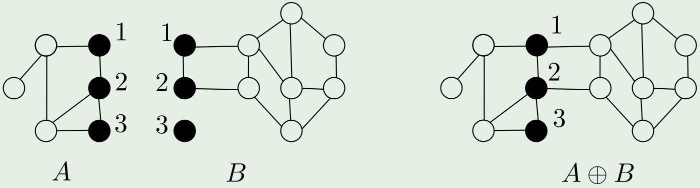
Boundaried graphs

- A distinguished set of vertices labeled $1, 2, \ldots, k$, is called the boundary of a (finite simple) graph.
- A boundary size k *factorization* of a graph G is two k-boundaried graphs A and B such that $G = A \oplus B$.

Example

The $\oplus$ operator on two 3-boundaried graphs A and B is illustrated below.



Path-decompositions

Definition

A *path-decomposition* of a graph $G = (V, E)$ is a sequence $X_1, X_2, \ldots, X_r$ of subsets of V that satisfy the following:

1. $\bigcup_{1 \le i \le r} X_i = V$,
2. for every edge $(u, v) \in E$, there exists an X_i such that $u \in X_i$ and $v \in X_i$, and
3. for $1 \le i < j < k \le r$, $X_i \cap X_k \subseteq X_j$.

Definition

The *pathwidth of a path-decomposition* $X_1, X_2, \ldots, X_r$ is $\max_{1 \le i \le r} |X_i| - 1$. The *pathwidth of a graph* G is the minimum pathwidth over all path-decompositions of G.

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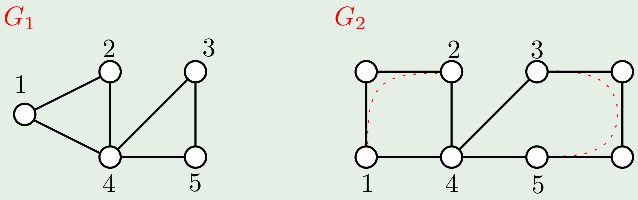
Graph embeddings

Definition

An *(homeomorphic) embedding* of a graph $G_1 = (V_1, E_1)$ in a graph $G_2 = (V_2, E_2)$ is an injection from vertices V_1 to V_2 such that the edges E_1 are mapped to disjoint paths of G_2.

Example



Topological order

Definition

The set of homeomorphic embeddings between graphs gives a partial order, called the *topological order*.

Definition

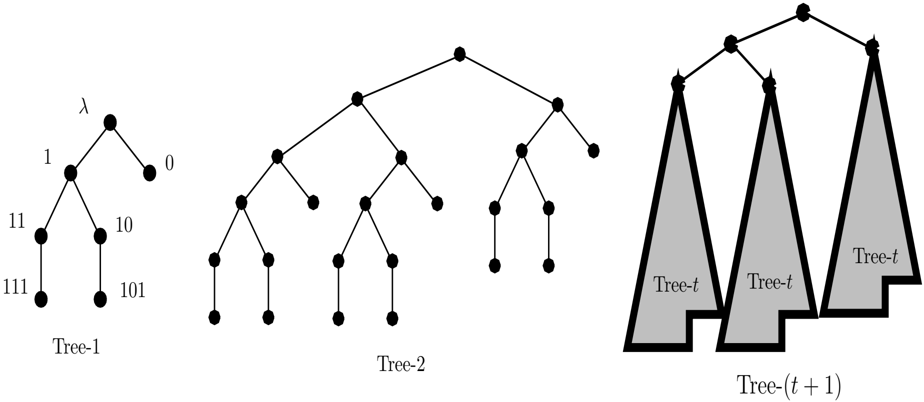
A *lower ideal* $\mathcal{J}$ in a partial order $(\mathcal{U}, \geq)$ is a subset of $\mathcal{U}$ such that if $X \in \mathcal{J}$ and $X \geq Y$ then $Y \in \mathcal{J}$. The *obstruction set* for $\mathcal{J}$ is the set of minimal elements of $\mathcal{U} - \mathcal{J}$.

Recursively generated tree obstructions

Some recursive rules for generating all topological tree obstructions of pathwidth t:

1. The tree K_2 is the only obstruction of pathwidth 0.
2. If T_1, T_2 and T_3 are any 3 tree obstructions for pathwidth t then the tree T consisting of a new degree 3 vertex attached to any vertex of T_1, T_2 and T_3 is a tree obstruction for pathwidth $t + 1$.

Embedding tree obstructions in binary trees.



This shows that the complete binary tree of height $h(t) = 2t + 1$ and order $f(t) = 2^{2t+1} - 1$ has pathwidth greater than t.

Main result

Theorem

Let H be an arbitrary undirected graph, and let t be a positive integer. One of the following two statements must hold:

1. *The pathwidth of H is at most $f(t) - 1$.*
2. *H can be factored: $H = A \oplus B$, where A and B are boundaried graphs with boundary size $f(t)$, the pathwidth of A is greater than t and less than $f(t)$.*

Proof idea of main result

- Assume we can embed a *guest tree* $B_{h(t)}$ in the *host graph* H then we know that the pathwidth of H is greater than t. (e.g. height $h(t) \geq \lg f(t)$)
- Refer to the vertices of $B_{h(t)}$ as tokens, and call tokens *placed* (or *unplaced*) if they are (not) mapped to vertices of H in the current partial embedding.
  A vertex v of H is *tokened* if a token maps to v.
- Let $P[i]$ denote the set of vertices of H that are tokened at time step i.
  The sequence $P[0], P[1], \ldots, P[s]$ will describe either a path-decomposition of H or of a factor A.

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Identifying tokens and tokened vertices

We recursively label the tokens of the guest binary tree by the following standard rules:

1. The root token of $B_{h(t)}$ is labeled by the empty string λ.
2. The left child token and right child token of a height h parent token $P = b_1 b_2 \cdots b_h$ are labeled $P \cdot 1$ and $P \cdot 0$, respectively.

The token placement algorithm is described as follows.

1. Initially consider that every vertex of H is colored blue.
2. A vertex of H has its color changed to red when a token is placed on it, and stays red if the token is removed.
3. Only blue vertices can be tokened, and so a vertex can only receive a token once.

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Linear-time algorithm (grow part)

function GrowTokenTree

1 **if** root token λ is not placed on H **then**

arbitrarily place λ on a blue vertex of H

endif

2 **while** there is a vertex $u \in H$ with token T and blue neighbor v, and token T has an unplaced child $T \cdot b$ **do**

2.1 place token $T \cdot b$ on v

endwhile

3 **return** {tokened vertices of H}

Linear-time algorithm (main program)

program PathDecompositionOrSmallFatFactor

1 $i \leftarrow 0$

2 $P[i] \leftarrow$ **call** GrowTokenTree

3 **until** $|P[i]| = f(t)$ **or** H has no blue vertices **repeat**

  3.1 pick a token T with an unplaced child token

  3.2 remove T from H

  3.3 **if** T had one tokened child **then**

    replace all tokens $T \cdot b \cdot S$ with $T \cdot S$

  endif

  3.4 $i \leftarrow i + 1$

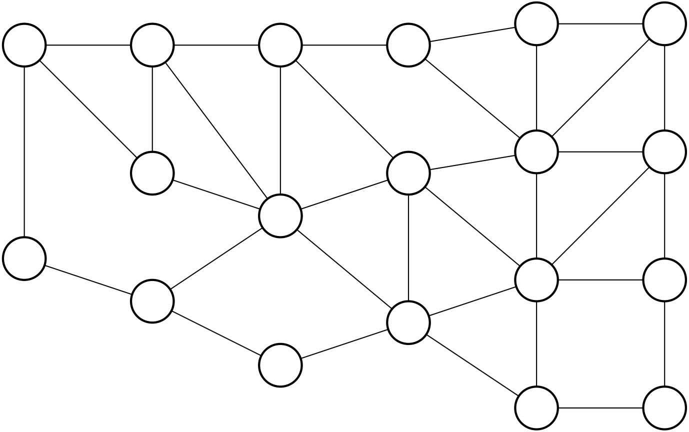
  3.5 $P[i] \leftarrow$ **call** GrowTokenTree

enduntil

done

Illustration of algorithm execution

Trying to embed a complete binary tree of height 3.



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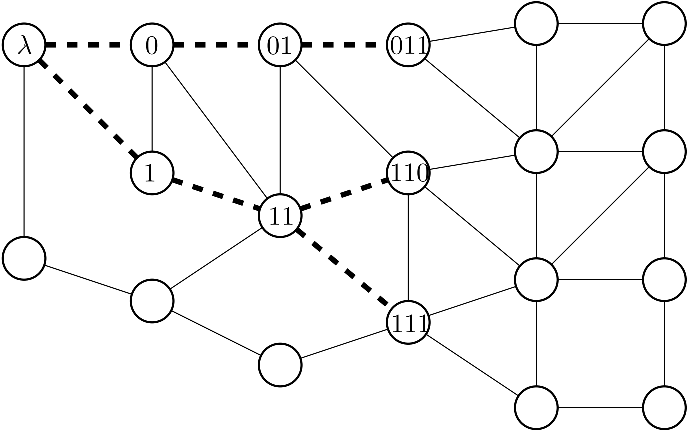

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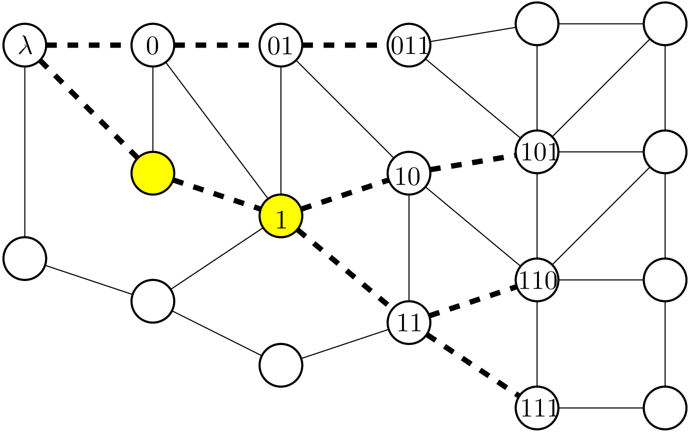

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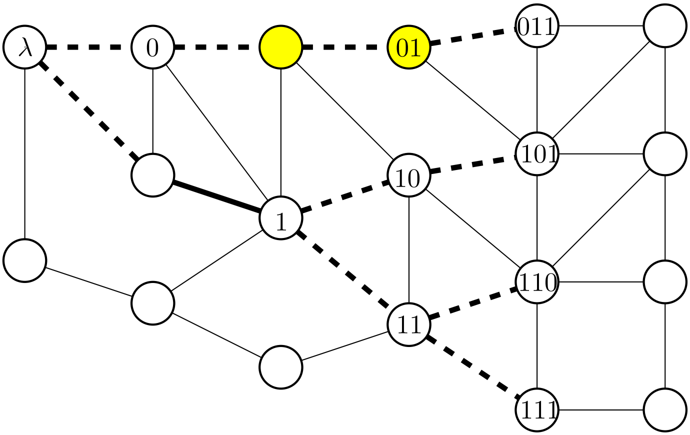

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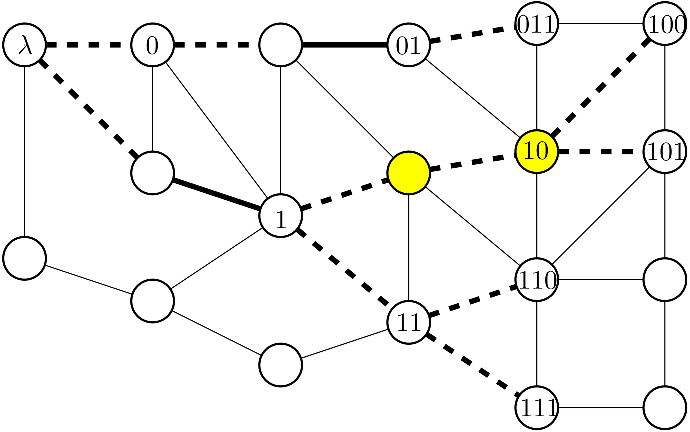

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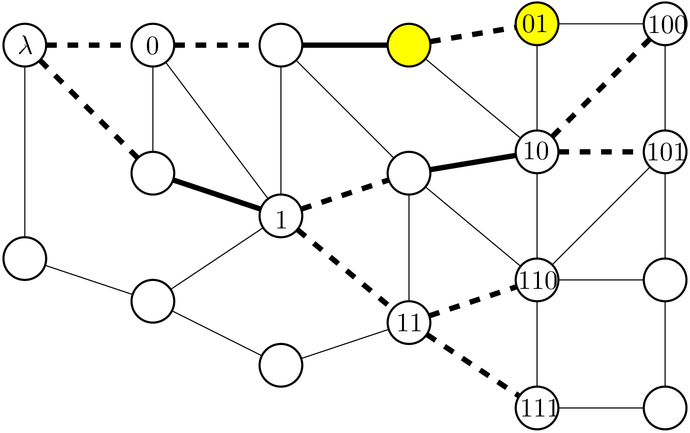

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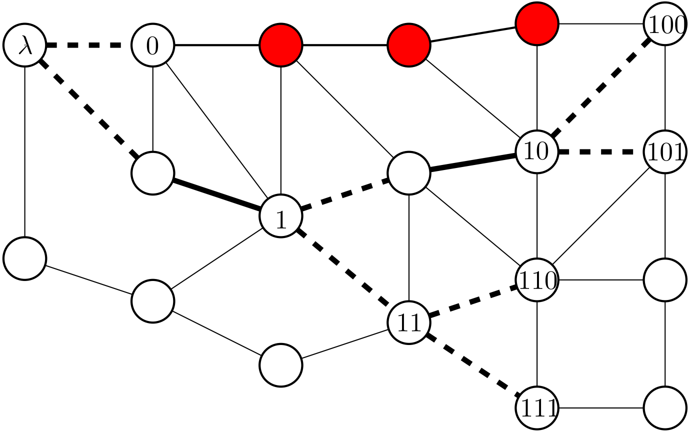

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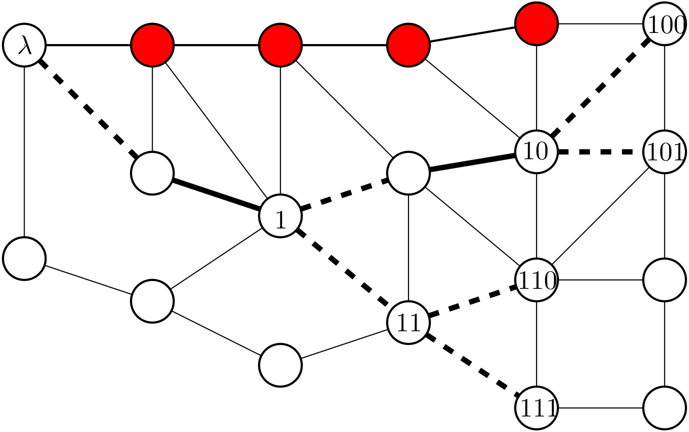

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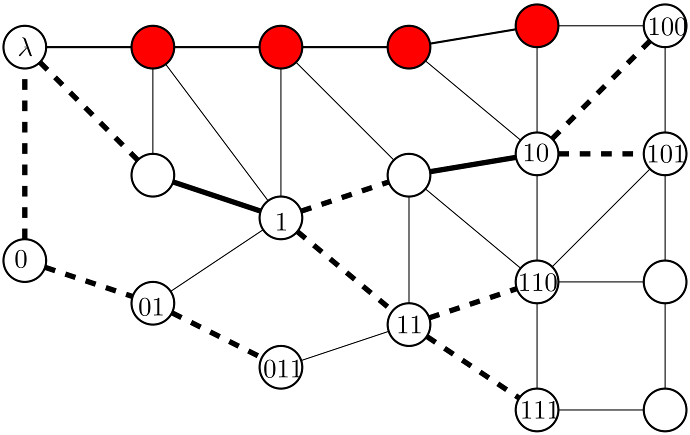

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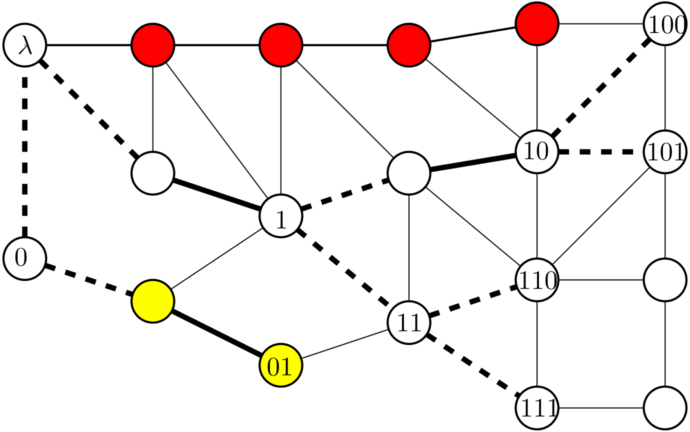

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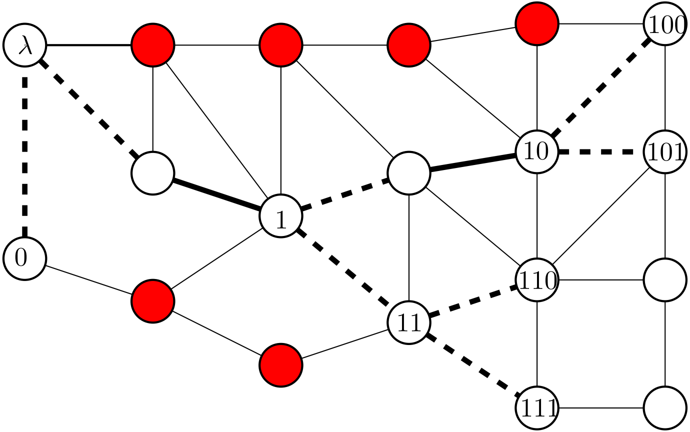

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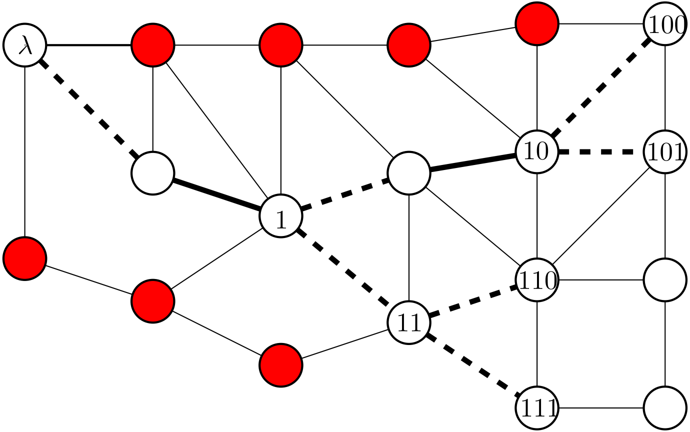

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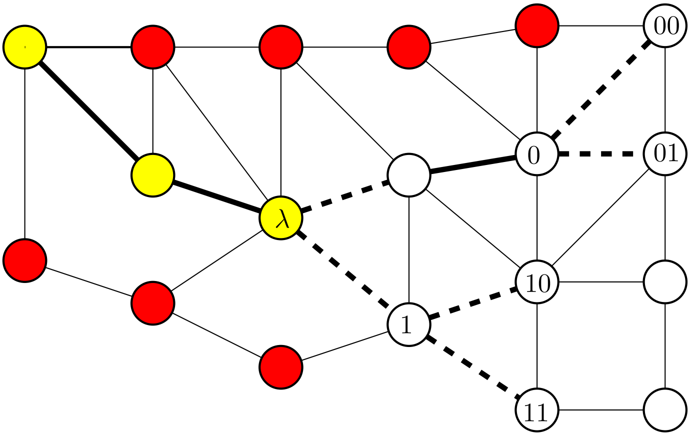

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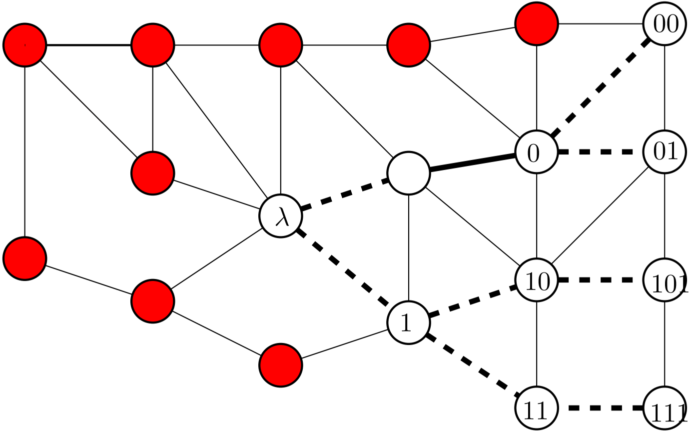

Why the algorithm is correct (1 of 2)

Some properties of the algorithm:

- The root token λ of $B_{h(t)}$ is placed at most once for each component of H. But can move in steps 3.2-3.3.
- GrowTokenTree only returns when either $B_{h(t)}$ is completely embedded or there are no blue neighbors for the unplaced tokens.
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Why the algorithm is correct (2 of 2)

Why $P[0], \ldots, P[s]$ is a path-decomposition of H or A?

- Since each vertex u is tokened at most once, the interpolation property holds.
- Let (u, v) be an edge and assume vertex u is tokened first. We only untoken a vertex when there is an unplaced child token step 3.2.
  Thus, vertex v will be tokened as a child token of u.
  Therefore, there is some $P[i]$ containing both u and v.

If all tokens of $B_{h(t)}$ are embedded into a subgraph of H we claim that A contains a subdivision of $B_{h(t)}$.
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Why the algoritm runs in linear time

- If graph H has more than $t \cdot n$ edges then the pathwidth is greater than t (i.e. input has $O(n)$ edges).
- All operations on $B_{h(t)}$ are constant time.
- In GrowTokenTree step 2 if we find an edge (u, v) where v is a red vertex, we can delete it.
  Also it is safe to remove (u, v) after step 2.1.
  Therefore, across all calls, each edge of H needs to be considered at most once.
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Other results

Corollary

Any subtree of the binary tree $B_{h(t)}$ that has pathwidth greater than t may be used in the pathwidth algorithm.

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Every graph with no minor isomorphic to forest F, where F is a minor of a complete binary tree B, has pathwidth at most $c = |B| - 2$.

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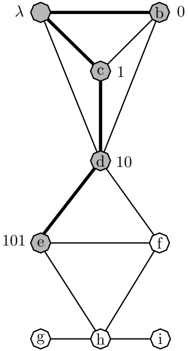
Corollary

Every graph with no minor isomorphic to forest F, where F is a minor of a complete binary tree B, has pathwidth at most $c = |B| - 2$.

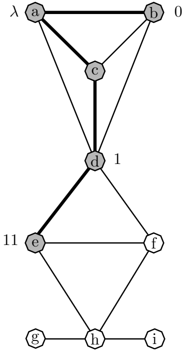
This is basically the main result of Bienstock, Robertson, Seymour and Thomas (1991) that for any forest F there is a constant c, such that any graph not containing F as a minor has pathwidth at most c.

Illustration of algorithm (revised)

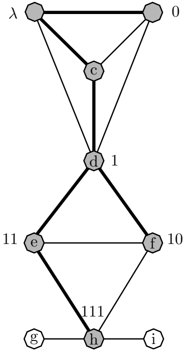
Trying to directly embed the Tree-1 obstruction.



After step 2,
$P[0] = \{a, b, c, d, e\}$



After 3.3, $T = 1$



After 3.5
$P[1] = \{a, b, d, e, f, h\}$

Illustration of algorithm (revised)

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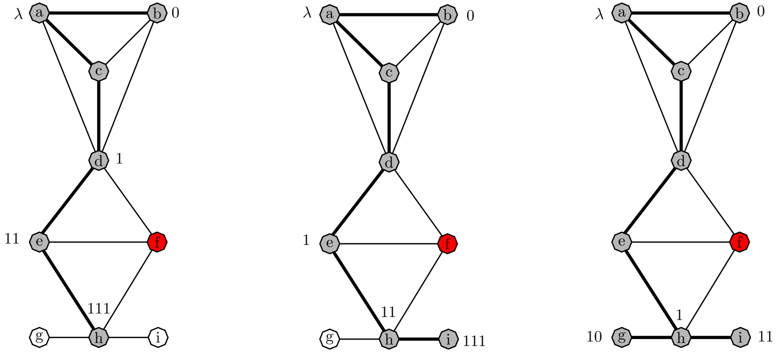

$T = 10$, after 3.5

$P[2] = \{a, b, d, e, h\}$

$T = 1$, after 3.5

$P[3] = \{a, b, e, h, i\}$

$T = 1$, after 3.5

$P[4] = \{a, b, g, h, i\}$

Summary

We have presented a simple linear-time algorithm (for each fixed constant t) that either establishes that the pathwidth of a graph is greater than t, or finds a path-decomposition of width at most $O(2^t)$.

- The width is equal to the number of tokens used. In practice this may be smaller than the complete binary tree.
- Can the width of the path-decomposition be bounded to the number of vertices in tree obstructions?
- There may be placement heuristics that can improve our performance on "typical" instances.

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Thank you!

Vertex separation

Definition

A layout L of a graph $G = (V, E)$ is a one to one mapping $L : V \rightarrow \{1, 2, \ldots, |V|\}$.

For a graph $G = (V, E)$ we conveniently write a layout L as a permutation of the vertices $(v_1, v_2, \ldots, v_n)$.
For any layout $L = (v_1, v_2, \ldots, v_n)$ of G let

$$V_i = \{v_j \mid j \leq i \text{ and } (v_j, v_k) \in E \text{ for some } k > i\}$$

for each $1 \leq i \leq n$.

Definition

The vertex separation of a graph G with respect to a layout L is $vs(L, G) = \max_{1 \leq i \leq |G|}\{|V_i|\}$.
The vertex separation of a graph G, denoted by $vs(G)$, is the minimum $vs(L, G)$ over all layouts L of G.