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Discrete Mathematics 307 (2007) 2484–2500
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Properties of vertex cover obstructions

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Received 20 December 2004; received in revised form 8 September 2006; accepted 18 January 2007
Available online 3 February 2007

Abstract

We study properties of the sets of minimal forbidden minors for the families of graphs having a vertex cover of size at most k. We denote this set by $\mathcal{O}(k\text{-Vertex Cover})$ and call it the set of obstructions. Our main result is to give a tight vertex bound of $\mathcal{O}(k\text{-Vertex Cover})$, and then confirm a conjecture made by Liu Xiong that there is a unique connected obstruction with maximum number of vertices for k-Vertex Cover and this graph is C_{2k+1}. We also find two iterative methods to generate graphs in $\mathcal{O}((k+1)\text{-Vertex Cover})$ from any graph in $\mathcal{O}(k\text{-Vertex Cover})$.

Keywords: Vertex cover; Obstructions; Forbidden graphs

1. Introduction

A common practice in graph theory is to characterize an infinite family of graphs by a finite set of minimal graphs that are not in the family. Here, one defines minimal with respect to some partial ordering of graphs. For example, Kuratowski's Theorem states that planar graphs are characterized by the two forbidden graphs $K_{3,3}$ and K_5, under the topological subgraph order. The *obstruction set* for planarity thus consists of these two graphs. In this paper we present some new properties about the obstructions to the families of graphs that have a vertex cover of size at most k, $k \geqslant 0$.

For the remainder of this section we formally define the graph families k-Vertex Cover, as those graphs having a minimum vertex cover of size at most k, and what it means to characterize them by a set of obstructions. In Section 2, we prove a conjecture that the cycle C_{2k+1} is the only largest connected obstruction for k-Vertex Cover, along with an appropriate theorem relating the maximum degree to the order of the obstructions. In Section 3, we investigate two good simple techniques for generating a large subset of the obstructions for $(k+1)$-Vertex Cover from the set of obstructions for k-Vertex Cover. Finally, we end the paper with some concluding remarks.

1.1. Preliminaries

The graph families of interest in this paper are based on the following classic problem.

Problem 1.1 (*Vertex Cover*). *Input*: Graph $G = (V, E)$ and a non-negative integer $k \leqslant |V|$.
Question: Is there a subset $V' \subseteq V$ with $|V'| \leqslant k$ such that V' contains at least one vertex from every edge in E?

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0012-365X/$ - see front matter
doi:10.1016/j.disc.2007.01.003

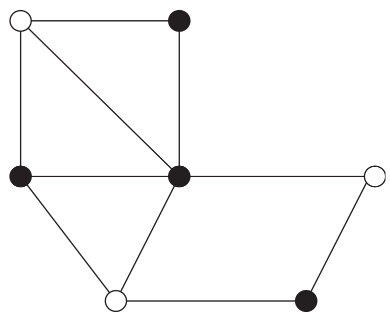

Fig. 1. A graph G with a minimum vertex cover in black.

A set V' in the above problem is called a *vertex cover* for the graph G. If for any vertex cover V'' for the graph G, $|V'| \leqslant |V''|$ always holds, then V' is called a *minimum vertex cover* of G (see example: Fig. 1). Note, for a given G, there may be more than one minimum vertex cover.

The vertex cover problem is one of the classic NP-complete decision problems highlighted in Gary and Johnson's book on the subject [11]. However, in practice, this problem seems to be practically solvable if k is a fixed constant (i.e., fixed parameter tractable [10]). The current best running time is $O(1.2852^k + kn)$, where n is the number of vertices of the input graph (see [1,7]). There has also been work on developing parallel techniques and the current status is that the problem is solvable for up to at least $k = 461$ [6].

The mathematical background for our structural finite characterization of k-VERTEX COVER is now presented.

A graph H is a *minor* of a graph G, denoted $H \leqslant_m G$, if a graph isomorphic to H can be obtained from G by a (possibly empty) sequence of operations chosen from:

1. delete an isolated vertex (i.e., vertex with degree equals zero),
2. delete an edge, or
3. contract an edge (i.e., superpose two vertices connected with an edge and remove any multiple edges or loops that form).

The *minor order* is the set of finite graphs ordered by $\leqslant_m$ and is easily seen to be a partial order. A family $\mathscr{F}$ of graphs is a *lower ideal*, under a partial order $\leqslant_p$, if whenever a graph $G \in \mathscr{F}$ implies that $H \in \mathscr{F}$ for any $H \leqslant_p G$ (i.e., a lower ideal $\mathscr{F}$ is a set closed downward under $\leqslant_p$). An *obstruction* G (often called a *forbidden minor*) for a lower ideal $\mathscr{F}$ is a minor-order minimal graph *not* in $\mathscr{F}$ (i.e., $G \notin \mathscr{F}$ and for all H, $H <_m G$ implies $H \in \mathscr{F}$).

The *Graph Minor Theorem* of Robertson and Seymour [13] states that *any set of graphs is a well-partial order under the minor order*. Or equivalently, every graph family that is closed under the minor order has finitely many obstructions (up to isomorphism). Thus, a complete set of obstructions describes a *finite characterization* for any minor-order lower ideal $\mathscr{F}$.

See [3,4] for a general (but somewhat difficult) procedure for computing forbidden minors for families of graphs, like k-VERTEX COVER (see [2]), that have a known pathwidth/treewidth upper bound for all of their obstructions. However, a more family-specific approach was used to compute the 6-VERTEX COVER obstructions [9]. We will soon justify and utilize a claim that the special graph family k-VERTEX COVER is also finitely characterizable within the *subgraph partial order* (which is not a well-partial order, in general).

1.2. Frequently used notation

For the following paper we use the following graph notation.

$E(G)$	all edges of a graph G
$V(G)$	all vertices of a graph G; the size of this set is called the *order* of a graph G
ϕ	The empty graph (i.e. the graph of order 0)

$N(u)$	all the vertex neighbors of vertex u in a specified graph
$G[V_x]$	an induced subgraph (V_x, E_x) of $G=(V,E)$, where $V_x \subseteq V$ and $E_x = \{(u,v) \mid (u,v) \in E$ and $u,v \in V_x\} \subseteq E$
$E(v)$	all incident edges of vertex v in a specified graph
$\mathrm{VC}(G)$	the non-negative integer $\lvert V'\rvert$, where V' denotes a minimum vertex cover of graph G
k-VERTEX COVER	the family of graphs that have a vertex cover of size at most k
$\mathcal{O}$(k-VERTEX COVER)	the finite set of obstructions of k-VERTEX COVER, where integer $k \geqslant 0$
O	denotes an arbitrary (connected or disconnected) graph in $\mathcal{O}$(k-VERTEX COVER)
O_c	denotes a connected graph in $\mathcal{O}$(k-VERTEX COVER)
O_d	denotes a disconnected graph in $\mathcal{O}$(k-VERTEX COVER)

1.3. A framework for characterizing vertex cover families

It is easy to see that k-VERTEX COVER is a lower ideal in the minor order (e.g. [2, Lemma 1]). In [9], Dinneen and Xiong built a computational model to generate the whole set of connected graphs in $\mathcal{O}$(k-VERTEX COVER), which is based on these steps: (1) Bound the search space of graphs within a reasonable interval for possible order. (2) For each fixed order, generate graphs with all possible combinations of edges, and then find efficient properties to eliminate the graphs that are not in $\mathcal{O}$(k-VERTEX COVER). (3) Decide if the remaining graphs are obstructions. To bound the search space, they set up an (exact) upper bound of $2k+1$ on the order of each connected obstruction O_c of $\mathcal{O}$(k-VERTEX COVER) (see [9, Theorem 10]). For the reader's convenience, we mention that all connected graphs of $\mathcal{O}$(k-VERTEX COVER) ($k \leqslant 6$) are listed in the appendices of [2,9] (also see [8]).

However, from a practical point of view, the search space for all possible combination of edges still grows exponentially even if we have set up an upper bound on the order of graphs in $\mathcal{O}$(k-VERTEX COVER). In the worst case, when the order increases up to $2k+1$, the search space size when considering all possible combination of edges peaks but it seems that only one connected graph of that order is in $\mathcal{O}$(k-VERTEX COVER). The original intention of this paper is to prove this conjecture: *There is a unique connected obstruction with maximum number of vertices for* k-VERTEX COVER *and this graph is* C_{2k+1}, as given in [9,14]. During the proof, we find a tighter vertex bound of graphs in $\mathcal{O}$(k-VERTEX COVER) when also considering the maximum degree of the graphs.

With respect to the definition of a minor, Dinneen and Xiong proved a simplified procedure for detecting an obstruction of k-VERTEX COVER to be: *A graph* $G=(V,E)$ *is in* $\mathcal{O}$(k-VERTEX COVER) *if and only if* (a) *for all* $v \in V$, *degree*$(v) \neq 0$ (*i.e.*, *no isolated vertices*); (b) $\mathrm{VC}(G)=k+1$ *and* $\mathrm{VC}(G\backslash\{e\})=k$, *for all* $e \in E$ (see [9, Theorem 4]).[1] They argued that if $G\backslash\{e\} \in k$-VERTEX COVER for all $e \in E(G)$, then any single edge contraction of G is also in k-VERTEX COVER. Hence, we can omit the third operation defining minor inclusion: "contract an edge"; the remaining two operations: "delete an isolated vertex" and "delete an edge" are sufficient and necessary for defining $\mathcal{O}$(k-VERTEX COVER). For this reason, we call condition (a) and (b) to be a our "*definition of an obstruction* for k-VERTEX COVER" when discussed later in this paper.

In this paper, we focus on studying all connected vertex cover obstructions, because any disconnected obstruction O_d of k-VERTEX COVER is a union of connected obstructions for vertex cover families with smaller values of k. Recall $(k-1)$-VERTEX COVER $\subset$ k-VERTEX COVER for all $k>1$ implies a hierarchy of graph families. More accurately, for any given O_d, with $s>1$ connected components, it is easy to see that $O_d = \bigcup_{j=1}^{s} G_j$, where each G_j is a connected obstruction for p_j-VERTEX COVER with $p_j = \mathrm{VC}(G_j)-1$. Furthermore, we conclude that

$$k+1=\mathrm{VC}(O_d)=\sum_{j=1}^{s}(p_j+1)=s+\sum_{j=1}^{s}p_j.$$

Thus $1<s\leqslant k+1$ and $0\leqslant p_1, p_2, \ldots, p_s<k$, which limits the number of components and gives us a process to enumerate all disconnected obstructions for k-VERTEX COVER if we know all the connected obstructions for k'-VERTEX COVER, $k'<k$.

[1] Condition (a) was mistakenly omitted in the statement of Theorem 4 of [9] since the context of discussion should have been restricted to connected graphs.

1.4. Checking membership in $\mathcal{O}$(k-Vertex Cover)

For any graph G without isolated vertices, a general algorithm to decide if G is in $\mathcal{O}$(k-Vertex Cover) is easy to describe. First let $GA(G)$ be a graph membership algorithm that returns true if and only if $\mathrm{VC}(G) \leqslant k$. Then to decide if a graph G is an obstruction, we check

$$\mathrm{VC}(G) > k \text{ and for each edge } e \in E(G), \quad \mathrm{VC}(G \backslash \{e\}) \leqslant k. \tag{1}$$

Condition (1) is equivalent to condition (b) of our definition of an obstruction for the family k-Vertex Cover. The reasons why we define $GA(G)$ to be a Boolean value of $\mathrm{VC}(G) \leqslant k$ rather than $\mathrm{VC}(G) = k$ are: Firstly, from programming point of view, the running time of deciding $\mathrm{VC}(G) \leqslant k$ may be shorter than deciding $\mathrm{VC}(G) = k$; Secondly, from theoretical point of view, sometimes condition (1) makes a proof of existence easier (see Section 3, extension method 1), because the weaker condition $\mathrm{VC}(G) \leqslant k$ does not ask for a constructive proof of a minimum vertex cover while condition $\mathrm{VC}(G) = k$ usually does.

Now, we explain that *condition* (1) *is equivalent to condition* (b) *of our definition of an obstruction for* k-Vertex Cover. Obviously, this definition of an obstruction for k-Vertex Cover satisfies condition (1); For any graph G satisfies condition (a) and (1), let $\widetilde{V}_{(u,v)}$ denote an arbitrary minimum vertex cover of $G \backslash \{(u, v)\}$, then $|\widetilde{V}_{(u,v)}| \leqslant k$. It is easy to see $u, v \notin \widetilde{V}_{(u,v)}$, otherwise $\widetilde{V}_{(u,v)}$ covers G, which contradicts $\mathrm{VC}(G) > k$. Therefore $\widetilde{V}_{(u,v)} \cup \{u\}$ covers G. We get $k + 1 \geqslant \mathrm{VC}(G \backslash \{(u, v)\}) + 1 = \mathrm{VC}(G) > k$, where the '1' denotes either u or v. That is, $\mathrm{VC}(G) = k + 1$ and for each edge $(u, v) \in E(G)$, $\mathrm{VC}(G \backslash \{(u, v)\}) = k$. Hence G is in $\mathcal{O}$(k-Vertex Cover).

2. Properties of vertex cover obstructions

We now present our first set of results about the k-Vertex Cover obstructions.

2.1. Preliminary remarks

This section presents some analysis about minimum vertex cover and application of the well-known *Hall's Marriage Theorem*, which is given in [12] (also see [5]). These results will contribute to the proof of an upper bound of all connected obstructions later on. The proof ideas of Statements 2.1–2.3 are mainly extracted from the proof given in:

Theorem 10 of Dinneen and Xiong [9]. *A connected obstruction for* k-Vertex Cover *has at most* $2k + 1$ *vertices*.

Statement 2.1. *For a graph* $G = (V, E)$ *with no isolated vertices, let* V_1 *denote a minimum vertex cover of* G, *then* $N(V \backslash V_1) = V_1$.

Proof. Divide V into two subsets V_1 and V_2, as indicated in Fig. 2, such that V_1 is a minimum vertex cover of G and $V_2 = V \backslash V_1$.

There is no edge between any pair of vertices in V_2, otherwise V_1 is not a vertex cover, so $N(V_2) \subseteq V_1$. Further, each vertex $v \in V_1$ has at least one neighbor in V_2, otherwise we move v from V_1 to V_2, then $V_1 \backslash \{v\}$ is a vertex cover of G with fewer vertices (this contradicts the assumption: V_1 is a minimum vertex cover of G). So $N(V_2) \supseteq V_1$. Therefore $N(V_2) = V_1$. □

Statement 2.2. *For a graph* $G = (V, E)$ *with no isolated vertices, let* V_1 *denote a minimum vertex cover of* G, *if there exists a subset* $S \subseteq V_2 = V \backslash V_1$ *such that* $|N(S)| < |S|$, *then we can always find*:

1. *A minimal subset* V_3, $V_3 \subseteq S$ *such that* $|N(V_3)| < |V_3|$ *and for all* $T \subset V_3$, $|N(T)| \geqslant |T|$.[2]
2. *The set* V_3 *also satisfies* $|N(V_3)| = |V_3| - 1$ *and for any* $v \in V_3$, $N(V_3 \backslash \{v\}) = N(V_3)$.

[2] In mathematical terminology, the critical limit V_3 must exist.

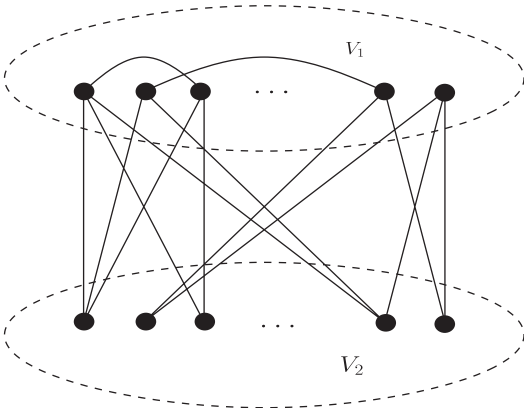

Fig. 2. Divide the vertex set of G into two subsets.

Procedure MinSubset(Vertices S, Graph G)
  For $i = 2$ **to** $|S|$
    For each $V_3 \subseteq S$ of size i
      If $|N(V_3)| < |V_3|$ **then return** V_3
    endFor
  endFor
end

Fig. 3. Find the minimum subset V_3 of Statement 2.2(1).

Proof. (1) If V_1 is a minimum vertex cover of G, then $N(V_2) \subseteq V_1$ (mentioned in proof of Statement 2.1). Because any $v \in V$, $|N(v)| \geqslant |\{v\}| = 1$, we can always find a V_3 which satisfies Statement 2.2(1) by exhausting all possible combinations during the growing of any single vertex v in S up to the whole vertex set of S (see Fig. 3).

Note, the returned V_3 of the procedure MinSubset is minimum, because any subset V' of S in order of k ($< |V_3|$) must satisfy $|N(V')| \geqslant |V'|$ (i.e., condition 'If' is always false while $i \leqslant k$). In worst case, $V_3 = S$.

(2) According to Statement 2.2(1), we delete any vertex $v \in V_3$, leaving $V_3' = V_3 \backslash \{v\}$, then any subset $T \subseteq V_3'$ satisfies $|N(T)| \geqslant |T|$. Let $T = V_3'$, then $|V_3| - 1 = |V_3'| \leqslant |N(V_3')| \leqslant |N(V_3)| < |V_3|$. Therefore $|N(V_3)| = |N(V_3')| = |V_3| - 1$. $\square$

A *matching* in a bipartite graph is a set of independent edges with no common end points.

Recall *Hall's marriage Theorem* [12]: A bipartite graph $B = (X_1, X_2, E)$ has a matching of cardinality $|X_1|$ if and only if for each subset $A \subseteq X_1$, $|N(A)| \geqslant |A|$.

Statement 2.3. *In a connected obstruction O_c, let V_1 denote a minimum vertex cover, then for each $S \subseteq V_2 = V \backslash V_1$, $|N(S)| \geqslant |S|$.*

Proof. We prove by way of contradiction. Assume there exists a subset $S \subseteq V_2$ such that $|N(S)| < |S|$, from Statement 2.2, we know:

1. There exists a minimal subset V_3, $V_3 \subseteq S \subseteq V_2$ such that $|N(V_3)| < |V_3|$ and for all $T \subset V_3$, $|N(T)| \geqslant |T|$.
2. Such V_3 satisfies $|N(V_3)| = |V_3| - 1$ and for any $v \in V_3$, $N(V_3 \backslash \{v\}) = N(V_3)$.

Define $V_3' = V_3 \backslash \{v\}$, $V_4 = N(V_3')$ (refer to Fig. 4). By applying Hall's Marriage Theorem, there is a matching of cardinality $|V_3'|$ in the induced bipartite subgraph $G_1 = (V_3', N(V_3'), E_{G_1})$ in O_c. Define $D_1 = O[V_3' \cup V_4]$. Obviously,

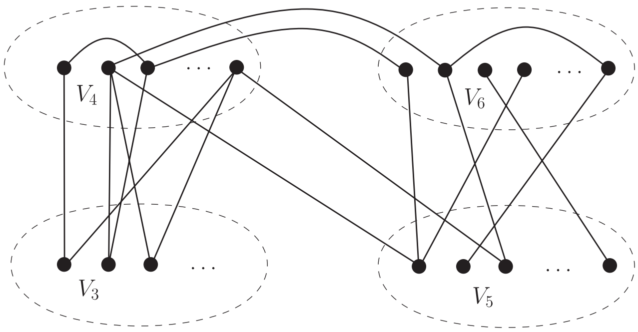

Fig. 4. Divide the vertex set of O into four subsets.

$G_1 \subseteq D_1$, because there might be edges among V_4. Then $\mathrm{VC}(D_1) \geqslant |V_3'| = |V_3| - 1 = |N(V_3)| = |N(V_3')| = |V_4|$ (see Statement 2.2(2)). Moreover, there are no edges among $V_3 \subseteq V_2$ (as mentioned in Statement 2.1), we get $\mathrm{VC}(D_1) \leqslant |V_4|$. Therefore,

$$\mathrm{VC}(D_1) = |V_4|. \tag{2}$$

Let $V_5 = V_2 \backslash V_3'$ and $V_6 = V_1 \backslash V_4$. Then Fig. 2 can be further divided as indicated in Fig. 4.

Because O_c is a connected graph, some edges must exist between V_4 and V_6 or between V_4 and V_5. Let us delete all edges between V_4 and V_5 and all edges between V_4 and V_6. Then, D_1 and $D_2 = O[V_5 \cup V_6]$ are two *isolated* connected components in the resulting graph.

Consider the graph D_2. Obviously, $\mathrm{VC}(D_2) \leqslant |V_6|$.

(i) $\mathrm{VC}(D_2) < |V_6|$: Since all deleted edges are also covered by V_4, V_4 together with a minimum vertex cover of D_2 must cover all edges of O_c. Thus from (2), we get $\mathrm{VC}(O_c) = |V_4| + \mathrm{VC}(D_2) < |V_4| + |V_6| = k + 1$. This contradicts our definition of an obstruction.

(ii) $\mathrm{VC}(D_2) = |V_6|$: Even if those edges between D_1 and D_2 were deleted, the rest of graph still needs $\mathrm{VC}(D_1 \cup D_2) = |V_4| + |V_6| = k + 1$ vertices to cover (see (2)). This also contradicts our definition of an obstruction.

Therefore, the assumption is incorrect, which means for all $S \subseteq V_2$, $|N(S)| \geqslant |S|$. □

2.2. *Vertex bound for an obstruction of $\mathcal{O}$(k-Vertex Cover)*

As was proved in Theorem 4 of [9], the operation of 'contracting edge(s)' can be omitted for the purpose of checking membership of $\mathcal{O}(k$-VERTEX COVER). Now, we modify our obstruction checking procedure of Section 1.4 to produce an obstruction $O \in \mathcal{O}(k$-VERTEX COVER) from any graph G, with $\mathrm{VC}(G) \geqslant k + 1$, only by deleting edges and isolated vertices of G.

Lemma 2.4. *For any graph G with $\mathrm{VC}(G) \geqslant k + 1$, there always exists a graph $F \in \mathcal{O}(k$-VERTEX COVER) such that $F \subseteq G$ (i.e., F is a subgraph of G).*

Proof. Fig. 5 lists a procedure that constructs an obstruction for k-VERTEX COVER from an input graph G. As mentioned in Section 1, $GA(G)$ returns true if and only if $\mathrm{VC}(G) \leqslant k$. That is, in Fig. 5, the first 'If' decides whether $\mathrm{VC}(G) \leqslant k$ while the second 'If' decides whether $\mathrm{VC}(G') > k$.

Now let us go through the recursive procedure `Generate_O`. First, we input a graph G that satisfies $\mathrm{VC}(G) \geqslant k + 1$.

(i) If G is an obstruction for k-VERTEX COVER, then from condition (1) (see Section 1), we know $\mathrm{VC}(G) > k$ (i.e., the first 'If' is false) and for each edge $e \in E(G)$, $\mathrm{VC}(G \backslash \{e\}) \leqslant k$ (i.e., the second 'If' is always false). Hence the original G is returned.

(ii) If G is not an obstruction for k-VERTEX COVER, then after deleting all isolated vertices from G, there must exist an edge $e \in E(G)$ such that $\mathrm{VC}(G') > k$ where $G' = G \backslash \{e\}$ Recursive calls to `Generate_O`(GA, G') leads to deleting

```
Graph Procedure Generate_O (GraphMembershipAlgorithm GA, Graph G)
    Delete all isolated vertices from G.
    If GA(G) = true then return φ
    For each edge e in G do
        G' = G \ {e}
        If GA(G') = false then
            return G = Generate_O(GA, G')
        endif
    endFor
    return G
end
```

Fig. 5. Procedure to generate an obstruction for k-Vertex Cover.

a sequence of edge(s) of the original G, while always removing isolated vertices, until every edge e of G satisfies $\mathrm{VC}(G\backslash\{e\})\leqslant k$. □

From Lemma 2.4, it is easy to see that the family of graphs k-Vertex Cover can be described by a complete set of forbidden subgraphs.

Corollary 2.5. *A graph $G \in k$-Vertex Cover if and only if for any obstruction O, $O \nsubseteq G$ (i.e., O is not a subgraph of G).*

That is, we can conclude that the set of forbidden subgraphs are the same as the set of forbidden minors for k-Vertex Cover.

In the remaining part of this section we present some properties of $\mathcal{O}(k$-Vertex Cover$)$ and facts about a minimum vertex cover of any obstruction O. Through a partition procedure (see Definition 2.10 and Lemma 2.11) of an obstruction O, we assemble all known statements and lemmas to prove one of the main results of this paper: a more useful upper bound on the order of any connected obstruction for k-Vertex Cover, which appears later as Theorem 2.12.

Lemma 2.6. *Given any edge $(u, v) \in E(O)$, for any minimum vertex cover V' of $O\backslash\{(u, v)\}$, $u \notin V'$ and $v \notin V'$.*

Proof. If not, the vertices of V' can cover the edges of O, which contradicts our definition of an obstruction. □

Lemma 2.7 (*Extension of Dinneen and Xiong [9, Lemma 6] Cattell–Dinneen*). *For any given obstruction O and two arbitrary different vertices $u_1, u_2 \in O$, $N(u_2) \nsubseteq N(u_1)$.*

Proof. We prove this by contradiction. Suppose there exists u_1 and u_2 in O such that

$$N(u_2) \subseteq N(u_1). \tag{3}$$

Without loss of generality, let $\mathrm{degree}(u_1) = j$ and $\mathrm{degree}(u_2) = i$ with $j \geqslant i$. See Fig. 6.

Define: $E' = \bigcup_{t=1}^{i} \{(u_1, v_t) \cup (u_2, v_t)\}$.

Now we delete one edge (u_1, v_t) for any fixed $t \in \{1, 2, \ldots, i\}$. From Lemma 2.6, we know $\{v_1, v_2, \ldots, v_{t-1}, v_{t+1}, \ldots, v_i, u_2\}$ must be contained in any minimum vertex cover V' of $O\backslash\{(u_1, v_t)\}$ for covering all edges of $E'\backslash\{(u_1, v_t)\}$. Hence

1. If $|V'| \leqslant k$, then we define $\widetilde{V} = \{v_t\} \cup V'\backslash\{u_2\}$. $\widetilde{V}$ is vertex cover of O and $|\widetilde{V}| \leqslant k$, which implies $\mathrm{VC}(O) \leqslant k$ (contradicts our definition of an obstruction).
2. If $|V'| \geqslant k+1$, then $\mathrm{VC}(O\backslash\{(u_1, v_t)\}) = |V'| \geqslant k+1$ which also contradicts our definition of an obstruction.

Therefore, the assumption (3) is incorrect and Lemma 2.7 must hold. □

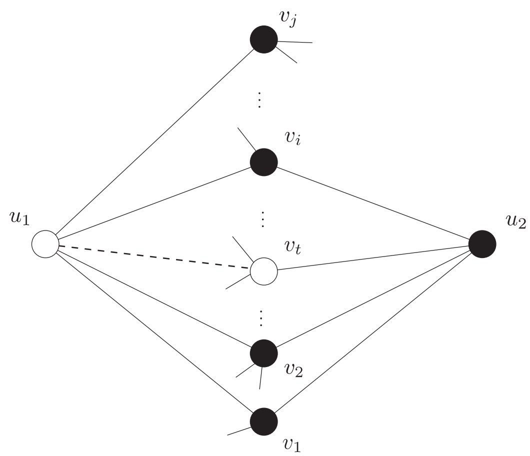

Fig. 6. The set of neighbors $N(u_1)=\{v_1, v_2, \ldots, v_j\}$ and $N(u_2)=\{v_1, v_2, \ldots, v_i\}$.

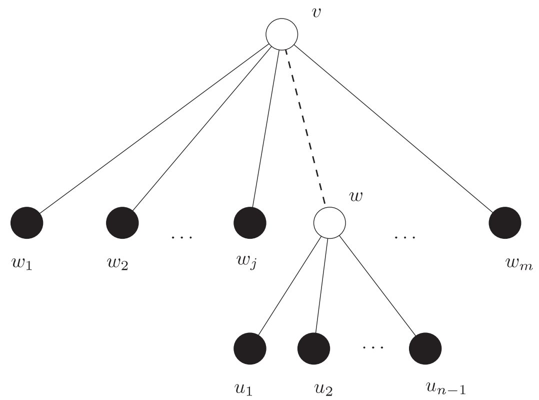

Fig. 7. Edge (w, v) and all neighbors of vertices w and v.

Lemma 2.8. *For any edge $(v, w) \in E(O)$ of an obstruction O*

(1) *There exists a minimum vertex cover V_1 of O, such that $N(v) \cup N(w) \backslash \{v\} \subseteq V_1$.*
(2) *There exists a minimum vertex cover V_1' of O, such that $N(w) \cup N(v) \backslash \{w\} \subseteq V_1'$.*

Proof. Without loss of generality, suppose degree$(v)=m$, degree$(w)=n$ (see Fig. 7). Defined $N(v)=\bigcup_{j=1}^{m}\{w_j\}$, where w_t is marked as w for some $1 \leqslant t \leqslant m$; $N(w)=\{v\} \cup \bigcup_{i=1}^{n-1}\{u_i\}$ (note, some of u_i, w_j might be of superposition in O).

Delete edge (v, w). According to Lemma 2.6, we know: in order to cover all edges (v, w_j) (where $j=1, 2, \ldots, t-1, t+1, \ldots, m$) and (w, u_i) (where $i=1, \ldots, n-1$), for any minimum vertex cover V' of $O\backslash(v, w)$, $\{N(v)\backslash\{w\}\} \cup \{N(w)\backslash\{v\}\} \subseteq V'$. Thus, from our definition of an obstruction, we know $V_1 = V' \cup \{w\}$ is a minimum vertex cover of O (i.e., $N(v) \cup N(w)\backslash\{v\} \subseteq V_1$).

Likewise, $V_1' = V' \cup \{v\}$ is also a minimum vertex cover of the same obstruction O (i.e., $N(v) \cup N(w)\backslash\{w\} \subseteq V_1'$). □

Lemma 2.9. *For any edge $(v, w') \in E(O_c)$ of a connected obstruction (for $k \geqslant 1$), there exists a minimum vertex cover V'' of O_c, such that $\{v, w'\} \subseteq V''$.*

Proof. According to our definition of an obstruction, any O_c for $k \geqslant 1$ contains at least three vertices. We know each O_c is a biconnected graph ([9, Lemma 5]). Hence for each vertex $v \in O_c$, degree$(v) \geqslant 2$. Otherwise, if there exists an $v \in O_c$ such that degree$(v) = 1$, then the single neighbor u of v is a cut-vertex.

Define u to be any vertex in $N(v)\backslash\{w'\}$. Then according to Lemma 2.6, we know that $w' \in V'$ for covering edge (v, w'). Then $\{v, w'\} \subseteq V' \cup \{v\}$ is a desired minimum vertex cover V'' for O_c. □

As mentioned earlier in Section 1, an arbitrary obstruction O is either a connected obstruction for k-VERTEX COVER or the union of more than one connected obstructions for other k'-VERTEX COVER families, $0 \leqslant k' < k$. Thus, for any given O, Lemma 2.9 holds for all edges in $O\backslash H$, where H represents the union of all K_2 components in O. Recall that for the excluded case $k = 0$ of Lemma 2.9, $\mathcal{O}$(0-VERTEX COVER) $=\{K_2\}$.

Now we use the following procedure to partition an arbitrary obstruction O step-by-step so as to find a deeper insight into the structure of O.

Definition 2.10 (*Vertex cover delete procedure (VCDP) for graph G*). Suppose $\widetilde{V} = \{u_1, u_2, \ldots, u_{k+1}\}$ is a minimum vertex cover of graph G.

Define $G_1 = G$

For $i = 1$ **to** $k + 1$
1. delete u_i together with all associated edges $E(u_i)$ in G_i
2. delete any isolated vertices in $G_i\backslash E(u_i)$
3. define the resulting graph as G_{i+1}

endFor

Note each G_i is an induced subgraph of G. For $G = O$, we get $|\widetilde{V}| = k + 1$, $G_i \neq \phi$ $(i = 1, 2, \ldots, k + 1)$ and $G_{k+2} = \phi$. The following Fig. 8 illustrates the *VCDP* procedure for an $O_c \in \mathcal{O}$(3-VERTEX COVER). We name each iteration of the **For loop**, a vertex cover delete (VCD) step.

Lemma 2.11. *At each VCD step of VCDP for an* $O \in \mathcal{O}$(k-VERTEX COVER),

(1) VC$(G_{j+1}) = k - j + 1$, *where* $j \in \{0, 1, \ldots, k\}$,
(2) *there exists* $F \in \mathcal{O}((k - i + 1)$-VERTEX COVER) *such that* $F \subseteq G_i$, *where* $i \in \{1, 2, \ldots, k + 1\}$.

Proof. (1) Because $\widetilde{V}$ is a minimum vertex cover with $|\widetilde{V}| = k + 1$ of $G_1 = O$, the set $\widetilde{V}\backslash\{u_1, u_2, \ldots, u_j\}$ is a vertex cover of G_{j+1}. So VC$(G_{j+1}) \leqslant k + 1 - j$.

If there exists a vertex cover V' with $|V'| =$ VC$(G_{j+1}) < k+1-j$. Then the set $V' \cup \{u_1, u_2, \ldots, u_j\}$ is a vertex cover of G_1, which contains $|V'| + j (< k + 1)$ vertices. This contradicts our assumption that $G_1 \in \mathcal{O}$(k-VERTEX COVER).

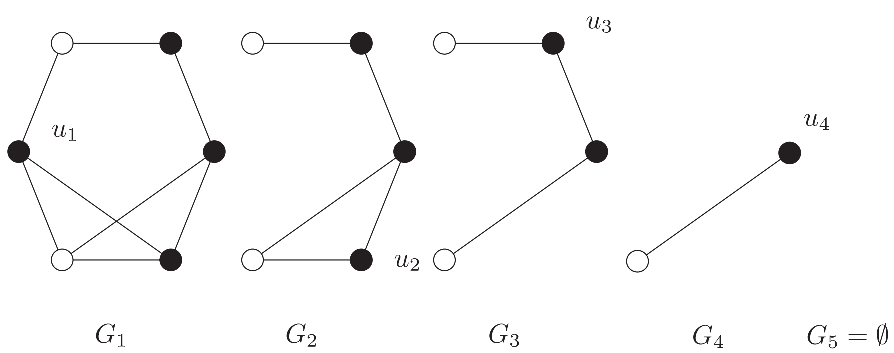

Fig. 8. Each step of *VCDP* for an obstruction O_c of 3-VERTEX COVER.

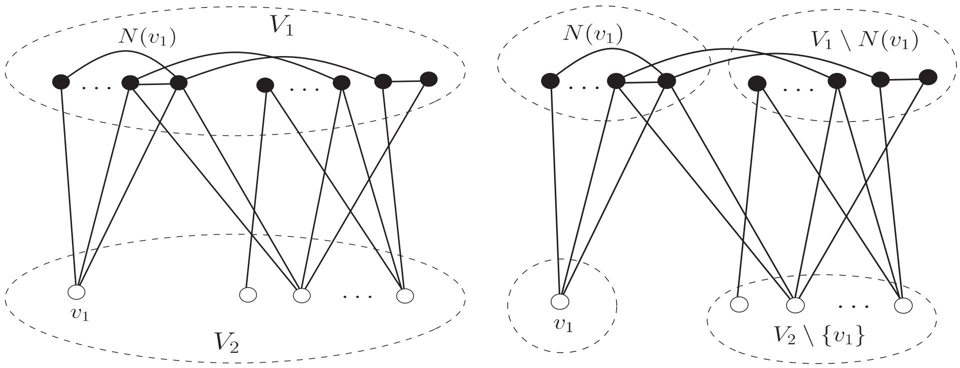

Fig. 9. (a) $N(V_2) = V_1$ and $N(v_1) \subseteq V_1$, $v_1 \in V_2$. (b) Illustration of (5).

(2) From Lemma 2.11(1), let $i = j+1$, we know $\mathrm{VC}(G_i) = k-(i-1)+1 = k-i+2$, where $i \in \{1, 2, \ldots, k+1\}$. Then from Lemma 2.4, we know Lemma 2.11(2) is correct. □

Now we will discuss the first main result of this paper. We will prove an upper bound on the order for all connected obstructions, and then give a vertex bound for all obstructions.

Theorem 2.12. *For any connected obstruction* $O_c \in \mathcal{O}(k\text{-VERTEX COVER})$ *and any* $v_1 \in V(O_c)$,

$$|O_c| \leqslant 2k + 3 - \mathrm{degree}(v_1).$$

Proof. Without loss of generality, for a given O_c and an arbitrary vertex $v_1 \in V(O_c)$, $\mathrm{VC}(O_c) = k+1$ and Lemma 2.8(1) holds for v_1 (i.e., let v_1 denote v and pick any $w \in N(v_1)$ for Lemma 2.8). Then $V(O_c)$ can be split into two subset V_1 and V_2, as indicated in Fig. 9(a), such that V_1 is a minimum vertex cover of size $k+1$, $v_1 \in V_2$, $N(v_1) \subseteq V_1$ and $V_2 = V \backslash V_1$. Obviously no edge exists between any pair of vertices in V_2, otherwise V_1 is not a vertex cover.

$$\text{Each vertex in } V_1 \text{ has at least one vertex in } V_2 \text{ as its neighbor.} \tag{4}$$

Otherwise it can be moved from V_1 to V_2. Namely, this vertex is not needed in the minimum vertex cover set.

From Lemma 2.7, we know for all $u \in \{N(N(v_1)) \backslash \{v_1\}\} \cap V_2$ (i.e., vertices in $V_2 \backslash \{v_1\}$ that are incident on $N(v_1)$), $N(u) \nsubseteq N(v_1)$. That is, there does not exist a vertex in $V_2 \backslash \{v_1\}$ whose neighbors are a subset of $N(v_1)$. Therefore, as illustrated in Fig. 9(b):

$$\text{For all } p \in V_2 \backslash \{v_1\}, \text{ there exists } q \in V_1 \backslash N(v_1), \text{ such that } (p, q) \in E(O_c). \tag{5}$$

We use the VCDP for this O_c to delete $N(v_1)$ in sequence. Then the remaining part is $G_{|N(v_1)|+1}$ (see Definition 2.10).

From (5), we know no vertex in $V_2 \backslash \{v_1\}$ becomes isolated vertex and has been deleted by these VCD steps; likewise, from (4), we know no vertex in $V_1 \backslash N(v_1)$ has been deleted by these VCD steps, because for each vertex of $V_1 \backslash N(v_1)$, there exists at least one neighbor in $V_2 \backslash \{v_1\}$. Hence $N(v_1) \cup \{v_1\} \cup V(G_{|N(v_1)|+1}) = V(O_c)$ and $V(G_{|N(v_1)|+1}) \cap (N(v_1) \cup \{v_1\}) = \phi$, where $V(G_{|N(v_1)|+1}) = \{V_1 \backslash N(v_1)\} \cup \{V_2 \backslash \{v_1\}\}$ (refer to Fig. 9(b)).

Assume $|G_{|N(v_1)|+1}| \leqslant 2(k - |N(v_1)| + 1)$, then $|O_c| \leqslant |N(v_1)| + 1 + (2k - 2|N(v_1)| + 2) = 2k - |N(v_1)| + 3$. Because $\mathrm{degree}(v_1) = |N(v_1)|$, this theorem would be proven.

Now, let us prove the assumption:

$$|G_{|N(v_1)|+1}| \leqslant 2(k - |N(v_1)| + 1). \tag{6}$$

To avoid confusion, we define $\mathrm{Nei}(H)$ to be all neighbors of set $H \subseteq V(G_{|N(v_1)|+1})$ within the graph $G_{|N(v_1)|+1}$ and let $N(H)$ denote all neighbors of set $H \subseteq V(O_c)$ within O_c as usual.

Any subset S of $V_2\backslash\{v_1\}$ in $G_{|N(v_1)|+1}$ can be classified into two categories:

Case 1: Any vertex $v \in S$, $v \notin N(N(v_1))$ (i.e., no vertex in S is a neighbor of $N(v_1)$).

So $N(S)\cup S \subseteq G_{|N(v_1)|+1}$ and $N(S)=\text{Nei}(S)$. From Statement 2.3, we know $|N(S)|\geqslant|S|$ in O_c. Hence $|\text{Nei}(S)|\geqslant|S|$ in graph $G_{|N(v_1)|+1}$.

Case 2: There exists a vertex $v \in S$, such that $v \in N(N(v_1))$.

We will prove that $|\text{Nei}(S)|\geqslant|S|$ must hold in $G_{|N(v_1)|+1}$ as well.

Prove by contradiction: From Lemma 2.11(1), we know $\text{VC}(G_{|N(v_1)|+1}) = k - |N(v_1)| + 1$. Hence $V_1\backslash N(v_1)$ is a minimum vertex cover of $G_{|N(v_1)|+1}$. According to VCDP, each G_i does not contain isolated vertices. From Statement 2.1, we know

$$\text{Nei}(V_2\backslash\{v_1\}) = V_1\backslash N(v_1). \tag{7}$$

From Statement 2.2(1), we know that for a minimum vertex cover $V_1\backslash N(v_1)$ of $G_{|N(v_1)|+1}$, if there exists a subset $S \subseteq V_2\backslash\{v_1\}$ such that $|\text{Nei}(S)| < |S|$, then in $G_{|N(v_1)|+1}$

$$\begin{aligned}&\text{there exists a minimal } V_3,\ V_3 \subseteq S, \text{ such that } |\text{Nei}(V_3)| < |V_3|\\ &\quad\text{and for all } T \subset V_3, |\text{Nei}(T)|\geqslant|T| \text{ (see Fig. 10).}\end{aligned} \tag{8}$$

Obviously, there must exists an $u \in V_3$ such that $u \in N(N(v_1))$ (u may or may not be v, because v is not necessarily included in any critical limit V_3). Otherwise, if for all $u \in V_3$, $u \notin N(N(v_1))$, then for such V_3, $|\text{Nei}(V_3)| < |V_3|$ of (8) which contradicts the above result of Case 1. Thus we can define a vertex w in $N(v_1) \cap N(u)$ (see Fig. 10).

Define $V_3' = V_3\backslash\{u\}$. From (8) and Hall's Marriage Theorem, we know in $G_{|N(v_1)|+1}$ there is a matching of cardinality $|V_3'| = |V_3| - 1$ in the induced bipartite subgraph $D = [V_3' \cup \text{Nei}(V_3')]$.

From Statement 2.2(2), for the graph $G_{|N(v_1)|+1}$, $|\text{Nei}(V_3)| = |\text{Nei}(V_3')| = |V_3| - 1$. Note $N(v_1) \cup N(V_3) = N(v_1) \cup \text{Nei}(V_3)$, because $\text{Nei}(V_3) \subset N(V_3) \subseteq N(v_1) \cup \text{Nei}(V_3)$. Hence, in O_c, if we delete set $A = N(v_1) \cup N(V_3) \subseteq V_1$ and all associated edges, the remaining graph is $G^A_{|N(v_1)\cup N(V_3)|+1} = G^A_{|N(v_1)|+|\text{Nei}(V_3)|+1} = G^A_{|N(v_1)|+|V_3|}$ (see Fig. 10), where superscript A specifies the subset of a minimum vertex cover deleted by VCD steps.

On the other hand, from Lemma 2.8(1), we know that for the defined w (see Fig. 10), there exists a minimum vertex cover V_1' of O_c, such that $\{u\} \cup N(v_1) \subseteq N(v_1) \cup N(w)\backslash\{v_1\} \subseteq V_1'$.

We delete partial minimum vertex cover $B = N(v_1) \cup \{u\}$ in O_c by VCD steps, the remaining graph is $G^B_{|N(v_1)|+2} \subset G_{|N(v_1)|+1}$ (see Fig. 10). For any minimum vertex cover of $G^B_{|N(v_1)|+2}$, in order to cover the matching of cardinality $|V_3| - 1$ within $D(\subseteq G^B_{|N(v_1)|+2})$, at least $|V_3| - 1$ vertices are needed inevitably.

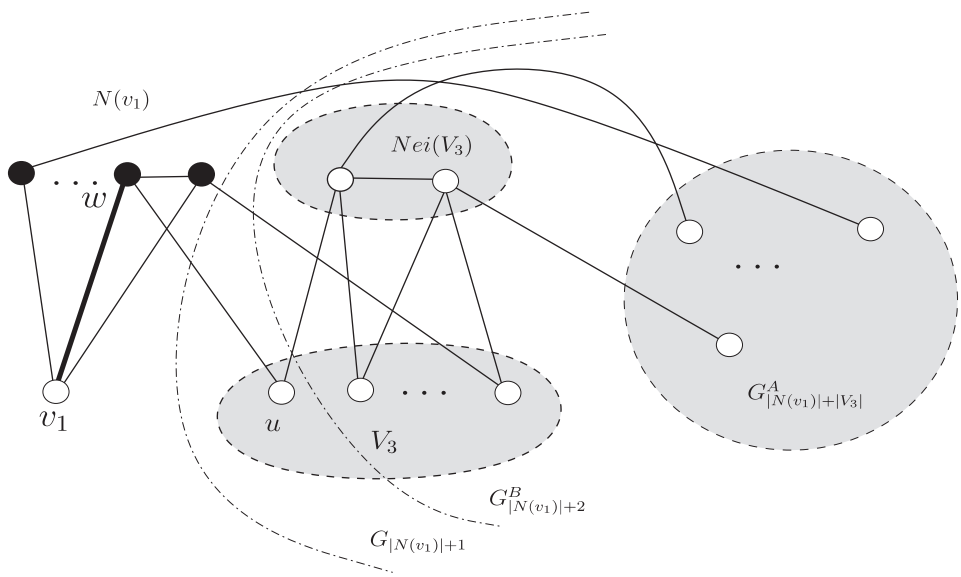

Fig. 10. Decompose O_c by VCD steps. The dashed lines (excluding end points) represent the scope of sets while the other lines represent edges.

Further, we delete the $|V_3|-1$ vertices $C=\{c_1,c_2,\ldots,c_{|V_3|-1}\}$, which is a subset of a minimum vertex cover of $G^B_{|N(v_1)|+2}$, where c_i is picked from the end points of ith independent edge of the matching. The resulting graph is $G^{B\cup C}_{|N(v_1)|+|V_3|+1}(\supseteq G^A_{|N(v_1)|+|V_3|})$, where $G^{B\cup C}_{|N(v_1)|+|V_3|+1}=G^A_{|N(v_1)|+|V_3|}$ if $c_i\in V_1$ holds for all $i=1,2,\ldots,|V_3|-1$. Note deleting any vertices in V_3 will not affect $G^A_{|N(v_1)|+|V_3|}$, because $N(V_3)\cap V(G^A_{|N(v_1)|+|V_3|})=\phi$ (see definition of A).

However, according to Lemma 2.11(1), we know $\mathrm{VC}(G^{B\cup C}_{|N(v_1)|+|V_3|+1})<\mathrm{VC}(G^A_{|N(v_1)|+|V_3|})$. When $G^{B\cup C}_{|N(v_1)|+|V_3|+1}\supset G^A_{|N(v_1)|+|V_3|}$ holds, the contradiction appears that the bigger graph has a smaller minimum vertex cover. When $G^{B\cup C}_{|N(v_1)|+|V_3|+1}=G^A_{|N(v_1)|+|V_3|}$ holds; there is a contradiction on the definition of a *minimum vertex cover*.

Therefore, in graph $G_{|N(v_1)|+1}$, any subset $S\subseteq V_2\backslash\{v_1\}$ of Case 2, $|\mathrm{Nei}(S)|\geqslant|S|$.

When we synthesize Case 1 and Case 2, we conclude that any subset S of $V_2\backslash\{v_1\}$ in $G_{|N(v_1)|+1}$, $|\mathrm{Nei}(S)|\geqslant|S|$. Particularly, let $S=V_2\backslash\{v_1\}$, from (7), we get

$$|V_2\backslash\{v_1\}|\leqslant|V_1\backslash N(v_1)|=k+1-|N(v_1)|.$$

Therefore, the above (6) holds due to

$$|G_{|N(v_1)|+1}|=|V_2\backslash\{v_1\}|+|V_1\backslash N(v_1)|\leqslant 2(k+1-|N(v_1)|).\qquad\square$$

Note Theorem 10 of [9] (i.e., $|O_c|\leqslant 2k+1$) is a special case of Theorem 2.12, because for any $v\in O_c$ of $\mathcal{O}(k$-VERTEX COVER) with $k\geqslant 1$, we have degree$(v)\geqslant 2$ (because O_c is biconnected).

As mentioned in Section 1.3, any disconnected obstruction O_d is a union of connected obstructions for smaller values of k: $O_d=\bigcup_{j=1}^{s}G_j$, where $p_j=\mathrm{VC}(G_i)-1$ and $\sum_{j=1}^{s}(p_j+1)=k+1$. So

$$\begin{aligned}|O_d|&=\sum_{j=1}^{s}|G_j|\\&\leqslant\sum_{j=1}^{s}(2p_j-\mathrm{degree}(v_j)+3)\\&=2(k+1)+\sum_{j=1}^{s}(1-\mathrm{degree}(v_j)),\quad\text{where } v_j\in V(G_j)\\&\leqslant 2k+2+1-\mathrm{degree}(v_1),\quad(\text{note for any } v_j\in V(G_j),\ \mathrm{degree}(v_j)\geqslant 1)\\&=2k+3-\mathrm{degree}(v_1).\end{aligned}$$

We can name any connected component of O_d to be the first connected obstruction G_1. Thus, we get an uniform vertex bound for any $O\in\mathcal{O}(k$-VERTEX COVER),

$$|O|\leqslant 2k-\mathrm{degree}(v_s)+3\quad\text{for all } v_s\in V(O).\tag{9}$$

The upper bound for all $O\in\mathcal{O}(k$-VERTEX COVER) is

$$|O|\leqslant 2k-\max_{v_s\in V(O)}\{\mathrm{degree}(v_s)\}+3.\tag{10}$$

Corollary 2.13. *If there exists a vertex $v_s\in V(O)$ with degree$(v_s)=k$, then $|O|=k+3$.*

Proof. From (9), we know for such an obstruction O, $|O|\leqslant 2k-k+3=k+3$.

It is proved in Lemma 8 of [9] that for any obstruction O, $|O|\geqslant k+2$ and $|O_c|=k+2$ if and only if O_c is K_{k+2} (i.e., a complete graph with $k+2$ vertices). Moreover, any disconnected $O_d\in\mathcal{O}(k$-VERTEX COVER) with $k+2$ vertices must be a subgraph of connected obstruction K_{k+2}, which is a contradiction. So for any O_d, $|O_d|>k+2$. Thus Lemma 8 of [9] can be stated as following:

$$\text{For any obstruction } O,\ |O| \geqslant k+2 \text{ and } |O| = k+2 \text{ if and only if } O \text{ is } K_{k+2}. \tag{11}$$

So $|O| = k+3$, if there exists $v_s \in V(O)$ with degree$(v_s) = k$. □

Obviously from (11), we also know that in an obstruction O, if there is a vertex whose degree equals k, then k must be the maximum degree of this obstruction. From (10), (11) and Corollary 2.13, we set up an upper bound and lower bound for all $O \in \mathcal{O}(k\text{-VERTEX COVER})$:

$$\begin{cases} k+3 \leqslant |O| \leqslant 2k - \text{maxDegree}(O) + 3 & \text{if maxDegree}(O) \leqslant k, \\ O = K_{k+2} & \text{if maxDegree}(O) = k+1. \end{cases}$$

2.3. The cycle conjecture confirmed

Theorem 2.12 also leads to another nice result which was first proposed as Conjecture 12 of [9]. The main idea of the following proof is to filter the redundant constructional possibilities by Theorem 2.12.

Theorem 2.14. *The cycle C_{2k+1} is the only (and largest) connected obstruction for the graph family k-VERTEX COVER, where $k \geqslant 1$.*

Proof. We have to prove two things:

(1) C_{2k+1} is in $\mathcal{O}(k\text{-VERTEX COVER})$.
(2) C_{2k+1} is the only and largest connected obstruction with $2k+1$ vertices.

Because each vertex $v \in V(C_{2k+1})$ is of degree 2, and k vertices in C_{2k+1} can cover at most $2k$ edges, there is still one edge uncovered. Hence $\text{VC}(C_{2k+1}) = k+1$.

We mark vertices of C_{2k+1}, as $v_1, v_2, \ldots, v_{2k+1}$ in sequence, then $\{v_1, v_2, v_4, v_6, \ldots, v_{2k}\}$ is a minimum vertex cover of C_{2k+1}. For each edge $e \in E(C_{2k+1})$, the graph $C_{2k+1} \backslash \{e\}$ is isomorphic to a path P_{2k+1}. We need at least k vertices to cover the $2k$ edges of P_{2k+1}. Hence $\text{VC}(P_{2k+1}) = k$.

Thus, from our definition of an obstruction, we know $C_{2k+1} \in \mathcal{O}(k\text{-VERTEX COVER})$.

From Theorem 10 of [9] (i.e., $|O_c| \leqslant 2k+1$), we know C_{2k+1} is the largest connected obstruction of k-VERTEX COVER. Now, we prove C_{2k+1} is the only one with $2k+1$ vertices.

Theorem 2.12 states that for all $v \in V(O_c)$, $|O_c| \leqslant 2k - \max\{\text{degree}(v)\} + 3$. This implies: if $\max\{\text{degree}(v)\} \geqslant 3$, then $|O_c| \leqslant 2k$. Hence, for all O_c, if $|O_c| = 2k+1$, then for all $v \in O_c$, degree$(v) \leqslant 2$. Note for all $v \in V(O_c)$ with $k \geqslant 1$, degree$(v) \geqslant 2$, since O_c is biconnected. Then we know that for any connected graph $G \in \mathcal{O}(k\text{-VERTEX COVER})$, if $|G| = 2k+1$, then

$$\text{for all } v \in V(G), \quad \text{degree}(v) = 2. \tag{12}$$

Since the graph G is connected we see that it must be a cycle. Hence C_{2k+1} is the unique connected graph with $2k+1$ vertices that satisfies (12). Recall all connected graph $G \in \mathcal{O}(k\text{-VERTEX COVER})$ with $2k+1$ vertices must satisfy (12). Thus C_{2k+1} is the only connected obstruction with $2k+1$ vertices. □

3. Generating obstructions of *k*-VERTEX COVER

In this section, we introduce two methods, namely extension method 1 and extension method 2, which generate graphs in $\mathcal{O}((k+1)\text{-VERTEX COVER})$ by transforming any graph in $\mathcal{O}(k\text{-VERTEX COVER})$ in constant time.

Definition 3.1 (*L transformation*). For a graph G, replacing any single edge of G with a path of length 3 (see Fig. 11), and keep the remaining part of G be unchanged. Let $L(G)$ denote the resulting graph.

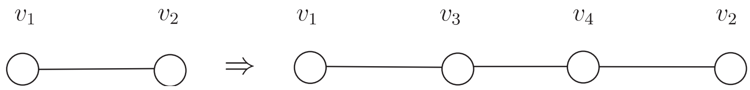

Fig. 11. An edge (v_1, v_2) of G before the L transformation and then afterwards.

Extension method 1: For any connected obstruction O_c for k-VERTEX COVER, ($k \geqslant 1$), the transformed graph $L(O_c)$ is in $\mathcal{O}((k+1)$-VERTEX COVER).

Explanation: Obviously, L transformation is transitive. In other words, applying the L transformation on O_c, t times, the resulting graph is in $\mathcal{O}((k+t)$-VERTEX COVER). If the L transformation is applied on symmetric edges of an O_c, then the resulting graphs are isomorphic.

Proof. Referring to Definition 3.1, we pick an edge from a given O_c and name it (v_1, v_2). Obviously $O_c \backslash \{(v_1, v_2)\} = L(O_c) \backslash \{(v_1, v_3), (v_3, v_4), (v_4, v_2), v_3, v_4\}$ for two new vertices v_3 and v_4.

According to our definition of an obstruction there exists a minimum vertex cover V' with $|V'| = k$ of $O_c \backslash \{(v_1, v_2)\}$, such that $v_1, v_2 \notin V'$ (see Lemma 2.6); there exists a minimum vertex cover V'' with $|V''| = k+1$ of O_c, such that $v_1, v_2 \in V''$ (see Lemma 2.9). Any minimum vertex cover V''' with $|V'''| = k$ of $O_c \backslash \{e\}$ (where $e \neq (v_1, v_2)$) must be in one of the three different cases: (1) $v_1 \in V'''$, $v_2 \notin V'''$; (2) $v_1 \notin V'''$, $v_2 \in V'''$; (3) $v_1 \in V'''$, $v_2 \in V'''$. Note: To cover edge (v_1, v_2) of $O \backslash \{e\}$, at least one of $\{v_1, v_2\}$ must be in V'''.

Now we prove $L(O_c) \in \mathcal{O}((k+1)$-VERTEX COVER).

1. $\mathrm{VC}(L(O_c)) \leqslant k+2$, because $V' \cup \{v_3, v_4\}$ cover the edges of $L(O_c)$ (see Fig. 11): Suppose, there is a set $\widetilde{V}$ of $k+1$ (or less) vertices to cover $E(L(O_c))$. In order to cover (v_3, v_4) of $L(O_c)$:
   (a) Both v_3 and v_4 are in $\widetilde{V}$: the remaining $k-1$ (or less) vertices $\widetilde{V} \backslash \{v_3, v_4\}$ cover the edges of $L(O_c) \backslash \{(v_1, v_3), (v_3, v_4), (v_4, v_2)\} = O_c \backslash \{(v_1, v_2)\} \cup \{v_3\} \cup \{v_4\} \supset O_c \backslash \{(v_1, v_2)\}$, which contradicts $\mathrm{VC}(O_c \backslash \{(v_1, v_2)\}) = k$.
   (b) Only one of $\{v_3, v_4\}$ is in $\widetilde{V}$ (generally assume it is v_3): the remaining vertices $\widetilde{V} \backslash \{v_3\}$ cover the edges of $L(O_c) \backslash \{(v_1, v_3), (v_3, v_4)\}$. Because $v_4 \notin \widetilde{V}$, in order to cover (v_4, v_2), we know $v_2 \in \widetilde{V}$. Therefore, these k (or less) vertices $\widetilde{V} \backslash \{v_3\}$ cover $E(O_c)$, which contradicts $\mathrm{VC}(O_c) = k+1$.

   Thus $\mathrm{VC}(L(O_c)) = k+2$.
2. Delete any edge e in $E(L(O_c))$: $e = (v_1, v_3)$: $\mathrm{VC}(L(O_c) \backslash \{e\}) \leqslant k+1$, because $V' \cup \{v_4\}$ covers $E(L(O_c)) \backslash \{e\}$,
   $e = (v_4, v_2)$: $\mathrm{VC}(L(O_c) \backslash \{e\}) \leqslant k+1$, because $V' \cup \{v_3\}$ covers $E(L(O_c)) \backslash \{e\}$,
   $e = (v_3, v_4)$: $\mathrm{VC}(L(O_c) \backslash \{e\}) \leqslant k+1$, because V'' covers $E(L(O_c)) \backslash \{e\}$,
   $e \neq \{(v_1, v_3), (v_3, v_4), (v_4, v_2)\}$: according to the above three different cases of possible minimum vertex cover V''' of $O_c \backslash \{e\}$, we construct a vertex cover for $L(O_c) \backslash \{e\}$ where: (1) $V''' \cup \{v_4\}$ covers $E(L(O_c)) \backslash \{e\}$; (2) $V''' \cup \{v_3\}$ covers $E(L(O_c)) \backslash \{e\}$; (3) $V''' \cup \{v_3\}$ covers $E(L(O_c)) \backslash \{e\}$.
   Hence $\mathrm{VC}(L(O_c) \backslash \{e\}) \leqslant k+1$ in all cases.

Obviously, there is no isolated vertices involved in the L transformation. Referring to an equivalent form of condition (b) (i.e., condition (1)) of our definition of an obstruction for k-VERTEX COVER in Section 1.4, we conclude $L(O_c) \in \mathcal{O}((k+1)$-VERTEX COVER). □

Extension method 2: For any obstruction $O = (V, E)$ for k-VERTEX COVER and $v \in V(O)$, the constructed graph $G = (V', E')$ where $V' = V \cup \{v'\}$ (a new vertex $v' \notin V$) and $E' = E \cup \{(v, v')\} \cup \{(v', u) \mid u \in N(v)\}$ is in $\mathcal{O}((k+1)$-VERTEX COVER) (see Fig. 12).

Proof. We prove this in terms of our definition of an obstruction.

(1) $\mathrm{VC}(G) = k+2$: Any minimum vertex cover $\widetilde{V}$ of O cannot cover all edges adjacent to v' in G, namely $E(v')$. Otherwise, in order to cover $E(v') = \{(v, v')\} \cup \{(u, v') \mid u \in N(v)\}$ in G, both v and $N(v)$ must be contained in a certain minimum vertex cover $\widetilde{V}$ of O. Therefore, k vertices $\widetilde{V} \backslash \{v\}$ cover O, which is a contradiction. From Lemma 2.8(1), there exists a minimum vertex cover V_1 of O, such that $N(v) \subseteq V_1$ and $v \notin V_1$. So $k+2$ vertices $\{v'\} \cup V_1$ is a vertex cover of G (i.e., $\mathrm{VC}(G) \leqslant k+2$). Now we prove $\mathrm{VC}(G) > k+1$ by contradiction.

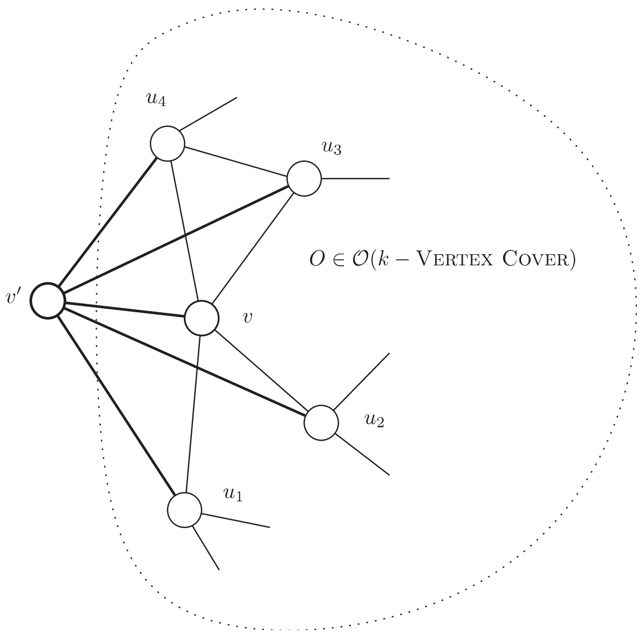

Fig. 12. Illustrating extension method 2.

Suppose there exist $k+1$ (or less) vertices U to cover the edges of G:

1. If $v' \notin U$ then $\{v\} \cup N(v) \subseteq U$. So k (or less) vertices $U \backslash \{v\}$ cover $E(O)$, which contradicts $\mathrm{VC}(O) = k+1$.
2. If $v' \in U$ then the remaining k (or less) vertices $U \backslash \{v'\}$ cover $E(O)$, which also contradicts our definition of an obstruction for k-VERTEX COVER.

Therefore, $\mathrm{VC}(G) = k+2$.

(2) For any $e \in E'$, $\mathrm{VC}(G \backslash \{e\}) = k+1$:

1. $e \in E(v')$: For any $u \in N(v)$, from Lemma 2.6, we know there exists a minimum vertex cover V' of $O \backslash \{(u, v)\}$ with $u, v \notin V'$, so $N(v) \backslash \{u\} \subseteq V'$ for covering each edge that is incident to v in $O \backslash \{(u, v)\}$. If $e = (v', u)$, $V' \cup \{v\}$ is a minimum vertex cover of O, which also covers $E(G) \backslash \{e\}$. Similarly, if $e = (v', v)$, $V' \cup \{u\}$ is a minimum vertex cover of O, which also covers $E(G) \backslash \{e\}$. So, $\mathrm{VC}(G \backslash \{e\}) \leqslant k+1$. Because $O \subseteq G \backslash \{e\}$, $\mathrm{VC}(G \backslash \{e\}) \geqslant \mathrm{VC}(O) = k+1$. Hence $\mathrm{VC}(G \backslash \{e\}) = k+1$.
2. $e \in E$ (i.e., any edge of O): For any minimum vertex cover $\widetilde{V}_1$ of $O \backslash \{e\}$, the $k+1$ vertices $\{v'\} \cup \widetilde{V}_1$ cover $G \backslash \{e\}$. Hence $\mathrm{VC}(G \backslash \{e\}) \leqslant k+1$. One the other hand, $O \backslash \{e\} \subseteq G \backslash \{e\}$. Hence $\mathrm{VC}(G \backslash \{e\}) \geqslant \mathrm{VC}(O \backslash \{e\}) = k$. Further, $\mathrm{VC}(G \backslash \{e\}) \neq k$. Otherwise, suppose $e = (v_1, v_2)$ and $\widetilde{V}''$ with $|\widetilde{V}''| = k$ covers the edges of $G \backslash \{e\}$, then $k+1$ vertices $\{v_1\} \cup \widetilde{V}''$cover G. (Contradicts the above analysis results: (1) $\mathrm{VC}(G) = k+2$.) Hence $\mathrm{VC}(G \backslash \{e\}) = k+1$.

There is no isolated vertices involved in extension method 2. Therefore, we conclude the new graph G is in $\mathcal{O}((k+1)$-VERTEX COVER). □

From the computations of $\mathcal{O}(k$-VERTEX COVER) that have been done for $k \leqslant 6$, we see that most of the connected obstructions are obtained by using one of these two extension methods. In fact, for $k = 6$ only 15% (28/188) of the connected obstructions are not found this way. A natural question comes up if one can find a sufficient set of extension methods to find all $\mathcal{O}(k$-VERTEX COVER), whenever we have all the obstructions for smaller families in the vertex cover hierarchy. A starting question is the following.

Question. Given any connected obstruction $O_c \in \mathcal{O}((k+1)$-VERTEX COVER), is there always an $O'_c \in \mathcal{O}(k$-VERTEX COVER) obtained from O_c by applying a sequence of edge contractions?

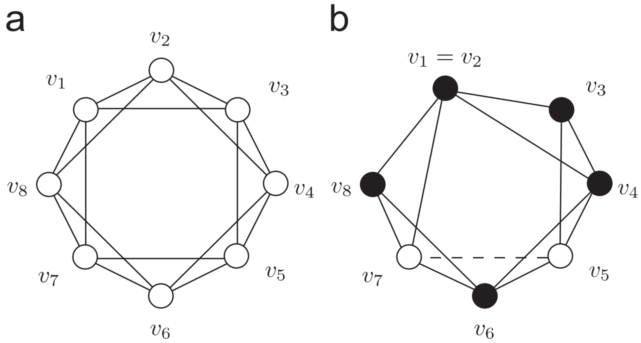

Fig. 13. (a) An $O \in \mathcal{O}$(5-Vertex Cover) and (b) after contracting edge (v_1, v_2).

Unfortunately, the answer to this question is "No". This means that extension methods that only expand edges (like extension methods 1 and 2) could not generate all of $\mathcal{O}(k$-Vertex Cover) from the set $\mathcal{O}((k-1)$-Vertex Cover). Any further extension methods must consist of more sophisticated operations of adding edges and vertices.

Counterexample. Let O_c be the graph displayed in Fig. 13(a), which is in $\mathcal{O}$(5-Vertex Cover). However, after contracting any edges of it, the resulting graph will not be in $\mathcal{O}$(4-Vertex Cover).

Proof. Analysis by way of symmetry. All cases of contracting edges can be classified into three categories as following:

(1) Contract one edge:

1. Contracting edge (v_1, v_2), we get the graph in Fig. 13(b), which is not in $\mathcal{O}$(4-Vertex Cover). Otherwise, if we delete edge (v_5, v_7), from Lemma 2.6, $N(v_5) \cup N(v_7)$ should be in any minimum vertex cover of resulting graph. But, there are five vertices, which contradicts our definition of O_c' being an obstruction of 4-Vertex Cover.
2. Contract edge (v_1, v_3). Similar analysis (i.e., delete edge (v_4, v_6)) will show that the resulting graph is not a member of $\mathcal{O}$(4-Vertex Cover) either.

(2) Contract any two edges e_1 and e_2:

All resulting graphs are of order 6, because each contraction reduces the order by one. However, the contract edge operations will not change the degrees of the vertices that are not involved. Thus, it must not be K_6, which is the only obstruction of 4-Vertex Cover of order 6 (recall claim (11) of Section 2.2). We know none of them is in $\mathcal{O}$(4-Vertex Cover), because all of them would be proper subgraphs of K_6.

(3) If we contract more than two edges, then the order of the resulting graph is strictly less than 6. Again all of them are proper subgraphs of K_6, so they are not in $\mathcal{O}$(4-Vertex Cover) as well. □

We end this section by mentioning that, using these two extension methods, we have computed a new lower bound on the size of $\mathcal{O}$(7-Vertex Cover): there are at least 1503 connected obstructions to go along with the exact count of 320 disconnected obstructions.

4. Conclusion

In this paper our main contributions are the following: (1) we confirmed a conjecture that there is an unique largest connected obstruction for each k-Vertex Cover, (2) established that the minor-order obstructions for k-Vertex Cover can be equivalently viewed as a finite set of forbidden subgraphs, and (3) presented two simple iterative methods for producing many obstructions for k-Vertex Cover.

In our quest to understand the properties of the vertex cover obstructions we have also discovered several areas to continue the study. First, can we exploit our new vertex bound (based on maximum degree) for obstructions of k-Vertex Cover (e.g. is the case for $k = 7$ now approachable)? Secondly, it would be nice to extend the number of available extension methods to generate more (if not all) obstructions within the vertex cover hierarchy of graph families. A final area of research, is to see if we can better characterize k-VERTEX COVER (or other graph families) by obstructions with respect to other graph partial orders.

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