# Algebraic Methods for Efficient Network Constructions

by

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to the required standard

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# ABSTRACT

The design of computer networks and parallel processor configurations is a topic of increasing importance. Network designs which efficiently support communications between nodes are crucial for many applications. Cost and physical limitations generally prevent the nodes in a network from having more than a fixed number of hardware connections to other nodes (that is, the nodes must have bounded degree). This fundamental constraint makes the design problem nontrivial. The topic of this thesis is an explanation of ways in which group theory can be used to design bounded-degree communication-efficient networks. Our methods have yielded a number of network designs that are the largest known for networks satisfying specified bounds on node degree and either diameter or broadcast time, for values of these parameters that are in the range of potential engineering significance.

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# Chapter 1

# Introduction

This thesis explores uses of group theory in the design of interconnection networks and multi-processor configurations. The progress of VLSI technology provides the hardware platform for ever-larger connected systems. Several fundamental design problems that deal with the topology of networks have emerged. A variety of design methodologies that address these problems have been studied [ABR, AHK, BE, BDQ, Wi]. The techniques developed in this thesis provide many of the presently best-known constructions for these design problems.

A basic constraint in many network design problems is a bound on the maximum node degree that is imposed by cost and fundamental engineering limitations. That is, network nodes can have at most a fixed number of communication lines connected to other nodes. At the same time, efficient network communications are crucial for many applications. We develop group-theoretic methods for designing large networks satisfying these practical constraints. Two basic design problems for which we provide record-breaking constructions are the following.

1. *The Degree/Diameter Problem.* Provide constructions of the largest possible networks satisfying bounds on maximum node degree and diameter. The diameter measures the maximum communication delay between any two nodes in a network. If each node can communicate simultaneously with all of its neighbors then the diameter also gives the maximum time needed to flood a message throughout the network.

2. *The Degree/Broadcast-Time Problem.* Provide constructions of the largest possible networks satisfying bounds on maximum node degree and broadcast time. In these networks a node can communicate with only one of its neighbors at a

time. Under this restriction the broadcast time is the maximum time needed for any node to disseminate a message throughout the network.

Generally a network's diameter is smaller than its broadcast time. This is intuitively clear since communications are one to many (for diameter) verses one to one (for broadcast time). Figure 1.1 shows the distinction between these two concepts in a simple ring architecture.

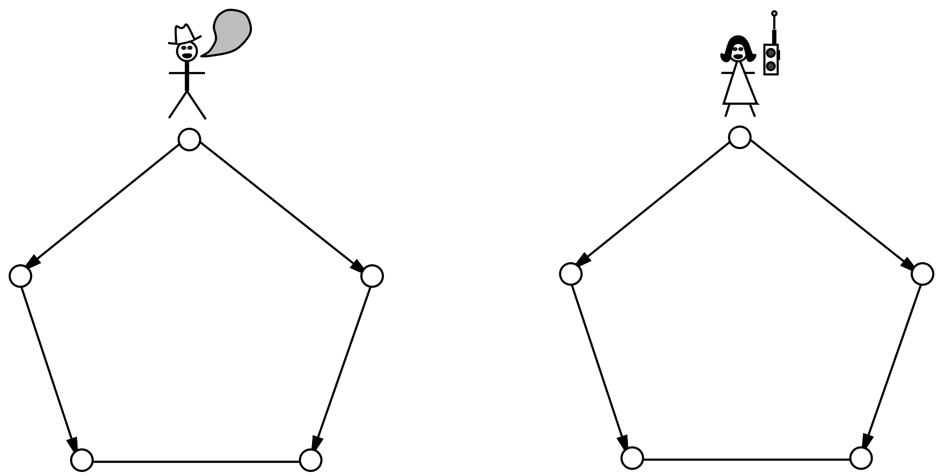

Figure 1.1: A comparison between (a) diameter and (b) broadcast time.

There are many advantages of using group theory in the design of connected systems. For one thing, our approach yields networks with the nice property of node symmetry. This allows message routing schemes to be node independent. For massive parallel-processors symmetry is a valuable, natural and useful organizational tool for meeting the difficult challenges of coordinating large number of computational units. Many of the developed (or proposed) parallel processors are node symmetric. In addition, most (!) node symmetric connected systems are (implicitly) based on Cayley (group) graphs. We will explore some of these group-theoretic descriptions. Other advantages of networks designed using group theory may include: (1) line symmetry, (2) hierarchical structure, and/or (3) high fault tolerance.

We view multi-processor configurations and interconnection networks in terms of graph theory where the vertices represent processors or nodes, and the edges represent connecting wires or communication lines. The Degree/Diameter problem for both directed and undirected networks is discussed in the next two chapters, and the

Degree/Broadcast-Time problem is investigated in Chapter 4. The last chapter briefly describes our algorithm for finding efficient networks based on groups. The remainder of this chapter reviews some basic concepts of algebra and graph theory.

## 1.1 Graph Theoretic Preliminaries

For convenience, we review several key concepts of graph theory that are useful in the sequel. Most of our definitions follow those in [CL]. See also [Ha].

**Definition:** A **graph** $G = (V, E)$ is a finite nonempty set $V$ of objects called **vertices** (the singular is **vertex**) together with a (possibly empty) set $E$ of unordered pairs of distinct vertices of $G$ called **edges**.

**Definition:** A **digraph** $G = (V, E)$ is a finite nonempty set $V$ of vertices together with a (possibly empty) set $E$ of ordered pairs of distinct vertices of $G$ called **arcs**.

**Definition:** The **order** of a graph (digraph) $G = (V, E)$ is $|V|$, sometimes denoted by $|G|$, and the **size** of this graph is $|E|$.

**Definition:** A **walk** in a graph (digraph) $G$ is a sequence of vertices $v_0 v_1 \ldots v_n$ such that for all $0 \leq i < n$ $(v_i, v_{i+1})$ is an edge (arc) in $G$. The **length** of the $v_0 v_n$-walk is the number $n$. A **path** is a walk in which no vertex is repeated.

**Definition:** A graph $G$ is **connected** if there is a path between vertices $u$ and $v$ for all $u$ and $v$ in $G$. A digraph $G$ is **strongly connected** if there is a path from vertex $u$ to vertex $v$ for all $u$ and $v$ in $G$.

In this thesis we are concerned only with connected graphs and strongly connected digraphs. Henceforth, we only use the terms *connected* and *strongly connected* when emphasis is needed.

**Definition:** In a graph, the **degree** of a vertex $v$, denoted by $deg(v)$, is the number of edges incident to $v$. For digraphs, the **out-degree** of a vertex $v$ is the number of arcs incident from $v$ and the **in-degree** of vertex $v$ is the number of arcs incident to $v$.

**Definition:** The **diameter** of a connected graph (strongly connected digraph) $G = (V, E)$ is the least integer $D$ such that for all vertices $u$ and $v$ in $G$ we have $d(u, v) \leq D$, where $d(u, v)$ denotes the **distance** from $u$ to $v$ in $G$, that is, the length of a shortest $uv$-path.

**Definition:** A $(\Delta, D)$ **graph** is a graph $G = (V, E)$ such that: (1) $deg(v) \leq \Delta$ for all vertices $v$ in $V$ and (2) graph $G$ has diameter less than or equal to $D$. A $(\Delta, D)$ **digraph** is similarly defined; in this case, all in-degrees and out-degrees must be bounded by $\Delta$. A $(\Delta, D)$ graph (digraph) is **optimal** if it has the maximum order possible for $(\Delta, D)$ graphs (digraphs).

In Chapter 4 we will similarly define $(\Delta, T)$ graphs for another graph invariant, the broadcast time $T$.

**Definition:** A graph (digraph) $G = (V, E)$ is **vertex symmetric** (or vertex transitive) if for each pair of vertices $u$ and $v$ in $G$ there exist a bijection $\theta : V \to V$ such that: (1) $\theta(u) = v$ and (2) $(a, b) \in E$ implies $(\theta(a), \theta(b)) \in E$, that is, $\theta$ preserves adjacencies. In other words, there is an automorphism of $G$ taking $u$ to $v$.

For an extensive annotated bibliography on the subject of symmetries in graphs see [FK].

## 1.2 Algebraic Preliminaries

This section presents the necessary background in algebra. The books [Fr, Ro, Sc] are standard references for the subject.

**Definition:** A **group** $(G, *)$ is a set $G$, together with a binary operation $*$ on $G$, such that the following axioms are satisfied:

1. The binary operation $*$ is associative. That is, for all elements $x, y$, and $z$ in $G$ we have $(x * y) * z = x * (y * z)$.

2. There is an element $e$ in $G$ such that $e * x = x * e = x$ for all elements $x$ in $G$. (This element $e$ is an **identity** element for $*$ on $G$.)

3. For each $x$ in $G$, there is an element $x^{-1}$ in $G$ with the property that $x^{-1} * x = x * x^{-1} = e$. (The element $x^{-1}$ is an **inverse** of $x$ with respect to $*$.)

A group $G$ is **abelian** if it is also commutative, that is $x * y = y * x$ for all elements $x$ and $y$ in $G$.

**Definition:** An element $x$ of a group $G$ with identity $e$ has **order** $r > 0$ if $x^r = e$ and no smaller positive power of $x$ is the identity. Further, if $r = 2$ then the element $x$ is an **involution**. The group $G$ is **cyclic** if some element (generator) has order $|G|$.

**Definition:** A group $(H, *)$ is a **subgroup** of a group $(G, *)$ if $H \subseteq G$ and the binary operator $*$ for $H$ is induced from $G$. The **index** of a subgroup $H$ of a finite group $G$ is $|G|/|H|$.

**Definition:** A **(commutative) ring** $(R, +, *)$ is a set $R$ together with two binary operations $+$ and $*$, which we call addition and multiplication, defined on $R$ such that the following axioms are satisfied:

1. $(R, +)$ is an abelian group.

2. Multiplication is associative (and commutative).

3. For all elements $x$, $y$, and $z$ in $R$, we have $x(y+z) = (xy) + (xz)$ and $(x+y)z = (xz) + (yz)$.

A multiplicative identity in a ring is an **unity** element.

**Definition:** Let $R$ be a ring with unity. An element $u$ in $R$ is a **unit** of $R$ if it has a multiplicative inverse in $R$. A **field** $(F, +, *)$ is a commutative ring with unity and every non-identity element, $F \setminus \{e\}$, of $F$ is a unit.

**Definition:** The units of a ring $R$ form the **group of units** of $R$ under multiplication and we denote this group by $U(R)$. If a generator exists for the group $U(R)$ then it is called a **primitive root**.

The network constructions described in this thesis use only a few types of groups. These finite groups are next described.

**Definition:** Given a finite field, $F = GF(p^i)$, the $n \times n$ nonsingular (invertible) matrices form the **general linear group**, $GL[n, F]$, under matrix multiplication.

**Definition:** The **symmetric group** $S_n$ is the collection of permutations of the set $\{0, 1, \ldots, n-1\}$ where group multiplication is defined by composition of permutations.

The next group construction is useful in designing efficient networks.

**Definition:** Let $(A, \cdot)$ and $(B, \circ)$ be groups, and suppose there exists a homomorphism $\sigma : A \rightarrow Aut(B)$ of A into the group of automorphisms of B. The set of all ordered pairs $\{(a, b) : a \in A, b \in B\}$ can be made into a group if we define products by

$$(a_1, b_1) * (a_2, b_2) = (a_1 \cdot a_2, b_1 \circ (\sigma(a_1))(b_2))$$

This group is called the **semi-direct product** of A and B, relative to the homomorphism $\sigma$, and denoted $A \times_\sigma B$.

Note that $\sigma(a)$, for all $a$ in $A$, is a bijective function from $B$ into itself and if $\sigma$ is the trivial homomorphism then we obtain the usual **direct (Cartesian) product**.

The following definition is the key to building graphs from groups. This construction was first given by Cayley in 1878 in [Ca1] and [Ca2] (also see [Ko, CL, Ha, Wh]).

**Definition:** Given a group $A$ and a set $S$ of generators for $A$ the **Cayley digraph** $G = (V, E)$, denoted by $< A, S >$, is constructed as follows: (1) the elements of the group A are the vertices $V$ of digraph $G$ and (2) an edge $(a, b)$ is in $E$ if and only if $ag = b$ for some generator $g$ in $S$. If we also require $S = S \cup S^{-1}$ then $G$ is a **Cayley graph**.

# Chapter 2

# Efficient Undirected Networks

Only a few undirected networks of maximum degree $\Delta$ and diameter $D$ are known to be optimal (that is, largest possible number of vertices). One is the Petersen graph, Figure 2.1, with $\Delta = 3$, $D = 2$ and order 10. Another is the Hoffman–Singleton graph with $\Delta = 7$, $D = 2$ and order 50, see [HS] and [CL]. Three other optimal constructions of $(3, 3)$, $(4, 2)$, and $(5, 2)$ graphs with orders $20, 15$, and 24, respectively, are described in [El]. The tree bound given in Section 2.2 provides (essentially) the only known upper bound on the order of optimal $(\Delta, D)$ graphs This bound is probably not tight.

This chapter addresses the following design problem. We examine several construction techniques for solving this problem.

- *The Degree/Diameter Problem* (undirected case). Provide constructions of the largest possible $(\Delta, D)$ graphs.

Group theory plays an important role in our constructions. For an overview of our results see Table 2.1. This table shows the orders of the largest graphs presently known for given values of degree and diameter. The entries in italics correspond to our Cayley graphs constructions employing semi-direct product groups (Section 3.1) and the bold entries indicate our Cayley graphs based on general linear groups (see [9a]). Some of the constructions for the degree four entries were first described in [CCD]. For references concerning the other entries in Table 2.1, the reader should consult the up-to-date report available from J-C. Bermond [Be].

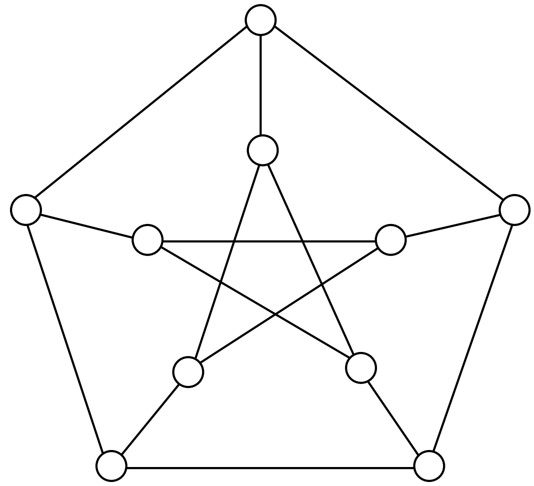

Figure 2.1: The Petersen graph.

The next section describes some advantages of Cayley graphs as designs for interconnection networks. Section 2.2 presents some upper bounds for the degree/diameter problem and, at the same time, indicates why we explored the groups listed in the introduction (semi-direct product groups, general linear groups, etc.). The last two sections give construction methods for finding large $(\Delta, D)$ graphs.

## 2.1 Advantages of Cayley Graphs

There are many properties of Cayley graphs that are useful. The most visible property possessed by Cayley graphs is that every Cayley graph is **regular**, all vertices have the same degree.

Many graph properties of a Cayley graph are easily inferred from its group generators. Walks in a Cayley graph $< A, S >$ correspond to words over the alphabet $S$ applied to some group element $x$ by group multiplication. A path from vertex $x$ to vertex $y$ corresponds (by translation) to a path from vertex $e$ to vertex $x^{-1}y$. Any word of generators equivalent to the identity element $e$ describes closed walks or **circuits** in the graph. A closed path is a **cycle**. It is conjectured that every Cayley graph $< A, S >$ is **Hamiltonian**, that is, there exists a cycle containing all the vertices of the graph (i.e., a word over $S$ equivalent to identity element $e$ of length $|A|$ yielding a cycle) [Ma].

We are mainly interested in the vertex symmetry property of Cayley graphs. Cayley graphs provide a vertex symmetric structure for any defining group, see [GT]

Table 2.1: The orders of the largest known graphs of degre

| $\Delta$ \ D | 2 | 3 | 4 | 5 | 6 | |
|---|---|---|---|---|---|---|
| 3 | 10 | 20 | 38 | 70 | 130 | |
| 4 | 15 | 41 | 95 | 364 | 734 | [illegible] |
| 5 | 24 | 70 | 184 | 532 | 2742 | [illegible] |
| 6 | 32 | 105 | *355* | *1088* | 7832 | *1* |
| 7 | 50 | 136 | *506* | *2460* | 10554 | *4* |
| 8 | 57 | *203* | *915* | *4108* | 39258 | *10* |
| 9 | 74 | 585 | *1254* | *6890* | 74954 | *21* |
| 10 | 91 | 650 | 1820 | **12144** | 132932 | *49* |
| 11 | 94 | 715 | 3200 | *16578* | 156864 | *91* |
| 12 | 133 | 780 | 4680 | *26268* | 354422 | *173* |
| 13 | 136 | 845 | 6560 | *33354* | 531440 | **272** |
| 14 | 183 | 910 | 8200 | *51302* | 804624 | 620 |

and [Sa]. From the engineer's point of view, an advantage of vertex symmetry in a network is that the same routing algorithms can be used by each node. In terms of hardware, vertex symmetric architectures are often cheaper to build since the same processor can be replicated throughout. That Cayley graphs are vertex symmetric has a fairly short and elegant proof, coming up next.

**Theorem 1** *Every Cayley graph $< A, S >$ is vertex symmetric.*
**Proof.** Let $a$ and $b$ be any two elements from the group $A$. We must show there is an adjacency-preserving automorphism $\phi$ of $A$ mapping $a$ to $b$. Define $\phi(x) = (ba^{-1})x$ for all $x \in A$. Clearly $\phi$ maps $a$ to $b$ since

$$\phi(a) = (ba^{-1})a = b(a^{-1}a) = be = b \text{ (associativity)}.$$

The map $\phi$ is injective since if $\phi(x) = \phi(y)$ then $(ba^{-1})x = (ba^{-1})y$ and so

$$x = (ba^{-1})^{-1}(ba^{-1})x = (ba^{-1})^{-1}(ba^{-1})y = y \text{ (inverses)}.$$

Similarly $\phi$ is surjective since for any $x$ in $A$, $\phi(ab^{-1}x) = (ba^{-1})(ab^{-1}x) = x$. So $\phi$ is a bijection. Finally, $\phi$ maps vertices adjacent to $a$ to vertices adjacent to $b$.

$$\phi(ag_i) = (ba^{-1})(ag_i) = bg_i \text{ for all } g_i \in S$$

$\square$

Note that the above proof uses all of the group axioms. This indicates that other Cayley-like graphs (e.g., based on semi-groups or monoids) might not have enough structure for vertex symmetry. For instance, a commutative monoid (set with an associative binary operator and an identity element) may yield a digraph that is not even regular. A concrete example is the $< (\{0, 1, 2\}, max), \{2\} >$ monoid digraph. Here, $(\{0, 1, 2\}, max)$ is the monoid with *maximum* as the binary operator over $\{0, 1, 2\}$ and the integer 2 is the generator for this digraph.

Can every vertex symmetric graph can be represented as a Cayley graph? The answer is no. The Petersen graph, Figure 2.1, is vertex symmetric but is not a Cayley graph. However, from the work of Sabidussi every vertex symmetric graph can be represented as a Cayley coset graph [Sa]. We will give the formal definition of Cayley coset graphs when they are used in the next chapter.

Similar to vertex symmetric graphs, **edge symmetric** graphs are desirable. That is, there exists a graph automorphism mapping any edge onto any other edge. This property guarantees that the load over communication links is distributed [AK2]. Akers and Krishnamurthy show that Cayley graphs are edge symmetric in certain instances [AK3]. Recall that every finite group can be represented as a subgroup of some permutation group $S_n$.

**Theorem 2** *Let $G$ be a Cayley graph defined on $S_n$ by a set of generators $S$. Then $G$ is edge symmetric if and only if for every pair of generators $g_i$ and $g_j$ there exists a permutation of the $n$ symbols that maps the set of generators into themselves, and, in particular, maps $g_i$ into $g_j$.*

In addition to being symmetric, some Cayley graphs possess a recursive decomposition property. For instance, the well-known $n$-cubes $Q_n$ (hypercubes) consist of 2 copies of $(n-1)$-cubes. Also the $n$-pancake graph, the Cayley graph

$$< S_n, \{ \begin{pmatrix} 0 & 1 & 2 & \dots & n-1 \\ 1 & 0 & 2 & \dots & n-1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & \dots & n-1 \\ 2 & 1 & 0 & \dots & n-1 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 & 2 & \dots & n-1 \\ n-1 & n-2 & n-3 & \dots & 0 \end{pmatrix} \} >,$$

can be broken down into $n$ copies of $(n-1)$-pancake graphs. In [AK3] a Cayley graph $< A, \{g_1, g_2, \dots, g_d\} >$ is defined to be **hierarchical** if the generators can be ordered $g_1, g_2, \dots, g_d$, so that for each $i$, $1 < i \leq d$, $g_i$ is outside the subgroup generated by the first $i-1$ generators. All Cayley graphs which are hierarchical have a recursive decomposition structure. Any hierarchical graph also has high **fault tolerance** (i.e., a graph that requires a large number of vertices to be removed before the graph becomes disconnected) [AK1].

Table 2.2: The Moore bound for the undirected $(\Delta, D)$ problem.

| $\Delta \setminus$ D | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| 3 | 10 | 22 | 46 | 94 | 190 | 382 | 766 |
| 4 | 17 | 53 | 161 | 485 | 1457 | 4373 | 13121 |
| 5 | 26 | 106 | 426 | 1706 | 6826 | 27306 | 109226 |
| 6 | 37 | 187 | 937 | 4687 | 23437 | 117187 | 585937 |
| 7 | 50 | 302 | 1814 | 10886 | 65318 | 391910 | 2351462 |
| 8 | 65 | 457 | 3201 | 22409 | 156865 | 1098057 | 7686401 |
| 9 | 82 | 658 | 5266 | 42130 | 337042 | 2696338 | 21570706 |
| 10 | 101 | 911 | 8201 | 73811 | 664301 | 5978711 | 53808401 |

## 2.2 The Moore and Abelian Bounds

The tree or Moore bound $m(\Delta, D)$ on the order of the largest possible $(\Delta, D)$ graphs is easily calculated as follows.

$$\begin{aligned} m(\Delta, D) &= 1 + \Delta + \Delta(\Delta-1) + \cdots \Delta(\Delta-1)^{D-1} \\ &= \frac{\Delta(\Delta-1)^D - 2}{\Delta - 2} \end{aligned}$$

To satisfy the reader's curiosity, a few values of $m(\Delta, D)$ are calculated in Table 2.2. It is interesting to compare these numbers with the orders of our constructions given in Table 2.1. (Best known $(\Delta, D)$ graphs are quite smaller than the current upper bounds.)

What groups yield the largest $(\Delta, D)$ graphs? Oddly, abelian groups, which are the easiest to use and most well-known, are the ones to avoid. The use of abelian groups for Cayley graphs imposes a stronger limit than the Moore bound on the maximum order of a $(\Delta, D)$ graph. The only place that these two bounds coincide is the $(3, 2)$ graph bound (achieved by the Petersen graph with 10 vertices). Given an abelian group with $\Delta$ generators the maximum order of a Cayley digraph $G$ with diameter $D$ is

$$|G| \leq \sum_{i=0}^{D} \binom{\Delta + i - 1}{i} = \binom{\Delta + D}{\Delta}$$

Comparing these *abelian Moore bounds* given in Table 2.3 for small $\Delta$ and $D$ with our current list of largest graphs, Table 2.1, we see that none of these bounds is as large

Table 2.3: An abelian Moore bound for directed Cayley graphs.

| $\Delta \setminus$ D | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| 3 | **10** | **20** | 35 | 56 | 84 | 120 | 165 |
| 4 | **15** | 35 | 70 | 126 | 210 | 330 | 495 |
| 5 | 21 | 56 | 126 | 252 | 462 | 792 | 1287 |
| 6 | 28 | 84 | 210 | 462 | 924 | 1716 | 3003 |
| 7 | 36 | 120 | 330 | 792 | 1716 | 3432 | 6435 |
| 8 | 45 | 165 | 495 | 1287 | 3003 | 6435 | 12870 |
| 9 | 55 | 220 | 715 | 2002 | 5005 | 11440 | 24310 |
| 10 | 66 | 286 | 1001 | 3003 | 8008 | 19448 | 43758 |

as any of the currently known $(\Delta, D)$ graph constructions, with 3 exceptions indicated in bold. A nonabelian group can provide better results, since for two generators $g_1$ and $g_2$ the products $g_1 g_2$ and $g_2 g_1$ may be distinct. Thus it is theoretically possible to come closer to the Moore bound when nonabelian groups are used as the underlying algebraic structure.

## 2.3 Cayley Graph Constructions

This section describes our main group construction for finding large $(\Delta, D)$ Cayley graphs. These graphs, listed in Appendix A, were created using semi-direct products of groups.

Given two cyclic groups, $Z_m$ and $Z_n$, one may form a semi-direct product group $G = Z_m \times_\sigma Z_n$ by defining an appropriate homomorphism $\sigma : Z_m \rightarrow Aut(Z_n)$. View $Z_n$ as a **commutative ring** with a group of units $U(Z_n)$. Let an element $r$ be chosen from $U(Z_n)$. Define a mapping $\sigma'(k) = (r^c)^k = r^{ck}$ where $c$ is chosen so that $r^{cm} = 1$. The group $G$ has its multiplication table defined by

$$(a_0, a_1) *_\sigma (b_0, b_1) = (a_0 + b_0 \bmod m,\ a_1 + \sigma'(a_0) \cdot b_1 \bmod n).$$

We see that $(\sigma(a))(b) = \sigma'(a) \cdot b$, for $a$ in $Z_m$ and $b$ in $Z_n$, is a suitable homomorphism. Note that $(0, 0)$ is the group identity for a semi-direct product group constructed as above and that $(-a_0,\ \sigma'(a_0)^{-1} \cdot (-a_1)) = (-a_0,\ \sigma'(-a_0) \cdot (-a_1))$ is the inverse of element $(a_0, a_1)$.

An example of a the above construction is given in Figure 2.2. This is a $(4, 2)$

Cayley graph with 12 vertices based on the group $Z_2 \times_\sigma Z_6$. A vertex of the graph is labeled $6a + b$ if the corresponding group element is $[a\ b]$, $a \in Z_2$ and $b \in Z_6$. The use of the first generator (and inverse), second generator, and last generator is represented by bold, hashed, and normal edges, respectively.

| (Δ, D) | Order | Group | Generators | Inverses | Order of Generator |
|---|---|---|---|---|---|
| (4, 2) | 12 | $Z_2 \times_\sigma Z_6$ | [ 0 1 ] | [ 0 5 ] | 6 |
| | | $\sigma'(1)$=5 | [ 1 0 ] | | 2 |
| | | | [ 1 3 ] | | 2 |

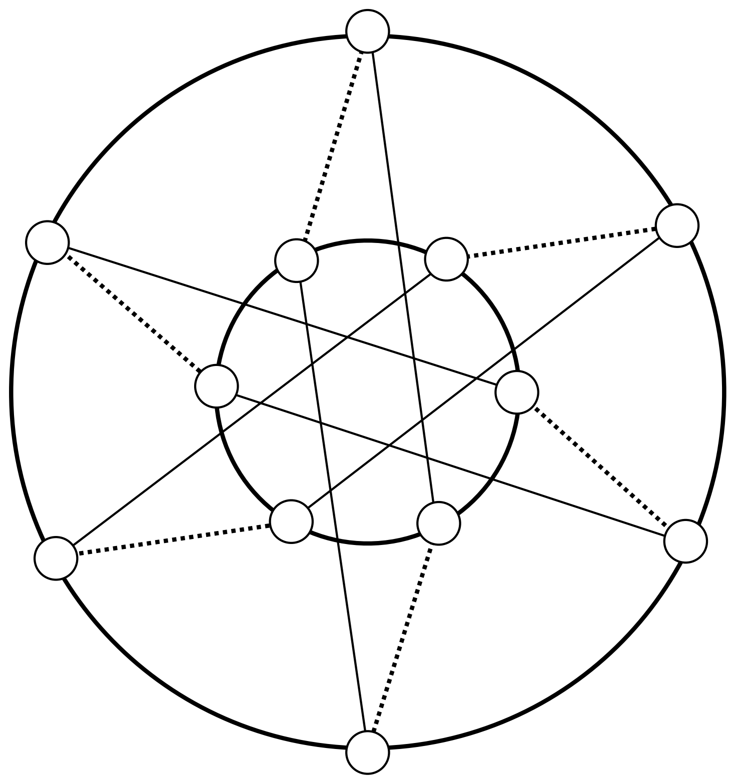

Figure 2.2: A $(4, 2)$ Cayley graph with 12 vertices.

Our search for constructions showed empirically that if $gcd(m, |U(Z_n)|) = gcd(m, \phi(n))$ is large, then the more likely one finds larger $(\Delta, D)$ graphs. Recall that if $n$ is a prime then $|U(Z_n)| = n - 1$. In this situation, we get several of the entries listed in the appendices. Further, if both $m$ and $n$ are prime (that is, using two fields) then we use a similar semi-direct product construction. See Section 3.4.2 for more specifics.

## 2.4 Other Undirected Network Constructions

This section reviews other current techniques for building interconnection networks which may or may not use Cayley graphs. The first few constructions are special cases of our general Cayley graph approach. These all use group theory in the design of symmetric interconnection networks. The last subsection describes a very recent method for creating large $(\Delta, D)$ graphs that is competitive with our group-theoretic approach.

### 2.4.1 Generalized cube-connected cycles

The first systematic use of Cayley graphs for the degree/diameter problem was due to Carlsson, Cruthirds, Sexton and Wright [CCSW]. The cube-connected cycles given in [PV] were shown to be Cayley graphs by these authors. These graphs are a special case of their generalized cube-connected cycles, a special class of Cayley graphs.

The cube-connected cycles, $n$-CCC, are similar to the $n$-cubes. The vertices are given as pairs $(i, V)$ where $i$ ranges between 0 and $n - 1$ and $V$ is a bit vector of length $n$. For edges, vertex $(i, V)$ is connected to vertex $(i', V')$ if and only if $i = i'$ and $V'$ differs in only the $i^{\text{th}}$ bit from $V$, or $|i - i'| = 1$ and $V = V'$. See Figure 2.3 for an example where $n = 3$.

Showing that a $n$-CCC is a Cayley graph follows from the algebraic specification of generalized cube-connected cycles, GCC. The groups used for these Cayley graphs are semi-direct products. Recalling the definition of semi-direct product groups, let $A = Z_m$ and $B = (Z_2)^k$. The homomorphism $\sigma$ will be defined in terms of linear transformations of the elements, viewed as bit vectors, of $B$. Let matrix $M$ be some $k \times k$ matrix over the field $GF(2)$ such that $M^m = I$ (identity matrix). We can now define group multiplication in $Z_m \times_\sigma (Z_2)^k$ as

$$(i_0, V_0) * (i_1, V_1) = (i_0 + i_1, V_0 + M^{i_0} V_1).$$

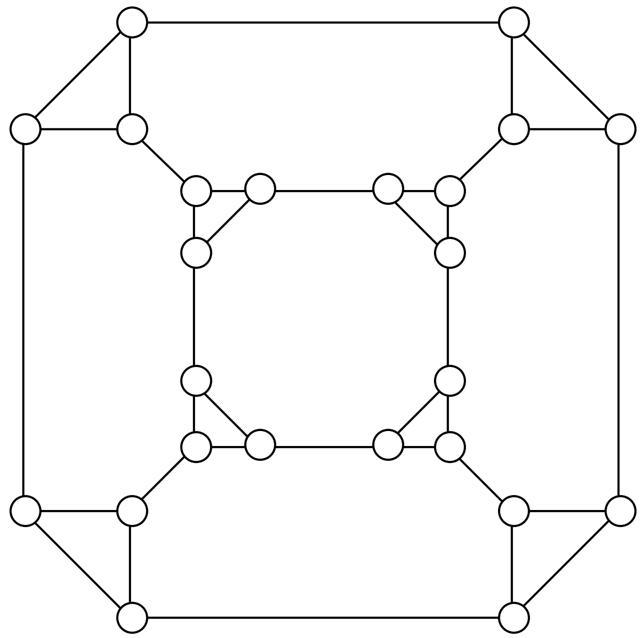

Figure 2.3: The 3-CCC presented as a Cayley graph.

We have all the ingredients needed to define GCCs; see [CCSW] for more information.

**Definition:** Let $G(m, k, M)$ be the group described above using $Z_m \times_\sigma (Z_2)^k$ and a $k \times k$ matrix $M$ (of 0's and 1's), where $M^m$ is the identity matrix. Then the **Generalized Cube-connected Cycles** is the connected component containing $(0, (0, \ldots, 0))$ of any Cayley graph $< G(m, k, M), S >$.

The $n$-CCC is formed with $m = k = n$,

$$M = \begin{bmatrix} 0 & \cdot & \cdot & \cdots & 0 & 1 \\ 1 & 0 & \cdot & \cdots & \cdot & 0 \\ 0 & 1 & 0 & \cdots & \cdot & \cdot \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \ddots & 0 & \cdot \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

and generators $S = \{(1, (0, \ldots, 0)), (n-1, (0, \ldots, 0)), (0, (1, 0, \ldots, 0))\}$.

Carlsson et. al. comment that several of their constructed graphs are competitive with the best known $(\Delta, D)$ graphs, $\Delta \leq 6$ [CCSW]. The results of this early paper have been surpassed by new developments and/or more computer time.

### 2.4.2 Star graphs

A relatively new type of interconnection network is given by Akers and krishnamurthy [AK2, AK3]. These star graphs described in this section do not give any of the largest $(\Delta, D)$ graphs, but they have nice algebraic structure and share or improve upon many of the properties that the $n$-cubes have.

Before describing how star graphs are constructed, we present Akers and Krishnamurhy's definition of Cayley graphs from **transposition trees**. A **transposition** is a permutation that leaves all the symbols but two fixed, and maps each of these onto the other. A **tree** is any connected graph without cycles. Now given a tree with $n$ vertices labeled $\{0, 1, \ldots, n-1\}$, we can view the edges as transpositions in the group $S_n$. These transpositions are the generators for some Cayley graph.

Many nice properties hold for a Cayley graph, $G$, built from some transposition tree.

1. $G$ has maximum degree $\Delta = n - 1$.

2. $G$ has $n!$ vertices.

3. $G$ has maximal fault tolerance, $n - 1$.

4. $G$ is bipartite (see [CL] for definition).

5. $G$ is a hierarchical Cayley graph.

These hierarchical Cayley graphs can be represented as $n$ identical copies of a Cayley graph of a transposition tree of order $n - 1$. Further, many properties of the Cayley graph with $n!$ vertices can be inferred by analyzing the transposition tree with $n$ vertices. If the tree with $n$ vertices is a **star**, all but one vertex has degree 1, then the resulting Cayley graph is called a **$n$-star graph**.

The diameter of the $n$-star graph is $\lfloor \frac{3(n-1)}{2} \rfloor$; see [AK3]. This compares favorably with the $n$-cubes (see Table 2.4). Routing algorithms for the $n$-star may be found in the references [AK2] and [AK3].

Table 2.4: A comparison between $n$-star graphs and $n$-cubes.

| $n$-star graph | | | $n$-cube | | |
|---|---|---|---|---|---|
| Degree | Diameter | Order | Degree | Diameter | Order |
| 3 | 4 | 24 | 4 | 4 | 16 |
| 4 | 6 | 120 | 5 | 5 | 32 |
| 5 | 7 | 720 | 6 | 6 | 64 |
| 6 | 9 | 5040 | 7 | 7 | 128 |
| 7 | 10 | 40320 | 8 | 8 | 256 |
| 8 | 12 | 362880 | 9 | 9 | 512 |
| | | | 10 | 10 | 1024 |
| | | | 11 | 11 | 2048 |
| | | | 12 | 12 | 4096 |

### 2.4.3 Connections between two cycles

Motivated by the structure of the Petersen graph, Figure 2.1, Bar-Yehuda and Etzion developed a method that takes two cycles of the same length $n$ as a backbone for building large $(\Delta, D)$ graphs [BE]. A periodic scheme of length $s$ where $s|n$ was used to connect the two cycles with cross edges. More formally, a graph $G = (V, E)$ with degree $\Delta$ and order $2n$ is created with a set $F$ of $\Delta - 2$ functions on the integers $0, 1, \ldots, s-1$. For each function $F_i,\ 1 \leq i \leq \Delta - 2$, it is required that

$$F_i(0) \bmod s,\ F_i(1) + 1 \bmod s,\ F_i(2) + 2 \bmod s, \ldots,\ F_i(s-1) + (s-1) \bmod s$$

is a permutation of the integers $0, 1, \ldots, s-1$. The vertices of $G$ are labeled $V = \{(i, 0) \mid 0 \leq i \leq n-1\} \cup \{(i, 1) \mid 0 \leq i \leq n-1\}$. The edge set $E$ contains the subsets $E_i,\ 0 \leq i \leq \Delta - 2$.

$$\begin{aligned} E_0 &= \{[(i,j), ((i+1) \bmod n, j)] \mid 0 \leq i \leq n-1, j = 0, 1\} \\ E_i &= \{[(j,0), ((j + F_i(j \bmod s)) \bmod n, 1)] \mid 0 \leq j \leq n-1\} \text{ for } 1 \leq i \leq \Delta - 2. \end{aligned}$$

The edges $E_0$ form the two cycles of length $n$ while the edges $E_i$ form a period of length $s$ joining the $(x, 0)$ vertices to the $(y, 1)$ vertices.

In checking the diameter of these connected-two-cycle graphs, it suffices to check each distance from $2s$ vertices to all other vertices. Computer searches where used to find the diameter of these graphs. A small example of this technique is their $(7, 3)$

graph with order 136 that is presented in Table 2.5.

Table 2.5: A (7,3) graph with order 136 formed by connections between 2 cycles.

| $i$ | $F_1(i)$ | $F_2(i)$ | $F_3(i)$ | $F_4(i)$ | $F_5(i)$ |
|---|---|---|---|---|---|
| 0 | 65 | 48 | 0 | 29 | 5 |
| 1 | 37 | 40 | 12 | 17 | 1 |
| 2 | 57 | 4 | 48 | 45 | 61 |
| 3 | 65 | 40 | 20 | 53 | 13 |

# Chapter 3

# Efficient Directed Networks

Related to the undirected network problem of the last chapter is a directed network problem. In this instance, communication lines can transmit messages in only one direction. In terms of graph theory, this problem can be stated as finding large digraphs that have diameter $D$ while the in-degree and out-degree of each vertex is less than or equal to $\Delta$.

- *The Degree/Diameter Problem* (directed case). Provide constructions of the largest possible $(\Delta, D)$ digraphs.

There are several constructions that share the same largest orders for $(\Delta, D)$ digraphs [K2, II, FMY]. Thus it may be reasonable to conjecture that these constructions are optimal. In this chapter we address the specific problem of finding efficient vertex-symmetric $(\Delta, D)$ digraphs. Cayley digraphs and Cayley coset digraphs (explained later) are useful for the symmetric case. This chapter will present some of our vertex-symmetric results along with the other classic approaches.

## 3.1 Large Vertex Symmetric $(\Delta, D)$ Digraphs

This section will discuss the current status on the largest known vertex symmetric $(\Delta, D)$ digraphs. Table 3.1 summarizes the best known maximum orders.

Appendix B contains Cayley digraph descriptions for some of these newly discovered largest-known digraphs. Table B.1 lists our $(\Delta, D)$ digraphs built from general linear groups (bold in Table 3.1) and Table B.2 lists our $(\Delta, D)$ digraphs built from semi-direct products of groups (italics in Table 3.1). Our class of largest known diameter two digraphs (Section 3.4.2) is also based on semi-direct products of groups.

Two largest known digraphs corresponding to the (2,6) and (2,7) entries in Table

Table 3.1: The orders of the largest known vertex symmetric

| $\Delta \backslash$ D | 2 | 3 | 4 | 5 | 6 | 7 | |
|---|---|---|---|---|---|---|---|
| 2 | 6 | **10** | **20** | *27* | 60 | 120 | |
| 3 | 12 | *27* | 60 | *155* | *333* | 1152 | |
| 4 | 20 | 60 | **136** | *420* | *1100* | 7200 | |
| 5 | 30 | 120 | 360 | *889* | *3197* | 28800 | |
| 6 | 42 | 210 | 840 | 2520 | *7224* | 88200 | **1** |
| 7 | 56 | 336 | 1680 | 6720 | 20160 | 225792 | **4** |
| 8 | 72 | 504 | 3024 | 15120 | 60480 | 508032 | **6** |
| 9 | 90 | 720 | 5040 | 30240 | 151200 | 1036800 | **18** |
| 10 | 110 | 990 | 7920 | 55440 | 332640 | 1960220 | 6 |

3.1 were found by Comellas and Fiol [Co]. The (2,7) digraph is the line digraph of the edge-symmetric (2,6) Cayley digraph $< S_5, \{ \left(\begin{smallmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 2 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 & 2 & 3 & 4 \\ 3 & 0 & 4 & 2 & 1 \end{smallmatrix}\right) \} >$.

Construction techniques for the other entries in Table 3.1 will be discussed in the next two sections.

Figure 3.1 shows one of the largest known digraphs that was built from general linear groups. This is our $(2,4)$ Cayley digraph with 20 vertices formed by taking a subgroup of $GL(2,5)$ generated by the two generators $\left[\begin{smallmatrix} 3 & 0 \\ 0 & 4 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 2 & 4 \\ 0 & 4 \end{smallmatrix}\right]$. The arcs of this digraph can be partitioned into two disjoint Hamiltonian cycles.

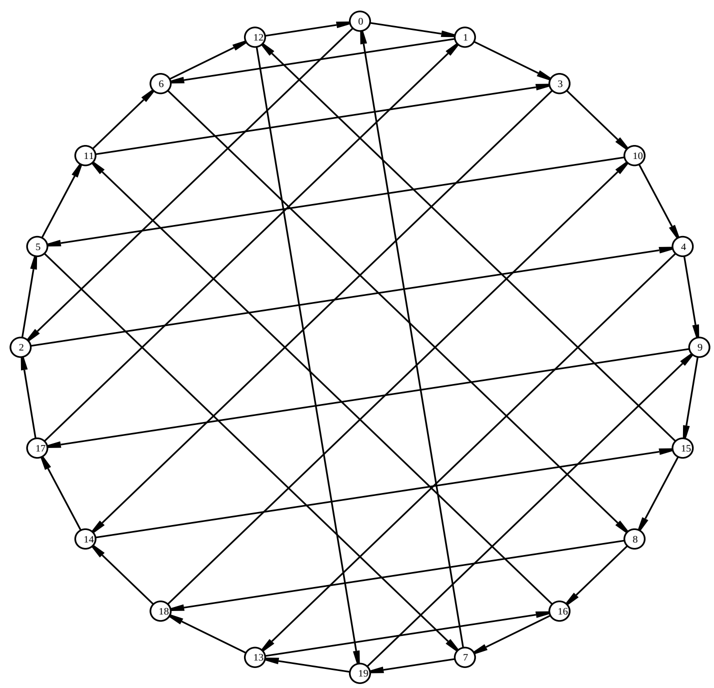


Figure 3.1: Largest known vertex symmetric $(2,4)$ digraph.

Figure 3.2 shows one of the largest known digraphs that was built from semi-direct product of groups. This is our (2,5) Cayley digraph with 27 vertices.

## 3.2 Cayley Coset Constructions

The main emphasis of this chapter is the use of group theory (cosets) to specify large symmetric digraphs. The digraphs mentioned in this section are the results of Faber

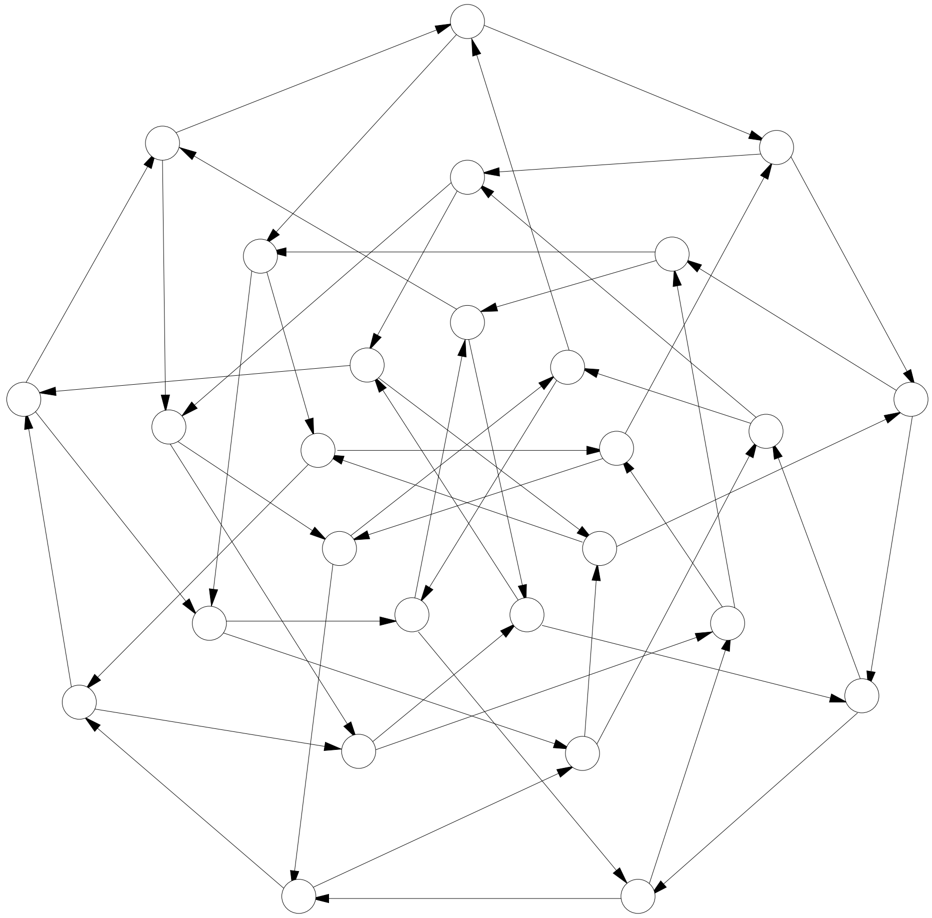

Figure 3.2: Largest known vertex symmetric $(2, 5)$ digraph.

and Moore [FM]. These Cayley coset digraphs have the largest known orders for the majority of $(\Delta, D)$ digraph values exhibited in Table 3.1. Cayley coset digraphs are defined as follows.

**Definition:** Let $G$ be a group, $H$ a subgroup of $G$, and $S$ a subset of $G$ such that:

1. $S$ is a set of distinct non-identity coset representatives of $H$ in $G$.

2. $HSH \subseteq SH$ (for well-defined arcs).

then we can form the **Cayley coset digraph** $< G, S, H >$ with vertices $\{gH \mid g \in G\}$ and an arc $(g_1H, g_2H)$ whenever $g_2H = g_1\delta H$ for some $\delta \in S$.

A couple of remarks should be made about the above definition. The first condition implies that $G = < S \cup H >$ (i.e., $G$ is generated by $S$ and $H$), and the second condition implies that the arcs are well-defined.

The proof that Cayley coset digraphs are vertex symmetric is similar to the argument given for Theorem 1. This important property of Cayley coset digraphs is the heart of the next result.

**Theorem 3** *For each $\Delta \geq D$, there exists a vertex symmetric digraph with degree $\Delta$, diameter $D$, and $\frac{(\Delta+1)!}{(\Delta+1-D)!}$ vertices.*

Faber and Moore's construction is now presented [FM]. Given the symmetric group $S_{\Delta+1}$, let the subgroup $H \subset S_{\Delta+1}$ be the set of elements $\{h \mid h \in S_{\Delta+1},\ h(i) = i$ for all $0 \leq i \leq D-1\}$. The set of generators (or coset representatives) $S = \{g_i \mid 1 \leq i \leq \Delta\}$ is chosen for each $i$ so that

$$\begin{aligned} g_i(0) &= i \\ g_i(j) &= \left\{ \begin{array}{ll} j-1 & \text{for } 1 \leq j \leq i \\ j & \text{for } j > i \end{array} \right\} \text{ if } 1 \leq j \leq D-1 \end{aligned}$$

We can denote the cosets of $H$ in $S_{\Delta+1}$ by the $k$-tuples $(a_0, a_1, \ldots, a_{D-1})$ with all $a_i$ distinct, since each coset $aH$ is completely determined by its action on $\{0, 1, \ldots, D-1\}$. Thus, the total number of cosets is $\frac{(\Delta+1)!}{(\Delta+1-D)!}$. Given any $(a_0, a_1, \ldots, a_{D-1})$ in at most $D$ steps using the generators as needed (finding in order $\{D-1, D-2, \ldots, 0\}$ by applying appropriate generators from $S$) the new representative becomes our coset identity $(0, 1, \ldots, D-1)$. Hence, the diameter is $D$ as claimed.

## 3.3 Composition of Cayley Coset Graphs

In order to find large symmetric digraphs with the diameter larger than the degree, Comellas and Fiol in [CF] have developed a method for constructing specific families of $(\Delta, D)$ digraphs using the Cayley coset digraphs described in Section 3.2. Their constructions require digraphs satisfying the following property.
**Definition:** A digraph $G = (V, E)$ is **k-reachable** if there exist a $uv$-path of length $k$ for all $u, v \in V$.

Faber and Moore's $(\Delta, D)$ digraphs are $D$-reachable for $D \geq 3$ so the new version of Conway and Guy's theorem, [CG], given by Comellas and Fiol can be applied.

**Theorem 4** *If there is a vertex-symmetric $\Delta$-regular $k$-reachable digraph with $N$ vertices then, for all $n$ and $m = \dot{n}$, there exists a vertex-symmetric $\Delta$-regular digraph with $mN^n$ vertices and diameter $kn + m - 1$.*

We present here only the construction which yields this theorem. Let $G = (V, E)$ be a vertex-symmetric $(\Delta, D)$ digraph. A new digraph $G' = (V', E')$ is built as follows. The vertex set $V'$ is this set of $(n+1)$-tuples, $(\alpha, p_0, p_1, \ldots, p_{n-1})$ with $\alpha$ in $Z_m$ and each $p_i$ from $V$. The arc set is determined by $(\alpha, p_0, \ldots, p_\alpha, \ldots, p_{n-1})$ being adjacent to $(\alpha+1, p_0, \ldots, q_\alpha, \ldots, p_{n-1})$ whenever $p_\alpha$ is adjacent to $q_\alpha$ in the digraph $G$. In this construction the indices of the vertices $p_i$ are always considered modulo $n$.

Using this theorem, Comellas and Fiol showed that the $(\Delta, 3)$ digraphs, $\Delta \geq 3$, can be used to construct large vertex symmetric digraphs. These are the $(\Delta, 7)$ digraphs with order $2 \cdot \left(\frac{(\Delta+1)!}{(\Delta-2)!}\right)^2$ and the $(\Delta, 11)$ digraphs with order $3 \cdot \left(\frac{(\Delta+1)!}{(\Delta-2)!}\right)^3$. Further, using the $(\Delta, 4)$ digraphs, $\Delta \geq 4$, they obtain the $(\Delta, 9)$ digraphs with order $2 \cdot \left(\frac{(\Delta+1)!}{(\Delta-3)!}\right)^2$ .

Another technique that not only increases the diameter but increases the degree is given in [CF]. This method is a corollary of the next theorem.

**Theorem 5** *If there is a vertex-symmetric $\Delta$-regular $k$-reachable digraph with $N$ vertices then, for all $n, m, b \in Z^+$ there exists a vertex-symmetric $(\Delta+1)$-regular digraph with $mN^n$ vertices and diameter $kn+d$ with $d$ being the diameter of a fixed 2-step digraph with $m$ vertices and steps 1 and $b$, see [FYAV].*

The design of these digraphs is similar to the previous theorem. With this construction vertex-symmetric $(\Delta, 10)$ digraphs with order $3 \cdot \left(\frac{\Delta!}{(\Delta-3)!}\right)^3$ can be created from $(\Delta-1, 3)$ Cayley coset digraphs.

The digraphs in this section and Section 3.2 give the majority of the entries listed in Table 3.1. However, the technique presented in this section is only useful for building large symmetric $(\Delta, D)$ digraphs with relatively large degree and diameter. Our use of Cayley digraphs from general linear groups and semi-direct products of cyclic groups, see Appendix B, yields larger $(\Delta, D)$ digraphs for some of the smaller cases.

## 3.4 Other Directed Network Techniques

This section gives a summary of other known $(\Delta, D)$ digraph constructions. Vertex symmetry was not a design requirement for most of the following approaches. Consequently, most of these methods give larger digraphs. Table 3.2 shows a few of the largest known $(\Delta, D)$ digraphs for this general degree/diameter problem.

### 3.4.1 Iterations of line digraphs

Fiol, Yebra, and De Miquel found that the following well-known technique of taking line digraphs also constructs large $(\Delta, D)$ digraphs. Line digraphs are created as follows. The vertices of a line digraph $L(G)$ are the arcs of the digraph $G$ and there is an arc in $L(G)$ for each walk of length 2 in $G$. This generalizes to the $k^{\text{th}}$ iterative construction where the vertices of a line digraph $L^k(G)$ are the walks of length $k$ in the digraph $G$ and the arcs of $L^k(G)$ are the walks of length $k+1$ in $G$.

The base case of Fiol, Yebra, and De Miquel's construction is the complete digraph $K^*_{\Delta+1}$ which is the $(\Delta, 1)$ digraph. Since taking the line digraph $L(G)$ of a digraph $G$ preserves the regularity of $G$ we know that $L(G)$ has the same maximum degree $\Delta$. Also from the definition, the diameter of $L(G)$ is only one greater than the diamter of $G$. An illustration of how the (2,3) digraph is created from the (2,2) digraph is shown in Figures 3.3 and 3.4.

Using this line digraph iteration technique one can see that the order of each ($\Delta$, D) digraph is $\Delta^D + \Delta^{D-1}$ [FMY]. The difference between this amount and the directed Moore bound is

$$\frac{\Delta^{D+1} - 1}{\Delta - 1} - (\Delta^D + \Delta^{D-1}) = \frac{\Delta^{D-1} - 1}{\Delta - 1}.$$

It is known that the directed Moore bound, except for $\Delta = 1$ or $D = 1$, can not be achieved [BT].

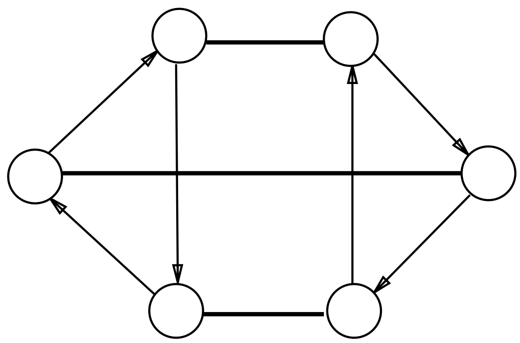

Figure 3.3: An optimal $(2, 2)$ directed network.

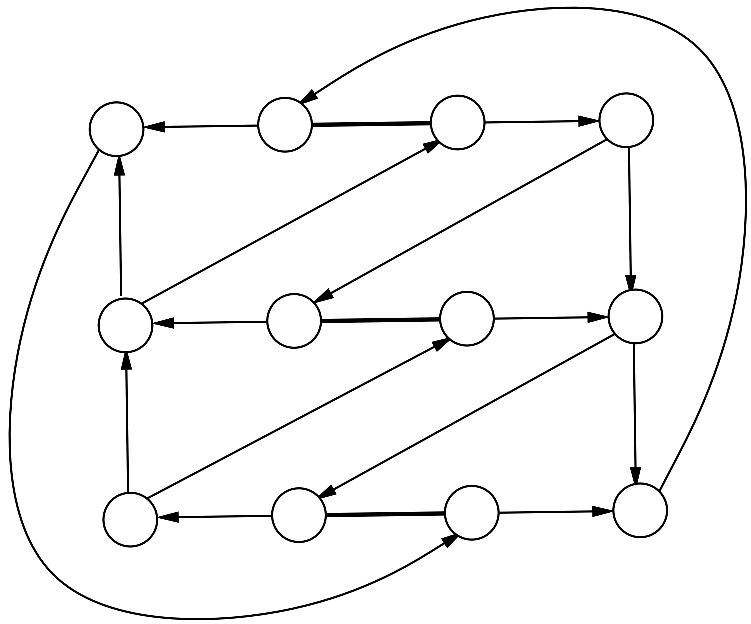

Figure 3.4: A line digraph iteration showing a $(2, 3)$ digraph of order 12.

### 3.4.2 Semi-direct products

This subsection shows an alternate construction for diameter 2 digraphs based on semi-direct products of groups. Since these constructions are Cayley digraphs, they will immediately be vertex symmetric. Some of the other digraphs of Section 3.4 are known not to be vertex symmetry.

Using our definition of semi-direct product groups, we now formally describe a class of groups for this subsection (and the appendices). Let group A be $GF^*(q_1)$ and group B be $GF^+(q_2)$ for fields with compatible orders $q_1$ and $q_2$ (compatible in the sense that the following homomorphism is definable). Let $r_1$ and $r_2$ be primitive roots for these fields. Define the mapping $\sigma : GF^*(q_1) \rightarrow Aut(GF^+(q_2))$ in terms of

$$\sigma'(r_1^j) = \begin{cases} r_2^{kj} & \text{if } q_2 - 1 = k(q_1 - 1) \\ r_2^j & \text{if } q_1 - 1 = k(q_2 - 1) \end{cases}$$

as $(\sigma(a))(b) = \sigma'(a) \cdot b$. This mapping is a group action of $A$ on $B$ by multiplication in $GF(q_2)$.

We next describe some Cayley digraphs with diameter 2 based on the above semi-direct product groups. The following $(\Delta, 2)$ digraphs also have the largest known order.

**Theorem 6** *If the degree* $\Delta$ *is equal to* $q-1$*, where* $q$ *is a power of a prime, then there exists a diameter 2 Cayley digraph of order* $\Delta(\Delta+1)$ *constructed from the group* $GF^*(q) \times_\sigma GF(q)$.

**Proof.** Let the $\Delta$ generators for the group be $g_i = (i, 1)$ where $i$ is from $GF^*(q)$. Since the digraph is vertex symmetric, we only need to check for paths of length two or less from the identity element $e = (1, 0)$. The neighborhood of $e$ is the set of vertices at a distance of exactly one from $e$. These vertices are simply $e * g_i = g_i$, the generators of the group. The products $g_i * g_j = (ij, i+1)$ for all $i, j \in GF^*(q)$ are the vertices at distance at most two from $e$. To see that all of the elements in the group have been reached, fix $i$ in $GF^*(q)$ and run $j$ through $GF^*(q)$. The set $S_i = \{(ij, i+1) \mid j \in GF^*(q)\}$ has $q-1 = \Delta$ distinct elements since $ij$ is unique for all distinct $j$. Notice that $g_i \notin \cup S_i$ for all generators $g_i$ since $i+1 \neq 1$ for all $i \in GF^*(q)$. Thus the total number of distinct elements reached at diameter 2 is $(q-1)(q-1)+(q-1) = (q-1)q$. This is also the order of the group. Therefore, we have a $(\Delta, 2)$ digraph with order $\Delta(\Delta+1)$. $\square$

From the theorem, we easily get the following collection, Table 3.3, of digraphs with diameter two.

### 3.4.3 The method of Imase and Itoh

Imase and Itoh's construction of digraphs is also easy to specify [II]. designing other efficient networks. Their digraphs are identical to those created by line digraph iterations and to the Kautz digraphs presented in Section 3.4.4.

For their digraph construction, let $\Delta$ be the degree and $D$ be the diameter of a requested $(\Delta, D)$ digraph. A digraph with order $n = \Delta^D + \Delta^{D-1}$ will be created. Let the vertices be labeled $0, 1, 2, \ldots, n-1$ and the arcs defined from vertex $i$ to vertex $j$ whenever

$$j = -id - q \pmod{n}, \; q = 1, 2, \ldots, d.$$

After some extensive analysis of paths of length $k$ from each node and several tricks one will see that the diameter is $D$ (see [II]).

### 3.4.4 Kautz digraphs

The earliest known large $(\Delta, D)$ digraphs were discovered by Kautz [K1, K2]. These $(\Delta, D)$ digraphs have the same order of $\Delta^D + \Delta^{D-1}$. Again, it is of interest to see different constructions for these large digraphs. Further, these various constructions give us a strong impression that these digraphs achieve the maximum possible order. It was shown that the diameter 2 digraphs are optimal.

The simple construction of the Kautz $(\Delta, D)$ digraphs follows. The vertices are labeled with words $a_1a_2 \ldots a_D$ where the $a_i$ is any one of $\Delta + 1$ symbols except that $a_i \neq a_{i+1}$ for $1 \leq i < D$. The arcs from vertex $a_1a_2 \ldots a_D$ are directed to the $\Delta$ vertices $a_2a_3 \ldots a_Da_{D+1}$, where $a_{D+1}$ is different from $a_D$. Figure 3.5 demonstrates this construction for the Kautz $(3, 2)$ digraph. Notice how the Kautz $(2, 2)$ digraph is embedded within the $(3, 2)$ digraph.

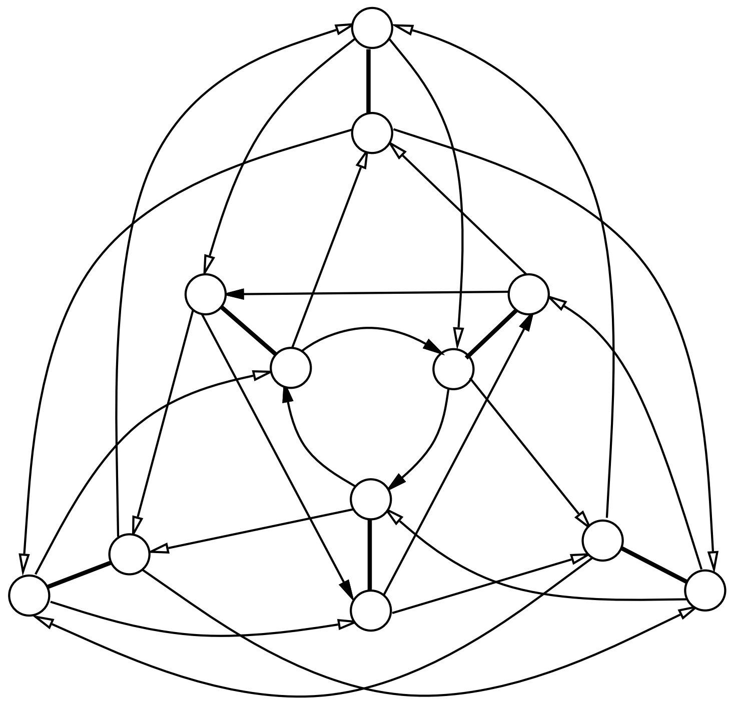

Figure 3.5: The Kautz $(3, 2)$ digraph with 12 vertices.

Table 3.2: The largest known digraphs for degree $\Delta$ and diameter $D$.

| $\Delta \setminus$ D | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| 2 | 6 | 12 | 24 | 48 | 96 | 192 | 384 |
| 3 | 12 | 36 | 108 | 324 | 972 | 2916 | 8748 |
| 4 | 20 | 80 | 320 | 1280 | 5120 | 20480 | 81920 |
| 5 | 30 | 150 | 750 | 3750 | 18750 | 93750 | 468750 |
| 6 | 42 | 252 | 1512 | 9072 | 54432 | 326592 | 1959552 |
| 7 | 56 | 392 | 2744 | 19208 | 134456 | 941192 | 6588344 |
| 8 | 72 | 576 | 4608 | 36864 | 294912 | 2359296 | 18874368 |
| 9 | 90 | 810 | 7290 | 65610 | 590490 | 5314410 | 47829690 |

Table 3.3: Some simple vertex-symmetric $(\Delta, 2)$ digraphs.

| $\Delta$ | 2 | 3 | 4 | 6 | 7 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|---|---|
| $q$ | 3 | $2^2$ | 5 | 7 | $2^3$ | $3^2$ | 11 | 13 |
| $\lvert\mathbf{G}\rvert$ | 6 | 12 | 20 | 42 | 56 | 72 | 110 | 156 |

# Chapter 4

# Efficient Broadcast Networks

This chapter presents the largest known broadcast networks satisfying bounds of maximum node degree and broadcast time. Broadcasting concerns the dissemination of a message originating at one node of a network to all other nodes with the restriction that each node can only forward the message to one of its neighbors at a time.

We now state a few definitions in preparation for our broadcast network constructions.

**Definition:** For a graph $G = (V, E)$ and vertex $v$ in $V$, let **broadcast(v)** denote the minimum time needed to broadcast a message originating at vertex $v$. The **broadcast time** $T$ of the graph $G$ is $max\{broadcast(v) \mid v \in V\}$.

A **broadcast (routing) scheme** for vertex $v$ with $T = broadcast(v)$ may be represented as a sequence $V_0 = \{v\}, E_1, V_1, E_2, \ldots, E_k, V_T$ such that each $V_i \subseteq V$, each $E_i \subseteq E$, $V_T = V$, and, for $1 \leq i \leq T$, (1) each edge in $E_i$ has exactly one endpoint in $V_{i-1}$, (2) no two edges in $E_i$ share a common endpoint, and (3) $V_i = V_{i-1} \cup \{v \mid (u, v) \in E_i\}$.

**Definition:** A $(\Delta, T)$ **broadcast graph** is a graph $G = (V, E)$ such that: (1) $deg(v) \leq \Delta$ for all vertices $v$ in $V$ and (2) $G$ has broadcast time less than or equal to $T$.

The broadcast network design problem is now expressed in terms of these definitions.

- *The Degree/Broadcast-Time Problem.* Provide constructions of the largest possible $(\Delta, T)$ broadcast graphs.

As with the network design problems considered in the previous chapters, our objective in designing $(\Delta, T)$ broadcast graphs is to maximize the number of nodes for fixed constraints of maximum vertex degree $\Delta$ and broadcast time $T$. Our problem is slightly different than those studied in [LP1] and [BHLP2] where the number of vertices were fixed and the goal was to minimize the number of edges so that the broadcast time $T = \lceil \log_2 |G| \rceil$.

Table 4.1 shows the current largest known broadcast graphs while Table 4.2 shows an upper bound. For the reader's convenience the bold entries in Table 4.1 show where upper bounds have been achieved. The asteriks in the table denote where computer searches were used. We call any $(\Delta, T)$ broadcast graph that is as large as possible an **optimal** broadcast graph (network). The following sections contain an upper bound analogous to the Moore bound and give group theoretic techniques for finding broadcast graphs.

Table 4.1: The largest known $(\Delta, T)$ broadcast networks.

| $\Delta \setminus$ T | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| 2 | **4** | **6** | **8** | **10** | **12** | **14** | **16** | **18** | **20** |
| 3 | | **8** | **14** | **24** | **40** | 60* | 84* | 126* | 156* |
| 4 | | | **16** | **30** | **56** | 90* | 148* | 253* | 272* |
| 5 | | | | **32** | **62** | 108* | 186* | 336* | 506 |
| 6 | | | | | **64** | **126** | 220* | 390* | 750* |
| 7 | | | | | | **128** | **254** | 440 | 816* |
| 8 | | | | | | | **256** | **510** | 880* |
| 9 | | | | | | | | **512** | **1022** |

## 4.1 Bounds on the Maximum Order of Broadcast Networks

This section presents a recurrence relation for an upper bound on the maximum number of nodes in a $(\Delta, T)$ broadcast graph. Let $f(\Delta, T)$ be the branch-out bound

Table 4.2: Some $(\Delta, T)$ broadcast network upper bounds.

| $\Delta \setminus$ T | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
| 3 | 4 | 8 | 14 | 24 | 40 | 66 | 108 | 176 | 286 |
| 4 | 4 | 8 | 16 | 30 | 56 | 104 | 192 | 354 | 652 |
| 5 | 4 | 8 | 16 | 32 | 62 | 120 | 232 | 448 | 864 |
| 6 | 4 | 8 | 16 | 32 | 64 | 126 | 248 | 488 | 960 |
| 7 | 4 | 8 | 16 | 32 | 64 | 128 | 254 | 504 | 1000 |
| 8 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 510 | 1016 |
| 9 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1022 |
| 10 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 |

with out-degree $\Delta$ and depth $T$ of a directed tree.

$$\begin{aligned} f(\Delta, 0) &= 1 \\ f(\Delta, T) &= \Sigma_{i=1}^{\min(\Delta,T)} f(\Delta, T-i) + 1 \end{aligned}$$

The broadcast bound $b(\Delta, T)$ for broadcast graphs easily follows from the above.

$$\begin{aligned} b(\Delta, T) &= \Sigma_{i=1}^{\min(\Delta,T)} f(\Delta-1, T-i) + 1 \\ &= 2 \cdot f(\Delta-1, T-1) \end{aligned}$$

Closed form expressions for this bound are somewhat complicated. The broadcast bound for maximum degree equal to 3 is shown below.

$$b(3,T) = \left(\frac{5-3\sqrt{5}}{5}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{T} + \left(\frac{5+3\sqrt{5}}{5}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{T} - 2 \quad \text{for all } T > 0$$

In general one can obtain a closed formula for the broadcast bound for fixed degree by looking, for example, at generating functions [Wl].

Once we have a broadcast network of degree $\Delta$ and broadcast time $T$ a lower bound for larger broadcast networks can be obtained from the following observation.

**Theorem 7** *Given a $(\Delta, T)$ broadcast graph $G$ then the order of the largest $(\Delta + 1, T+1)$ broadcast graph is greater than or equal to $2 \cdot |G|$.*

**Proof.** Take two copies of the $(\Delta, T)$ broadcast graph and take the **Cartesian product**. That is, add an edge between each vertex $v$ and $v'$ in two identical graphs

$G$ and $G'$. Now, routing is done by using one of the $(v, v')$ edges during the first broadcast time. The remaining $\Delta$ broadcasts stay localized in the individual graphs $G$ and $G'$. By our assumption of $G$ being a $(\Delta, T)$ broadcast graph, the new graph is a $(\Delta+1, T+1)$ broadcast graph with twice as many vertices. □

One should note that if the original graph was a Cayley graph using the group $A$ then the constructed $(\Delta+1, T+1)$ broadcast graph is simply a Cayley graph on the direct product $Z_2 \times A$. The generators for this new graph are $(1, 0)$ and $\{(0, g_i) \mid g_i$ is a generator for $A\}$. This theorem is believed to be sharp only when the maximum degree is less than or equal to the broadcast time.

## 4.2 Algebraic Construction Techniques

This section explains what broadcast graphs correspond to the various entries listed in Table 4.1. The entries below the diagonal are omitted since, as will be seen, they trivially follow from the $(\Delta, \Delta)$ broadcast graphs.

There are no known Cayley graph constructions for two of the optimal $(\Delta, T)$ broadcast graphs. Both the $(3, 6)$ broadcast graph with order 40 and the $(4, 6)$ broadcast graph with order 56 were taken from [BHLP1]. After a little investigation, we found that these two broadcast graphs do satisfy our definition of $(\Delta, T)$ broadcast graphs. The $(3, 6)$ broadcast graph is presented in Figure 4.1.

All of the remaining entries in Table 4.1 are based on Cayley graphs. As mentioned earlier, we found the entries flagged with an asterisk by computer searches (see Section 5.2). These graphs are based on semi-direct product of cyclic groups (see Appendix C). To obtain the largest known (5,10) and (7,9) broadcast graphs we applied Theorem 7 to our constructed (4,9) and (6,8) broadcast graphs, respectively.

The remaining optimal broadcast graphs are covered by the following theorems.

**Theorem 8** *The ring network with $2T$ vertices, the cycle $C_{2T}$, is an optimal $(2, T)$ broadcast graph.*

**Proof.** Given $C_{2T}$, route by simply broadcasting to a neighbor that has not seen a message, that is, forward messages in the opposite direction of previous senders. Since at each time step two more vertices will receive a message, the required time constraint $T$ easily follows. □

It is trivially seen that $< Z_{2T}, \{1, -1\} >$ is a Cayley graph representing $C_{2T}$. For a particular originating vertex, there are exactly two broadcast schemes depending

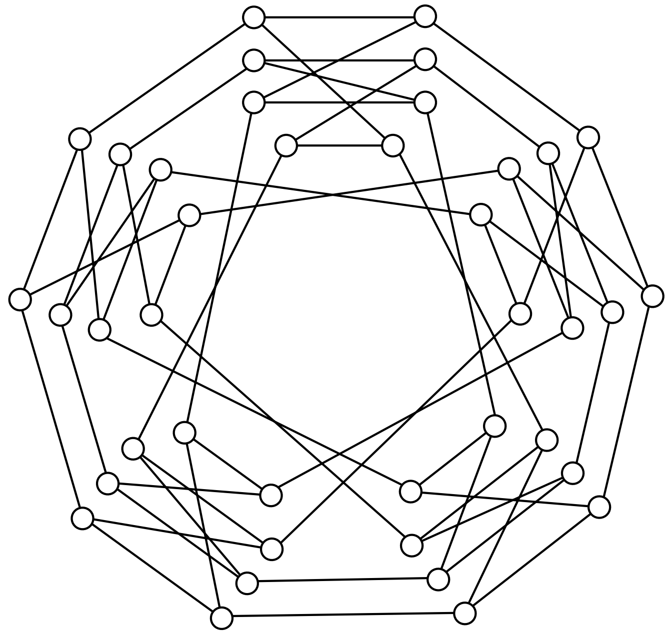

Figure 4.1: A $(3,6)$ broadcast graph of 40 vertices.

on what generator is used first. The graph $C_{2T}$ is one instance where the broadcast time $T$ equals the diameter $D$. However, this property does not hold for the graph $C_{2D+1}$ (recall Figure 1.1).

We now describe some more elaborate Cayley graph constructions which give rise to efficient broadcast graphs. The next well-known theorem (folklore) follows from using the complete graph $K_2$ on two vertices along with Theorem 7. The proof given below, however, explicitly features a Cayley graph construction.

**Theorem 9** *The hypercube $Q_\Delta$ is an optimal $(\Delta, \Delta)$ broadcast graph.*
**Proof.** The hypercubes $Q_n$ can be represented as a Cayley graph using the abelian group $(Z_2)^n$ with generators $\{e_i \mid 1 \leq i \leq n\}$ where $e_i = (\overbrace{0,\ldots,0}^{i-1},1,\overbrace{0,\ldots,0}^{n-i})$. Since Cayley graphs are vertex symmetric, it suffices to show that $broadcast((0,\ldots,0)) = n$. For the routing scheme, all vertices with the broadcast message route by using generator $e_i$ at time $i$. The first message is sent to vertex $(0,\ldots,0)+e_1 = (1,0,\ldots,0)$. After the $i^{\text{th}}$ broadcast time all of the following vertices will have seen the message.

$$\{(x_1, x_2, \ldots, x_i, 0, \ldots, 0) \mid x_j = \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right\} \text{ for } 1 \leq j \leq i\}$$

At time $n$ every vertex in the hypercube will have received the message. Hence, the broadcast time is equal to the degree. Since $2^n$ is the order of $Q_n$ and also equals the upper bound for vertices in a broadcast network, the hypercube $Q_n$ is an optimal broadcast graph. □

**Corollary 10** *There exists an optimal* $(\Delta,\ T)$ *broadcast graph for all* $T \leq \Delta$.
**Proof.** The upper bound for a (Δ, T) broadcast graph for all $T \leq \Delta$ is $2^T$. Since the hypercubes of Theorem 9 give broadcast graphs of same order with less maximum degree, the hypercube $Q_T$ is also an optimal (Δ, T) broadcast graph. □

The above corollary indicates that broadcast graphs with a smaller time constraint than maximum degree are not very interesting. In this case, a better problem might be to minimize the number of edges while still obtaining an optimal $(\Delta, T)$ broadcast graph.

Some progress above the diagonal in our $(\Delta, T)$ broadcast graph table has been made. The following theorem by the author in collaboration with Faber shows that optimal broadcast graphs exist where the broadcast time is one greater than the maximum degree.

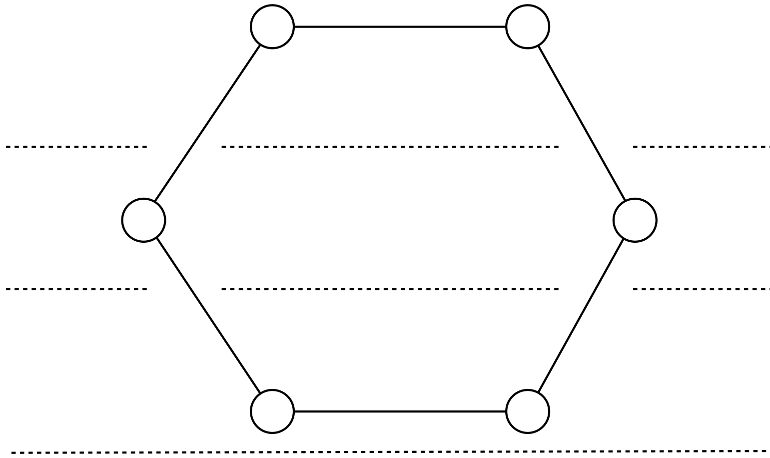

Figure 4.2: Illustrating the $(2,3)$ broadcast graph construction.

**Theorem 11** *The Cayley graphs from the dihedral groups* $D_{n_\Delta}(n_\Delta = 2^\Delta - 1)$ *with generators* $w, wx^1, wx^3, \ldots, wx^{2^{\Delta-1}-1}$ *where* $w^2 = e, wxw^{-1} = x^{-1}$ *and* $x^{2^\Delta - 1} = e$, *form optimal* $(\Delta, \Delta+1)$ *broadcast graphs.*

**Proof.** First note that each of the generators are involutions since

$$(wx^i)^2 = (wx^i)(wx^i) = (wx^i w)x^i = x^{-i}x^i = e.$$

So these generators do indeed form an undirected Cayley graph. The routing for these Cayley graphs will be done by using generator $i$ modulo $\Delta$ at time $i$, starting with $w$ as the first generator $(i = 0)$.

Before proving the general version of this theorem we illustrate it by building the simple $(2,3)$ broadcast graph (Figure 4.2). For this graph, the generators are $w$ and $wx$. Let $V_i = \{v \mid \text{ vertex } v \text{ has seen message at time } i\ \}$. Start broadcasting at the identity, $V_0 = \{e = w^0 = x^0\}$. Using $w$ as the first generator yields $V_1 = V_0 \cup \{w\} = \{e, w\}$. Using $wx$ as the second generator gives

$$V_2 = V_1 \cup \{wx, w^2x = x\} = \{e, w, x, wx\}.$$

Finally reusing the generator $w$ for the third and final broadcast time yields

$$V_3 = V_2 \cup \{wxw = x^{-1} = x^2, xw = wx^{-1} = wx^2\} = D_3.$$

This shows that the broadcast time is 3.

Now the complete theorem is proved. Start with a Cayley graph $D_{n_\Delta}$ with the generators $\{wx^{2^i-1} \mid 0 \le i \le \Delta - 1\}$. With resemblance to the above Cayley graph $D_{n_2}$, we can determine each $V_i$ for $i \le \Delta$. These are represented in the usual group notation.

$$\begin{aligned}
V_0 &= \{e = w^0 = x^0\} \\
V_1 &= V_0 \cup \{w\} \\
V_2 &= V_1 \cup \{wx^1, x^1\} \\
V_3 &= V_2 \cup \{wx^3, x^3, wx^{-1+3}, x^{-1+3}\} \\
&= \textstyle\bigcup_{i=0}^{3} \{wx^i, x^i\} \\
&\ \vdots \\
V_k &= V_{k-1} \cup V_{k-1} \cdot wx^{2^{k-1}-1} \\
&= \textstyle\bigcup_{i=0}^{2^{k-2}-1} \{wx^i, x^i\} \cup \{wx^{2^{k-1}-1-i}, x^{2^{k-1}-1-i}\} \\
&= \textstyle\bigcup_{i=0}^{2^{k-1}-1} \{wx^i, x^i\} \\
&\ \vdots \\
V_\Delta &= V_{\Delta-1} \cup V_{\Delta-1} \cdot wx^{2^{\Delta-1}-1} \\
&= \textstyle\bigcup_{i=0}^{2^{\Delta-1}-1} \{wx^i, x^i\}
\end{aligned}$$

Finally our Cayley graph has one extra time to route. Reusing the generator $w$, the last accumulation set is

$$\begin{aligned}
V_{\Delta+1} &= V_\Delta \cup (V_\Delta \setminus V_1) \cdot w \\
&= \textstyle\bigcup_{i=0}^{2^{\Delta-1}-1} \{wx^i, x^i\} \cup \bigcup_{i=1}^{2^{\Delta-1}-1} \{wx^i w, x^i w\} \\
&= \textstyle\bigcup_{i=0}^{2^{\Delta-1}-1} \{wx^i, x^i\} \cup \bigcup_{i=1}^{2^{\Delta-1}-1} \{wx^{n_\Delta - i}, x^{n_\Delta - i}\} \\
&= \textstyle\bigcup_{i=0}^{2^{\Delta}-1} \{wx^i, x^i\} = D_{n_\Delta}.
\end{aligned}$$

After $\Delta + 1$ broadcasts all the vertices have seen the message.

These broadcast networks are optimal since

$$|D_{n_\Delta}| = 2(2^\Delta - 1) = 2^{\Delta+1} - 2$$

is the upper bound. $\square$

The above theorem gives a new infinite class of optimal broadcast networks. Dihedral groups are a special case of semi-direct products, ($D_n \simeq Z_2 \times_\sigma Z_n$). Setting the group action $\sigma'(1) = -1$ (see Section 2.3) in our computer program was a contributing factor for the authors' arrival at this theorem.

One may wonder if these same dihedral groups can be used for other broadcast graphs. The answer is yes and one graph is illustrated in Figure 4.3. The corresponding routing scheme for this figure is listed in Table 4.3. In both Table 4.3 and Figure 4.3 the group elements are assigned an unique number for readability purposes (mainly for the figure where an element $[a\ b]$ is mapped to $12a + b$). At this time no complete classification of optimal $(\Delta, \Delta + 2)$ broadcast graphs is known.

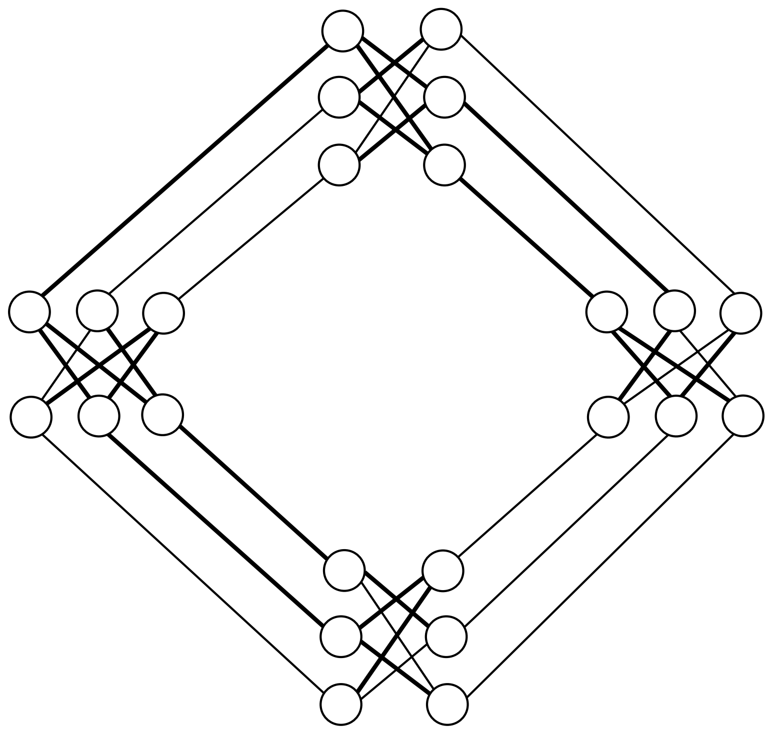

Figure 4.3: An optimal $(3, 5)$ broadcast network.

Table 4.3: One routing scheme for our $(3,5)$ broadcast network.

| Cayley group graph: $Z_2 \times_\sigma Z_{12} \simeq D_{12}$ | |
|---|---|
| Generators | Unique Label |
| [ 1 7 ] | 19 |
| [ 1 11 ] | 23 |
| [ 1 0 ] | 12 |

| Routing scheme from identity element 0. | |
|---|---|
| $0 \to 12$ | 1st broadcast: $\|V_1\| = 2$ |
| $0 \to 23$<br>$12 \to 11$ | 2nd broadcast: $\|V_2\| = 4$ |
| $0 \to 19$<br>$11 \to 13$<br>$12 \to 7$<br>$23 \to 1$ | 3rd broadcast: $\|V_3\| = 8$ |
| $1 \to 22$<br>$7 \to 17$<br>$11 \to 20$<br>$13 \to 10$<br>$19 \to 5$<br>$23 \to 8$ | 4th broadcast: $\|V_4\| = 14$ |
| $1 \to 18$<br>$5 \to 14$<br>$7 \to 16$<br>$8 \to 15$<br>$10 \to 21$<br>$13 \to 6$<br>$17 \to 2$<br>$19 \to 4$<br>$20 \to 3$<br>$22 \to 9$ | 5th broadcast: $\|V_5\| = 24$ |

## 4.3 Further Remarks

We have shown that group theory is useful in designing broadcast networks. To avoid becoming too biased, we now give an example of an optimal non-vertex-symmetric broadcast graph. Figure 4.4 contains a (3,4) broadcast graph with 12 vertices that requires 3 different routing schemes. For comparison our (3,4) broadcast graph (dihedral group $D_7$ of Theorem 11) is given in Figure 4.5. The reader may recognize this broadcast graph as the well-known Heawood graph.

Research on broadcast networks is relatively new compared to the other network problems given in Chapters 2-3. Cayley graphs yielded the largest known graphs in all our network problems. Another possible use of Cayley graphs would be in designing directed broadcast networks. A similar directed graph problem is given in [LP2] with the object to minimize arcs. In Liestman and Peters' paper only standard graph constructions were exhibited. With our success with Cayley graphs, some type of a group theoretical investigation might be useful.

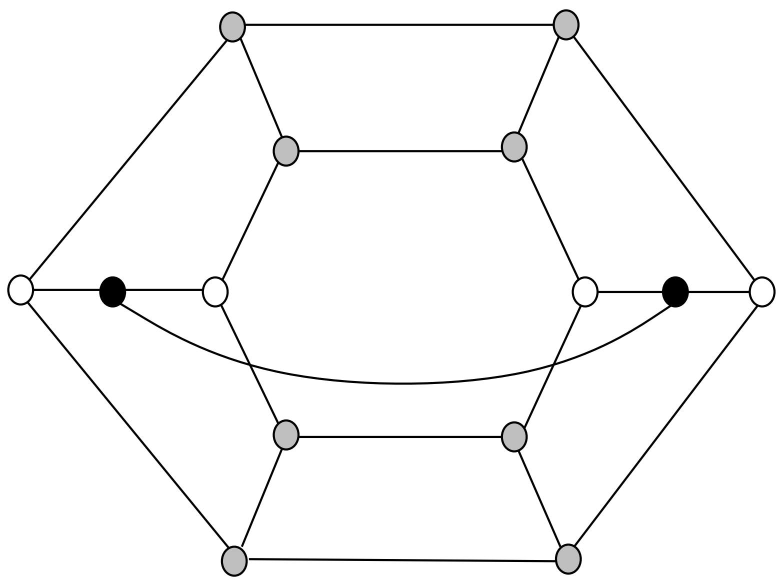

Figure 4.4: A non-symmetric $(3, 4)$ broadcast network.

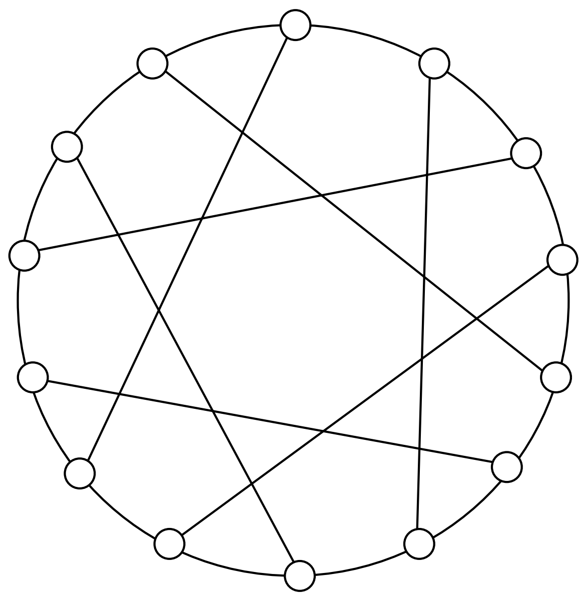

Figure 4.5: The vertex-symmetric Heawood graph is a $(3, 4)$ broadcast network.

# Chapter 5

# A Brief Description of Programs

This chapter describes how our computer search for efficient networks was conducted. Only the basic algorithms will be discussed. Section 5.1 contains the principle algorithm we used for calculating the diameter of Cayley graphs. Section 5.2 concludes with some of our heuristics used for finding Cayley broadcast graphs.

## 5.1 Calculating the Diameter of Cayley Graphs

An algorithm for finding the diameter of our undirected and directed networks is presented. Since our networks are based on a group structure, one would like to avoid representing these graphs explicitly (e.g., adjacency matrices). In [9a, AK2, CCD, Ch1, BDV] and other papers the diameter of a Cayley graph $< G, S >$ is shown to be

$$\max_{a \in G}(\min_{d}\{g_{i_1} g_{i_2} \cdots g_{i_d} = a \mid g_{i_k} \in S\}).$$

The diameter of a Cayley graph can be determined in $O(|G|)$ time while also using a linear amount of memory. The diameter equals the longest path from any root in a breadth-first search tree of a Cayley graph. A feasible algorithm to compute the diameter of a Cayley graph is shown in Figure 5.1. In our program the group identity is taken as the originating root vertex. The vertices at the current depth of the breadth-first spanning tree are stored in a queue while a list of all reached vertices is kept in a bit vector. The algorithm stops when no new nodes can be added to the accumulating bit vector.

For the majority of our computer searches we use both linked lists and hash tables as depicted in Figure 5.1. In some cases, to conserve space, the use of linked lists

**Input:** Group $A$ and generator set $S$.
**Output:** Diameter of Cayley graph $< A, S >$
.

1. Set up data structures.
   (a) Algebraic data type for group $A$.
   (b) Group element bitmap.
   (c) Group element linked list (queue).
2. Start in initial state.
   (a) Add_to_queue(identity $e$).
   (b) Diameter $\leftarrow$ 0.
   (c) Prev_nodes $\leftarrow$ 1.
   (d) New_nodes $\leftarrow$ 0.
3. While (Prev_nodes $> 0$) do the following.
   (a) Get_from_queue(element $x$).
   (b) Decrement Prev_nodes.
   (c) For all generators $g_i$ in $S$ do the following.
      i. Element $y \leftarrow x \cdot g_i$.
      ii. If (not In_bitmap?($y$)) then do the following.
         A. Add_to_queue($y$).
         B. Add_to_bitmap($y$).
         C. Increment New_nodes.
4. If (New_nodes $> 0$) then do the following.
   (a) Increment Diameter.
   (b) Prev_nodes $\leftarrow$ New_nodes.
   (c) New_nodes $\leftarrow$ 0.
   (d) Go back to step 3.
5. Clean up and exit with Diameter.

Figure 5.1: An algorithm for computing diameter of a Cayley graph.

was replaced by hash tables in our implementation. One instance was for our (14,8) graph with 30 million vertices. In this case, the memory available on a non-virtual 16-Mbyte Cray X-MP was scrutinized by allowing only two bits of work space for each of the vertices.

Our fast algorithm has the luxury of working with vertex-symmetric Cayley graphs. For comparison, the diameter of non-vertex-symmetric graphs can be computed in $O(n^3)$ time by Floyd's algorithm. However, for large graphs this amount of computer resource may be infeasible.

## 5.2 Finding Broadcast Times of Cayley Graphs

There is no known easy way to determine the broadcast time of a Cayley graph. In fact for arbitrary bounded-degree graphs ($\Delta \geq 3$), the problem of finding the broadcast time is NP-complete, [Di]. To speed up our search time we chose some heuristics to find upper bounds on the broadcast time of Cayley graphs. The graphs given in Table 4.1 are guaranteed to have the listed broadcast times. However, some of the large graphs may have smaller broadcast times for some undiscovered routing scheme. Three different methods were used to find broadcast Cayley graphs. These heuristics are next presented.

Suppose $< A, S >$ is a Cayley graph with generators $S = \{g_0, g_1, \ldots, g_{k-1}\}$. The simplest broadcasting method is based on an indexed order of the generators. Calls are placed to neighboring vertices using generator $i$ modulo $k$ at time $i$. To determine an upper bound on the broadcast time simply simulate the broadcast scheme with a message originating at the identity element. This assumes that the group $A$ has an easily computable multiplication. The broadcast proceeds until all elements of the group have been generated or until a predetermined maximum broadcast time constraint has been reached.

It is beneficial to separate generators and their inverses in the above index scheme. If not, every call at some broadcasting time will send a message back to a previous sender. (Hence, a wasted call will exist.) Another remedy to this generator ordering problem is to pick generators that are involutions. Our $(\Delta, \Delta+1)$ broadcast graphs used this approach.

Our second method is to choose a set of permutations $\{\pi_i \mid i = 0, 1, \ldots, t\}$ of the generator set $S$. A vertex receiving the message at time $i$ places calls to its neighbors in the order given (by multiplication) by the sequence of generators in $\pi_i$. Like the previous routing scheme, we determine by simulation an upper bound on the broadcast time.

Our last method does not yield easy routing schemes based on orderings of generators but it does find a broadcast tree for the identity element. Since Cayley graphs are vertex symmetric this broadcast scheme can be translated and used for any originating vertex. In building this tree a vertex with a message simply picks a random generator from $S$ and uses this generator to broadcast to a neighbor. Each vertex knows which neighbors it has sent a message, so future picked generators can be restricted. Like above, the broadcast simulation proceeds until all elements of the group have been generated or until a maximum broadcast time constraint has been reached. Most of the large broadcast graphs in Table 4.1 used this technique.

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# Appendix A

# Undirected (Δ,D) graph Results

Table A.1: Some new largest known (Δ, D) Cayley graphs.

| (Δ, D) | Order | Group | Generators | Inverses | Order of Generator |
|---|---|---|---|---|---|
| (4,7) | 1081 | $Z_{23} \times_\sigma Z_{47}$ | [ 9 39 ] | [ 14 4 ] | 23 |
| | | $\sigma'(1)$=25 | [ 6 14 ] | [ 17 41 ] | 23 |
| (4,8) | 2943 | $Z_{27} \times_\sigma Z_{109}$ | [ 6 21 ] | [ 21 31 ] | 9 |
| | | $\sigma'(1)$=97 | [ 17 44 ] | [ 10 19 ] | 27 |
| (4,9) | 7439 | $Z_{43} \times_\sigma Z_{173}$ | [ 33 72 ] | [ 10 145 ] | 43 |
| | | $\sigma'(1)$=16 | [ 39 38 ] | [ 4 140 ] | 43 |
| (4,10) | 15657 | $Z_{51} \times_\sigma Z_{307}$ | [ 11 98 ] | [ 40 109 ] | 51 |
| | | $\sigma'(1)$=275 | [ 36 91 ] | [ 15 1 ] | 17 |

| (Δ, D) | Order | Group | Generators | Inverses | Order of Generator |
|---|---|---|---|---|---|
| (5,7) | 4380 | $Z_{60} \times_\sigma Z_{73}$ | [ 38 60 ] | [ 22 42 ] | 30 |
| | | $\sigma'(1)$=3 | [ 29 47 ] | [ 31 68 ] | 60 |
| | | | [ 30 0 ] | | 2 |
| (5,8) | 12246 | $Z_{78} \times_\sigma Z_{157}$ | [ 22 39 ] | [ 56 10 ] | 39 |
| | | $\sigma'(1)$=25 | [ 42 72 ] | [ 36 43 ] | 13 |
| | | | [ 39 0 ] | | 2 |
| (5,9) | 44310 | $Z_{70} \times_\sigma Z_{633}$ | [ 50 617 ] | [ 20 280 ] | 21 |
| | | $\sigma'(1)$=2 | [ 28 167 ] | [ 42 43 ] | 15 |
| | | | [ 35 0 ] | | 2 |
| (5,10) | 123992 | $GF^*(89) \times_\sigma GF(1409)$ | [ 50 1287 ] | [ 73 375 ] | 22 |
| | | | [ 46 723 ] | [ 60 955 ] | 88 |
| | | | [ 88 0 ] | | 2 |

| (Δ, D) | Order | Group | Generators | Inverses | Order of Generator |
|---|---|---|---|---|---|
| (6,4) | 335 | $Z_5 \times_\sigma Z_{71}$ | [ 4 17 ] | [ 1 5 ] | 5 |
| | | $\sigma'(1)$=54 | [ 1 6 ] | [ 4 63 ] | 5 |
| | | | [ 2 14 ] | [ 3 54 ] | 5 |
| (6,5) | 1088 | $Z_{64} \times_\sigma Z_{17}$ | [ 33 1 ] | [ 31 11 ] | 64 |
| | | $\sigma'(1)$=3 | [ 30 14 ] | [ 34 10 ] | 32 |
| | | | [ 57 10 ] | [ 7 9 ] | 64 |
| (6,6) | 14878 | $Z_{86} \times_\sigma Z_{173}$ | [ 38 5 ] | [ 48 103 ] | 43 |
| | | $\sigma'(1)$=4 | [ 2 23 ] | [ 84 31 ] | 43 |
| | | | [ 49 162 ] | [ 37 28 ] | 86 |
| (6,8) | 53368 | $Z_{56} \times_\sigma Z_{953}$ | [ 35 226 ] | [ 21 304 ] | 8 |
| | | $\sigma'(1)$=86 | [ 31 865 ] | [ 25 661 ] | 56 |
| | | | [ 44 853 ] | [ 12 662 ] | 14 |
| (6,9) | 221100 | $Z_{132} \times_\sigma Z_{1675}$ | [ 128 598 ] | [ 4 1202 ] | 825 |
| | | $\sigma'(1)$=2 | [ 7 1483 ] | [ 125 19 ] | 132 |
| | | | [ 87 1278 ] | [ 45 579 ] | 44 |

| (Δ, D) | Order | Group | Generators | Inverses | Order of Generator |
|---|---|---|---|---|---|
| (7,4) | 506 | $Z_{22} \times_\sigma Z_{23}$ | [ 4 13 ] | [ 18 14 ] | 11 |
| | | $\sigma'(1)$=5 | [ 8 10 ] | [ 14 8 ] | 11 |
| | | | [ 10 17 ] | [ 12 16 ] | 11 |
| | | | [ 11 0 ] | | 2 |
| (7,5) | 2460 | $Z_{60} \times_\sigma Z_{41}$ | [ 54 35 ] | [ 6 24 ] | 10 |
| | | $\sigma'(1)$=36 | [ 13 9 ] | [ 47 16 ] | 60 |
| | | | [ 3 2 ] | [ 57 1 ] | 20 |
| | | | [ 30 0 ] | | 2 |
| (7,7) | 41024 | $Z_{64} \times_\sigma Z_{641}$ | [ 36 235 ] | [ 28 555 ] | 16 |
| | | $\sigma'(1)$=77 | [ 59 198 ] | [ 5 249 ] | 64 |
| | | | [ 45 478 ] | [ 19 220 ] | 64 |
| | | | [ 32 0 ] | | 2 |
| (7,8) | 150150 | $Z_{150} \times_\sigma Z_{1001}$ | [ 95 870 ] | [ 55 615 ] | 330 |
| | | $\sigma'(1)$=2 | [ 28 365 ] | [ 122 166 ] | 75 |
| | | | [ 23 665 ] | [ 127 525 ] | 150 |
| | | | [ 75 0 ] | | 2 |

| $(\Delta, D)$ | Order | Group | Generators | Inverses | Order of Generator |
|---|---|---|---|---|---|
| (8,3) | 203 | $Z_7 \times_\sigma Z_{29}$ | [ 4 21 ] | [ 3 27 ] | 7 |
| | | $\sigma'(1)$=16 | [ 1 20 ] | [ 6 6 ] | 7 |
| | | | [ 3 7 ] | [ 4 28 ] | 7 |
| | | | [ 3 6 ] | [ 4 24 ] | 7 |
| (8,4) | 915 | $Z_{15} \times_\sigma Z_{61}$ | [ 2 22 ] | [ 13 49 ] | 15 |
| | | $\sigma'(1)$=16 | [ 6 46 ] | [ 9 16 ] | 5 |
| | | | [ 2 45 ] | [ 13 42 ] | 15 |
| | | | [ 7 14 ] | [ 8 56 ] | 15 |
| (8,5) | 4108 | $Z_{52} \times_\sigma Z_{79}$ | [ 3 15 ] | [ 49 58 ] | 52 |
| | | $\sigma'(1)$=27 | [ 47 66 ] | [ 5 43 ] | 52 |
| | | | [ 24 28 ] | [ 28 49 ] | 13 |
| | | | [ 31 23 ] | [ 21 50 ] | 52 |
| (8,7) | 104808 | $Z_{264} \times_\sigma Z_{397}$ | [ 3 177 ] | [ 261 257 ] | 88 |
| | | $\sigma'(1)$=125 | [ 116 325 ] | [ 148 262 ] | 66 |
| | | | [ 93 161 ] | [ 171 36 ] | 88 |
| | | | [ 169 373 ] | [ 95 374 ] | 264 |
| (8,8) | 481179 | $Z_{321} \times_\sigma Z_{1499}$ | [ 137 795 ] | [ 184 978 ] | 321 |
| | | $\sigma'(1)$=1294 | [ 134 1164 ] | [ 187 1123 ] | 321 |
| | | | [ 11 639 ] | [ 310 348 ] | 321 |
| | | | [ 154 938 ] | [ 167 836 ] | 321 |

| (Δ, D) | Order | Group | Generators | Inverses | Order of Generator |
|---|---|---|---|---|---|
| (9,4) | 1254 | Subgroup of | [101 6 ] | [ 67 4 ] | 66 |
| | | $GF^*(199) \times_\sigma GF(19)$ | [147 8 ] | [ 88 12 ] | 66 |
| | | with index 3 | [191 16 ] | [174 17 ] | 66 |
| | | | [117 2 ] | [182 5 ] | 33 |
| | | | [198 0 ] | | 2 |
| (9,5) | 6890 | $Z_{130} \times_\sigma Z_{53}$ | [ 128 41 ] | [ 2 33 ] | 65 |
| | | $\sigma'(1)$=4 | [ 89 42 ] | [ 41 36 ] | 130 |
| | | | [ 16 35 ] | [ 114 8 ] | 65 |
| | | | [ 119 45 ] | [ 11 26 ] | 130 |
| | | | [ 65 0 ] | | 2 |
| (9,7) | 217622 | $Z_{466} \times_\sigma Z_{467}$ | [ 291 202 ] | [ 175 154 ] | 466 |
| | | $\sigma'(1)$=2 | [ 178 219 ] | [ 288 208 ] | 233 |
| | | | [ 422 463 ] | [ 44 160 ] | 233 |
| | | | [ 246 326 ] | [ 220 352 ] | 233 |
| | | | [ 233 0 ] | | 2 |
| (9,9) | 4965098 | Subgroup of | [ 16 2739 ] | [ 1950 2893 ] | 557 |
| | | $GF^*(4457) \times_\sigma GF(4457)$ | [ 2519 1897 ] | [ 3337 3108 ] | 557 |
| | | with index 4 | [ 3138 4366 ] | [ 1970 990 ] | 557 |
| | | | [ 65 1922 ] | [ 480 39 ] | 557 |
| | | | [ 4456 0 ] | | 2 |

| (Δ, D) | Order | Group | Generators | Inverses | Order of Generator |
|---|---|---|---|---|---|
| (10,7) | 490052 | Subgroup of | [ 212 13 ] | [ 658 1150 ] | 101 |
| | | $GF^*(1213) \times_\sigma GF(1213)$ | [ 118 487 ] | [ 257 993 ] | 202 |
| | | with index 3 | [ 1121 1009 ] | [ 857 156 ] | 404 |
| | | | [ 343 738 ] | [ 679 1080 ] | 101 |
| | | | [ 1042 104 ] | [ 986 561 ] | 101 |
| (10,8) | 2399049 | $Z_{1341} \times_\sigma Z_{1789}$ | [ 1254 942 ] | [ 87 81 ] | 447 |
| | | $\sigma'(1)$=1296 | [ 33 1151 ] | [ 1308 130 ] | 447 |
| | | | [ 59 313 ] | [ 1282 230 ] | 1341 |
| | | | [ 1331 715 ] | [ 10 1725 ] | 1341 |
| | | | [ 915 59 ] | [ 426 435 ] | 447 |

| (Δ, D) | Order | Group | Generators | Inverses | Order of Generator |
|---|---|---|---|---|---|
| (11,5) | 16578 | $Z_{54} \times_\sigma Z_{307}$ | [ 20 230 ] | [ 34 165 ] | 27 |
| | | $\sigma'(1)$=139 | [ 19 1 ] | [ 35 53 ] | 54 |
| | | | [ 16 167 ] | [ 38 270 ] | 27 |
| | | | [ 25 51 ] | [ 29 208 ] | 54 |
| | | | [ 17 306 ] | [ 37 139 ] | 54 |
| | | | [ 27 0 ] | | 2 |
| (11,7) | 914414 | $GF^*(479) \times_\sigma GF(1913)$ | [ 395 439 ] | [ 268 609 ] | 478 |
| | | | [ 350 95 ] | [ 453 74 ] | 239 |
| | | | [ 19 755 ] | [ 353 555 ] | 478 |
| | | | [ 295 1783 ] | [ 164 1746 ] | 478 |
| | | | [ 437 183 ] | [ 57 559 ] | 478 |
| | | | [ 478 0 ] | | 2 |

| (Δ, D) | Order | Group | Generators | Inverses | Order of Generator |
|---|---|---|---|---|---|
| (12,5) | 26268 | $Z_{132} \times_\sigma Z_{199}$ | [ 92 173 ] | [ 40 9 ] | 33 |
| | | $\sigma'(1)$=27 | [ 85 55 ] | [ 47 137 ] | 132 |
| | | | [ 82 105 ] | [ 50 32 ] | 66 |
| | | | [ 21 56 ] | [ 111 196 ] | 44 |
| | | | [ 90 98 ] | [ 42 191 ] | 22 |
| | | | [ 20 96 ] | [ 112 117 ] | 33 |
| (12,7) | 1732514 | $Z_{658} \times_\sigma Z_{2633}$ | [ 179 1184 ] | [ 479 2385 ] | 658 |
| | | $\sigma'(1)$=81 | [ 456 1224 ] | [ 202 2151 ] | 329 |
| | | | [ 237 670 ] | [ 421 1665 ] | 658 |
| | | | [ 256 941 ] | [ 402 725 ] | 329 |
| | | | [ 111 605 ] | [ 547 1433 ] | 658 |
| | | | [ 206 2179 ] | [ 452 692 ] | 329 |

| (Δ, D) | Order | Group | Generators | Inverses | Order of Generator |
|---|---|---|---|---|---|
| (13,5) | 33354 | $Z_{306} \times_\sigma Z_{109}$ | [ 195 13 ] | [ 111 40 ] | 102 |
| | | $\sigma'(1)$=4 | [ 202 54 ] | [ 104 33 ] | 153 |
| | | | [ 293 91 ] | [ 13 79 ] | 306 |
| | | | [ 88 52 ] | [ 218 40 ] | 153 |
| | | | [ 193 9 ] | [ 113 49 ] | 306 |
| | | | [ 250 22 ] | [ 56 84 ] | 153 |
| | | | [ 153 0 ] | | 2 |
| (13,8) | 13689528 | $GF^*(2617) \times_\sigma GF(5233)$ | [ 1821 1151 ] | [ 240 832 ] | 2616 |
| | | | [ 315 448 ] | [ 2509 2181 ] | 1308 |
| | | | [ 526 3285 ] | [ 403 4717 ] | 2616 |
| | | | [ 1880 3472 ] | [ 1012 2219 ] | 2616 |
| | | | [ 1373 3811 ] | [ 2333 4607 ] | 872 |
| | | | [ 2572 3406 ] | [ 756 1935 ] | 2616 |
| | | | [ 2616 0 ] | | 2 |

| (Δ, D) | Order | Group | Generators | Inverses | Order of Generator |
|---|---|---|---|---|---|
| (14,5) | 51302 | $Z_{226} \times_\sigma Z_{227}$ | [ 175 173 ] | [ 51 190 ] | 226 |
| | | $\sigma'(1)$=2 | [ 132 207 ] | [ 94 39 ] | 113 |
| | | | [ 165 12 ] | [ 61 71 ] | 226 |
| | | | [ 76 184 ] | [ 150 33 ] | 113 |
| | | | [ 38 168 ] | [ 188 192 ] | 113 |
| | | | [ 205 41 ] | [ 21 55 ] | 226 |
| | | | [ 210 24 ] | [ 16 19 ] | 113 |
| (14,8) | 29992052 | $GF^*(5477) \times_\sigma GF(5477)$ | [ 271 2654 ] | [ 384 5063 ] | 2738 |
| | | | [ 915 3003 ] | [ 2107 4091 ] | 1369 |
| | | | [ 1730 724 ] | [ 687 1019 ] | 5476 |
| | | | [ 1335 2648 ] | [ 3598 2476 ] | 5476 |
| | | | [ 4524 4652 ] | [ 4977 3752 ] | 5476 |
| | | | [ 4103 1442 ] | [ 4612 4051 ] | 2738 |
| | | | [ 3574 4723 ] | [ 1865 4098 ] | 1369 |

# Appendix B

# Symmetric (Δ,D) digraph Results

Table B.1: Largest known symmetric (Δ, D) digraphs built from linear groups.

| (Δ, D) | Order | Non-symmetric Largest Order | Group : Index | Generators : Orders |
|---|---|---|---|---|
| ( 2, 3) | 10 | 12 | GL[2,5]:2 | $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}: 5, \begin{bmatrix} 4 & 4 \\ 0 & 1 \end{bmatrix}: 2$ |
| ( 2, 4) | 20 | 24 | GL[2,5]:1 | $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}: 4, \begin{bmatrix} 2 & 4 \\ 0 & 4 \end{bmatrix}: 4$ |
| ( 2, 8) | 171 | 384 | GL[2,19]:2 | $\begin{bmatrix} 1 & 2 \\ 0 & 9 \end{bmatrix}: 9, \begin{bmatrix} 1 & 5 \\ 0 & 17 \end{bmatrix}: 9$ |
| ( 2,11) | 737 | 3072 | GL[2,67]:6 | $\begin{bmatrix} 9 & 59 \\ 0 & 1 \end{bmatrix}: 11, \begin{bmatrix} 59 & 55 \\ 0 & 1 \end{bmatrix}: 11$ |
| ( 2,12) | 1320 | 6144 | GL[2,11]:10 | $\begin{bmatrix} 1 & 5 \\ 5 & 4 \end{bmatrix}: 12, \begin{bmatrix} 5 & 10 \\ 5 & 8 \end{bmatrix}: 11$ |

| (Δ, D) | Order | Non-symmetric Largest Order | Group : Index | Generators : Orders |
|---|---|---|---|---|
| ( 3, 8) | 1751 | 8748 | GL[2,103]:1 | $\begin{bmatrix} 61 & 1 \\ 0 & 1 \end{bmatrix}: 17$ , $\begin{bmatrix} 79 & 91 \\ 0 & 1 \end{bmatrix}: 17$ , $\begin{bmatrix} 72 & 95 \\ 0 & 1 \end{bmatrix}: 17$ |
| ( 3, 9) | 3502 | 26244 | GL[2,103]:3 | $\begin{bmatrix} 100 & 80 \\ 0 & 1 \end{bmatrix}: 17$ , $\begin{bmatrix} 66 & 34 \\ 0 & 1 \end{bmatrix}: 17$ , $\begin{bmatrix} 22 & 101 \\ 0 & 1 \end{bmatrix}: 34$ |
| ( 3,10) | 8736 | 78732 | GL[2,13]:3 | $\begin{bmatrix} 8 & 7 \\ 8 & 12 \end{bmatrix}: 7$ , $\begin{bmatrix} 7 & 4 \\ 0 & 3 \end{bmatrix}: 12$ , $\begin{bmatrix} 2 & 8 \\ 5 & 1 \end{bmatrix}: 14$ |
| ( 3,11) | 24360 | 236196 | GL[2,29]:14 | $\begin{bmatrix} 15 & 24 \\ 26 & 3 \end{bmatrix}: 7$ , $\begin{bmatrix} 23 & 20 \\ 1 & 11 \end{bmatrix}: 5$ , $\begin{bmatrix} 0 & 13 \\ 20 & 21 \end{bmatrix}: 15$ |
| ( 3,12) | 61560 | 708588 | GL[2,19]:2 | $\begin{bmatrix} 11 & 12 \\ 8 & 11 \end{bmatrix}: 18$ , $\begin{bmatrix} 0 & 4 \\ 18 & 14 \end{bmatrix}: 18$ , $\begin{bmatrix} 17 & 4 \\ 10 & 3 \end{bmatrix}: 60$ |

| $(\Delta, \mathrm{D})$ | Order | Non-symmetric Largest Order | Group : Index | Generators : Orders |
|---|---|---|---|---|
| ( 4, 4) | 136 | 320 | GL[2,17]:2 | $\begin{bmatrix} 15 & 13 \\ 0 & 1 \end{bmatrix}: 8$, $\begin{bmatrix} 4 & 14 \\ 0 & 1 \end{bmatrix}: 4$, $\begin{bmatrix} 13 & 13 \\ 0 & 1 \end{bmatrix}: 4$, $\begin{bmatrix} 9 & 13 \\ 0 & 1 \end{bmatrix}: 8$ |
| ( 4, 8) | 9792 | 81920 | GL[2,17]:8 | $\begin{bmatrix} 11 & 11 \\ 3 & 0 \end{bmatrix}: 8$, $\begin{bmatrix} 3 & 9 \\ 12 & 8 \end{bmatrix}: 8$, $\begin{bmatrix} 10 & 5 \\ 9 & 1 \end{bmatrix}: 36$, $\begin{bmatrix} 15 & 11 \\ 14 & 0 \end{bmatrix}: 16$ |
| ( 4, 9) | 32928 | 327680 | GL[3,7]:3 | $\begin{bmatrix} 2 & 0 & 3 \\ 5 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}: 3$, $\begin{bmatrix} 3 & 4 & 6 \\ 3 & 6 & 5 \\ 0 & 0 & 1 \end{bmatrix}: 6$, $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 6 & 4 \\ 0 & 0 & 1 \end{bmatrix}: 14$, $\begin{bmatrix} 4 & 4 & 1 \\ 6 & 4 & 5 \\ 0 & 0 & 1 \end{bmatrix}: 16$ |
| ( 4,10) | 105456 | 1310720 | GL[3,13]:3 | $\begin{bmatrix} 1 & 1 & 12 \\ 0 & 7 & 1 \\ 0 & 0 & 2 \end{bmatrix}: 12$, $\begin{bmatrix} 1 & 11 & 6 \\ 0 & 9 & 5 \\ 0 & 0 & 2 \end{bmatrix}: 12$, $\begin{bmatrix} 1 & 4 & 9 \\ 0 & 7 & 8 \\ 0 & 0 & 3 \end{bmatrix}: 12$, $\begin{bmatrix} 1 & 7 & 0 \\ 0 & 12 & 5 \\ 0 & 0 & 1 \end{bmatrix}: 26$ |
| ( 4,12) | 688800 | 20971520 | GL[2,41]:4 | $\begin{bmatrix} 3 & 31 \\ 14 & 35 \end{bmatrix}: 28$, $\begin{bmatrix} 36 & 28 \\ 16 & 19 \end{bmatrix}: 40$, $\begin{bmatrix} 15 & 8 \\ 21 & 15 \end{bmatrix}: 40$, $\begin{bmatrix} 3 & 6 \\ 0 & 35 \end{bmatrix}: 40$ |

| $(\Delta, \mathrm{D})$ | Order | Non-symmetric Largest Order | Group : Index | Generators : Orders |
|---|---|---|---|---|
| ( 5, 8) | 50616 | 468750 | GL[2,37]:36 | $\begin{bmatrix} 31 & 32 \\ 14 & 30 \end{bmatrix}: 19$, $\begin{bmatrix} 12 & 3 \\ 11 & 9 \end{bmatrix}: 36$, $\begin{bmatrix} 16 & 26 \\ 26 & 3 \end{bmatrix}: 19$, $\begin{bmatrix} 19 & 16 \\ 33 & 22 \end{bmatrix}: 36$, $\begin{bmatrix} 21 & 18 \\ 20 & 26 \end{bmatrix}: 19$ |
| ( 5, 10) | 688800 | 1178750 | GL[2,41]:4 | $\begin{bmatrix} 34 & 1 \\ 8 & 8 \end{bmatrix}: 35$, $\begin{bmatrix} 19 & 9 \\ 32 & 24 \end{bmatrix}: 60$, $\begin{bmatrix} 16 & 15 \\ 20 & 28 \end{bmatrix}: 40$, $\begin{bmatrix} 28 & 0 \\ 28 & 17 \end{bmatrix}: 40$, $\begin{bmatrix} 17 & 16 \\ 21 & 15 \end{bmatrix}: 40$ |
| ( 5, 12) | 6408480 | $5^{12} + 5^{11}$ | GL[2,79]:6 | $\begin{bmatrix} 76 & 3 \\ 5 & 45 \end{bmatrix}: 78$, $\begin{bmatrix} 68 & 68 \\ 53 & 37 \end{bmatrix}: 13$, $\begin{bmatrix} 49 & 11 \\ 57 & 10 \end{bmatrix}: 2054$, $\begin{bmatrix} 43 & 73 \\ 62 & 44 \end{bmatrix}: 1040$, $\begin{bmatrix} 60 & 35 \\ 48 & 52 \end{bmatrix}: 1040$ |

| (Δ, D) | Order | Non-symmetric Largest Order | Group : Index | Generators : Orders |
|---|---|---|---|---|
| ( 6, 8) | 151848 | 1959552 | GL[2,37]:12 | |

$$\begin{bmatrix} 34 & 21 \\ 6 & 4 \end{bmatrix}: 57, \begin{bmatrix} 7 & 10 \\ 31 & 18 \end{bmatrix}: 19, \begin{bmatrix} 7 & 14 \\ 1 & 18 \end{bmatrix}: 19,$$

$$\begin{bmatrix} 21 & 34 \\ 16 & 7 \end{bmatrix}: 36, \begin{bmatrix} 8 & 32 \\ 7 & 20 \end{bmatrix}: 36, \begin{bmatrix} 19 & 15 \\ 16 & 1 \end{bmatrix}: 36$$

| (Δ, D) | Order | Non-symmetric Largest Order | Group : Index | Generators : Orders |
|---|---|---|---|---|
| ( 7, 8) | 410640 | 6588344 | GL[2,59]:29 | |

$$\begin{bmatrix} 33 & 57 \\ 10 & 3 \end{bmatrix}: 58, \begin{bmatrix} 57 & 42 \\ 57 & 12 \end{bmatrix}: 30, \begin{bmatrix} 34 & 17 \\ 54 & 1 \end{bmatrix}: 58, \begin{bmatrix} 6 & 43 \\ 19 & 38 \end{bmatrix}: 15,$$

$$\begin{bmatrix} 50 & 9 \\ 28 & 44 \end{bmatrix}: 58, \begin{bmatrix} 14 & 0 \\ 9 & 21 \end{bmatrix}: 58, \begin{bmatrix} 45 & 52 \\ 27 & 5 \end{bmatrix}: 60$$

| (Δ, D) | Order | Non-symmetric Largest Order | Group : Index | Generators : Orders |
|---|---|---|---|---|
| ( 8, 8) | 680760 | $8^8 + 8^7$ | GL[2,61]:20 | |

$$\begin{bmatrix} 44 & 54 \\ 5 & 30 \end{bmatrix}: 15, \begin{bmatrix} 27 & 21 \\ 53 & 53 \end{bmatrix}: 30, \begin{bmatrix} 24 & 36 \\ 23 & 32 \end{bmatrix}: 31, \begin{bmatrix} 31 & 1 \\ 29 & 60 \end{bmatrix}: 20,$$

$$\begin{bmatrix} 2 & 31 \\ 35 & 17 \end{bmatrix}: 93, \begin{bmatrix} 55 & 20 \\ 41 & 34 \end{bmatrix}: 366, \begin{bmatrix} 42 & 23 \\ 25 & 14 \end{bmatrix}: 60, \begin{bmatrix} 24 & 52 \\ 37 & 7 \end{bmatrix}: 93$$

| (Δ, D) | Order | Non-symmetric Largest Order | Group : Index | Generators : Orders |
|---|---|---|---|---|
| ( 9, 8) | 1822176 | $9^8 + 9^7$ | GL[2,37]:1 | |

$$\begin{bmatrix} 35 & 10 \\ 25 & 1 \end{bmatrix}: 36, \begin{bmatrix} 30 & 0 \\ 6 & 31 \end{bmatrix}: 36, \begin{bmatrix} 16 & 25 \\ 30 & 36 \end{bmatrix}: 36,$$

$$\begin{bmatrix} 6 & 8 \\ 4 & 29 \end{bmatrix}: 12, \begin{bmatrix} 35 & 9 \\ 13 & 28 \end{bmatrix}: 9, \begin{bmatrix} 36 & 27 \\ 23 & 17 \end{bmatrix}: 36,$$

$$\begin{bmatrix} 18 & 18 \\ 5 & 28 \end{bmatrix}: 18, \begin{bmatrix} 17 & 3 \\ 13 & 19 \end{bmatrix}: 18, \begin{bmatrix} 7 & 17 \\ 17 & 20 \end{bmatrix}: 36$$

Table B.2: Largest known symmetric (Δ, D) digraphs built from semi-direct products of cyclic groups.

| (Δ, D) | Order | Group | Generators | Order of Generator |
|---|---|---|---|---|
| (2,5) | 27 | $Z_3 \times_\sigma Z_9$ | [ 0 1 ] | 9 |
| | | $\sigma'(1)$=2 | [ 1 2 ] | 9 |
| (3,3) | 27 | $Z_3 \times_\sigma Z_9$ | [ 1 1 ] | 9 |
| | | $\sigma'(1)$=2 | [ 0 1 ] | 9 |
| | | | [ 1 8 ] | 9 |
| (3,5) | 155 | $Z_5 \times_\sigma Z_{31}$ | [ 2 16 ] | 5 |
| | | $\sigma'(1)$=3 | [ 1 18 ] | 5 |
| | | | [ 2 13 ] | 5 |
| (3,6) | 333 | $Z_9 \times_\sigma Z_{37}$ | [ 1 3 ] | 9 |
| | | $\sigma'(1)$=2 | [ 2 2 ] | 9 |
| | | | [ 4 22 ] | 9 |
| (4,5) | 420 | $Z_{12} \times_\sigma Z_{35}$ | [ 9 10 ] | 28 |
| | | $\sigma'(1)$=2 | [ 7 14 ] | 12 |
| | | | [ 3 19 ] | 28 |
| | | | [ 10 12 ] | 6 |
| (4,6) | 1100 | $Z_{20} \times_\sigma Z_{55}$ | [ 2 42 ] | 10 |
| | | $\sigma'(1)$=2 | [ 3 16 ] | 20 |
| | | | [ 18 5 ] | 10 |
| | | | [ 11 33 ] | 20 |
| (5,5) | 889 | $Z_7 \times_\sigma Z_{127}$ | [ 6 13 ] | 7 |
| | | $\sigma'(1)$=3 | [ 2 109 ] | 7 |
| | | | [ 3 42 ] | 7 |
| | | | [ 2 0 ] | 7 |
| | | | [ 3 4 ] | 7 |
| (5,6) | 3197 | $Z_{23} \times_\sigma Z_{139}$ | [ 18 90 ] | 23 |
| | | $\sigma'(1)$=2 | [ 4 68 ] | 23 |
| | | | [ 15 61 ] | 23 |
| | | | [ 22 89 ] | 23 |
| | | | [ 10 126 ] | 23 |
| (6,6) | 7224 | $Z_{42} \times_\sigma Z_{172}$ | [ 4 44 ] | 21 |
| | | $\sigma'(1)$=3 | [ 11 27 ] | 42 |
| | | | [ 27 140 ] | 14 |
| | | | [ 26 65 ] | 84 |
| | | | [ 14 169 ] | 12 |
| | | | [ 13 147 ] | 42 |

# Appendix C

# Broadcast ($\Delta$,D) graph Results

Table C.1: A collection of largest known ($\Delta$, D) broadcast graphs.

| ($\Delta$, D) | Order | Group | Generators | Inverses | Order of Generator |
|---|---|---|---|---|---|
| (3,4) | 14 | $Z_2 \times_\sigma Z_7$ | [ 1 0 ] | | 2 |
| | | $\sigma'(1)$=6 | [ 1 1 ] | | 2 |
| | | | [ 1 3 ] | | 2 |
| (3,5) | 24 | $Z_2 \times_\sigma Z_{12}$ | [ 1 0 ] | | 2 |
| | | $\sigma'(1)$=11 | [ 1 1 ] | | 2 |
| | | | [ 1 3 ] | | 2 |
| (3,7) | 60 | $Z_{12} \times_\sigma Z_5$ | [ 6 2 ] | | 2 |
| | | $\sigma'(1)$=2 | [ 7 4 ] | [ 5 2 ] | 12 |
| (3,8) | 84 | $Z_6 \times_\sigma Z_{14}$ | [ 1 9 ] | [ 5 11 ] | 6 |
| | | $\sigma'(1)$=3 | [ 3 0 ] | | 2 |
| (3,9) | 126 | $Z_{18} \times_\sigma Z_7$ | [ 13 1 ] | [ 5 2 ] | 18 |
| | | $\sigma'(1)$=3 | [ 9 0 ] | | 2 |
| (3,10) | 156 | $Z_{12} \times_\sigma Z_{13}$ | [ 7 1 ] | [ 5 7 ] | 12 |
| | | $\sigma'(1)$=2 | [ 6 0 ] | | 2 |

| $(\Delta$, D) | Order | Group | Generators | Inverses | Order of Generator |
|---|---|---|---|---|---|
| (4,5) | 30 | $Z_2 \times_\sigma Z_{15}$ | [ 1 0 ] | | 2 |
| | | $\sigma'(1)$=14 | [ 1 1 ] | | 2 |
| | | | [ 1 3 ] | | 2 |
| | | | [ 1 7 ] | | 2 |
| (4,7) | 90 | $Z_6 \times_\sigma Z_{45}$ | [ 3 30 ] | [ 3 15 ] | 6 |
| | | $\sigma'(1)$=4 | [ 0 13 ] | [ 0 32 ] | 45 |
| (4,8) | 148 | $Z_4 \times_\sigma Z_{37}$ | [ 1 20 ] | [ 3 28 ] | 4 |
| | | $\sigma'(1)$=31 | [ 2 27 ] | | 2 |
| | | | [ 2 0 ] | | 2 |
| (4,9) | 253 | $Z_{11} \times_\sigma Z_{23}$ | [ 6 7 ] | [ 5 6 ] | 11 |
| | | $\sigma'(1)$=2 | [ 9 7 ] | [ 2 18 ] | 11 |
| (4,10) | 272 | $Z_8 \times_\sigma Z_{34}$ | [ 6 7 ] | [ 5 6 ] | 11 |
| | | $\sigma'(1)$=3 | [ 7 15 ] | [ 1 1 ] | 8 |

| $(\Delta$, D) | Order | Group | Generators | Inverses | Order of Generator |
|---|---|---|---|---|---|
| (5,6) | 62 | $Z_2 \times_\sigma Z_{31}$ | [ 1 0 ] | | 2 |
| | | $\sigma'(1)$=30 | [ 1 1 ] | | 2 |
| | | | [ 1 3 ] | | 2 |
| | | | [ 1 7 ] | | 2 |
| | | | [ 1 15 ] | | 2 |
| (5,7) | 108 | $Z_6 \times_\sigma Z_{18}$ | [ 4 9 ] | [ 2 9 ] | 6 |
| | | $\sigma'(1)$=5 | [ 3 3 ] | | 2 |
| | | | [ 3 17 ] | | 2 |
| | | | [ 3 0 ] | | 2 |
| (5,8) | 186 | $Z_6 \times_\sigma Z_{31}$ | [ 2 14 ] | [ 4 23 ] | 3 |
| | | $\sigma'(1)$=26 | [ 3 6 ] | | 2 |
| | | | [ 3 10 ] | | 2 |
| | | | [ 3 0 ] | | 2 |
| (5,9) | 336 | $Z_6 \times_\sigma Z_{56}$ | [ 5 8 ] | [ 1 32 ] | 6 |
| | | $\sigma'(1)$=3 | [ 2 23 ] | [ 4 41 ] | 24 |
| | | | [ 3 0 ] | | 2 |

| (Δ, D) | Order | Group | Generators | Inverses | Order of Generator |
|---|---|---|---|---|---|
| (6,7) | 126 | $Z_2 \times_\sigma Z_{63}$ | [ 1 0 ] | | 2 |
| | | $\sigma'(1)$=62 | [ 1 1 ] | | 2 |
| | | | [ 1 3 ] | | 2 |
| | | | [ 1 7 ] | | 2 |
| | | | [ 1 15 ] | | 2 |
| | | | [ 1 31 ] | | 2 |
| (6,8) | 220 | $Z_{20} \times_\sigma Z_{11}$ | [ 4 2 ] | [ 16 4 ] | 5 |
| | | $\sigma'(1)$=2 | [ 17 6 ] | [ 3 7 ] | 20 |
| | | | [ 3 10 ] | [ 17 7 ] | 20 |
| (6,9) | 390 | $Z_6 \times_\sigma Z_{65}$ | [ 4 47 ] | [ 2 28 ] | 15 |
| | | $\sigma'(1)$=4 | [ 3 23 ] | | 2 |
| | | | [ 1 39 ] | [ 5 39 ] | 6 |
| | | | [ 3 0 ] | | 2 |
| (6,10) | 750 | $Z_{10} \times_\sigma Z_{75}$ | [ 2 58 ] | [ 8 62 ] | 75 |
| | | $\sigma'(1)$=4 | [ 1 48 ] | [ 9 63 ] | 10 |
| | | | [ 9 51 ] | [ 1 21 ] | 10 |

| (Δ, D) | Order | Group | Generators | Inverses | Order of Generator |
|---|---|---|---|---|---|
| (7,8) | 254 | $Z_2 \times_\sigma Z_{127}$ | [ 1 0 ] | | 2 |
| | | $\sigma'(1)$=126 | [ 1 1 ] | | 2 |
| | | | [ 1 3 ] | | 2 |
| | | | [ 1 7 ] | | 2 |
| | | | [ 1 15 ] | | 2 |
| | | | [ 1 31 ] | | 2 |
| | | | [ 1 63 ] | | 2 |
| (7,9) | 420 | $Z_{30} \times_\sigma Z_{14}$ | [ 17 3 ] | [ 13 5 ] | 30 |
| | | $\sigma'(1)$=3 | [ 20 12 ] | [ 10 8 ] | 3 |
| | | | [ 24 10 ] | [ 6 4 ] | 35 |
| | | | [ 15 0 ] | | 2 |
| (7,10) | 420 | $Z_{16} \times_\sigma Z_{51}$ | [ 14 34 ] | [ 2 17 ] | 24 |
| | | $\sigma'(1)$=5 | [ 6 18 ] | [ 10 42 ] | 8 |
| | | | [ 5 26 ] | [ 11 20 ] | 16 |
| | | | [ 8 0 ] | | 2 |

| (Δ, D) | Order | Group | Generators | Inverses | Order of Generator |
|---|---|---|---|---|---|
| (8,9) | 510 | $Z_2 \times_\sigma Z_{255}$ | [ 1 0 ] | | 2 |
| | | $\sigma'(1)$=254 | [ 1 1 ] | | 2 |
| | | | [ 1 3 ] | | 2 |
| | | | [ 1 7 ] | | 2 |
| | | | [ 1 15 ] | | 2 |
| | | | [ 1 31 ] | | 2 |
| | | | [ 1 63 ] | | 2 |
| | | | [ 1 127 ] | | 2 |
| (8,10) | 880 | $Z_{20} \times_\sigma Z_{44}$ | [ 1 23 ] | [ 19 7 ] | 20 |
| | | $\sigma'(1)$=3 | [ 2 35 ] | [ 18 1 ] | 20 |
| | | | [ 14 19 ] | [ 6 9 ] | 20 |
| | | | [ 0 10 ] | [ 0 34 ] | 22 |